CUBULATED HOLONOMY

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Abstract. Holonomy is well-known in the continuous context in studying parallel transport maps, yet here we explore its discrete analogue in "rolling" polyhedra in loops over combinatorial symmetrical surfaces. Extending previous research conducted at Duke University initially introducing "combinatorial holonomy" in terms of the tetrahedron rolled on triangulated surfaces, we apply their approach to the holonomy of the cube on cubulated surfaces. Further, we recognize parameters of surfaces for the holonomy of the cube as contained within the alternating group A_4 and discover small subgroups outside of A_4 but contained in S_4. Our generalizations allow us to formally categorize families of cubulated annuli and further, families of cubulated tori as having holonomy isomorphic to either A_4 or S_4.

1. Introduction

1.1. **The Combinatorial Approach.** We begin with a review on the holonomy of combinatorial surfaces and the combinatorial holonomy group, as defined by previous research at Duke University. While the holonomy of *continuous* surfaces is a well-explored theme throughout mathematics, it is not until recently that the consequence has been studied *discretely*. Via parallel transport maps, traditional holonomy is often studied in "sliding" a basis along a Riemannian manifold. Here we "roll" our highly symmetrical structures (i.e. polyhedra) in loops on its combinatorial surface counterpart to realize symmetries as induced by the holonomy of the connection. Specifically, we use the combinatorial fundamental group and local contributions to compute the holonomy group of surfaces. Previous research considered the tetrahedron as rolled on triangulated surfaces, and we hope to generalize these results to the cube on cubulated surfaces throughout this paper.

1.2. **Conclusions and Future Directions.** Combinatorial holonomy offers a novel, discrete perspective to the study of holonomy formerly only pursued in the continuous context. Advancing previous research, we have extrapolated combinatorial holonomy as initially considered in the tetrahedral case to the cube on various cubulated surfaces. Further, we succeeded in classifying cubulated annuli and built upon this discovery to categorize families of cubulated tori as having A_4 or S_4 holonomy. As we found that it is relatively easy to find A_4 or S_4 holonomy, yet difficult to find subgroups outside of A_4 and still in S_4, a natural next step may be to attempt to uncover such subgroups on other surfaces. Further, we would like to approach the question of shortest loops for given symmetries of surfaces in understanding elements of the holonomy group. As all previous research in combinatorial holonomy has studied tori in some respect, it would be of particular interest to investigate higher genus surfaces and the possible families of surfaces or symmetries which may arise

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there. Alternatively, one could define rolling and consider symmetries of higher dimensional n-cubes to continue research with cubulated surfaces.

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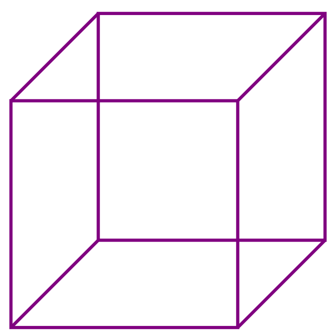
2. Symmetries of the Cube and Cubulated Symmetrical Surfaces

2.1. **Symmetry Group of the Cube.** We begin then with an overview of the symmetry group of the cube, with n-fold axes of rotational symmetry through opposite vertices, opposite edges, and opposite faces forming respective conjugacy classes. Consider first the identity element in its own conjugacy class, as well as the four 4-fold axes through opposite vertices containing 8 symmetries order 3. Further, there are the three 4-fold axes through opposite centroids, giving 9 elements of symmetry (3 of order 2 and 6 of order 4) and the six 2-fold axes through edges giving 6 elements of order 2. Hence we have a total of 24 rotational isometry-preserving symmetries of our cube $\mathcal{C}$ as isomorphic to S_4. The symmetric group S_4 on four elements can be seen as a group permuting the four diagonals of the cube. Note in particular S_4 as all possible orientation-preserving symmetries of the cube and A_4 as its maximal normal subgroup, as it is within these two subgroups that we will most often discover symmetries[Goo15].



2.2. **Cubulated Surfaces.** Many of the following definitions are abstracted from the previous work and revised to define rolling cubes $\mathcal{C}$ on cubulated surfaces S as will be explored throughout this paper.

Definition 2.1. (Cubulated surface.) *We define a cubulated surface $S = (V, E, F)$ as nonempty sets:*

- *V of vertices v.*
- *E of edges $e = \{v_1, v_2\}$ such that v_1, v_2 are distinct elements of V.*
- *F of square faces f of form $f = \{v_1, v_2, v_3, v_4\}$ such that all vertices and all edges are contained in a face f each with 4 distinct vertices and 4 distinct edges forming a four cycle graph.*

We also require that:

- *Every edge e is contained in either 1 or 2 faces $f \in F$.*
- *Every vertex v is contained in a face $f \in F$.*
- *Every 2 faces are disjoint or intersect at a single edge e.*

Definition 2.2. (Adjacency of faces) *A single edge e which is contained in 2 faces f_1 and f_2 asserts that e is an interior edge by which f_1 and f_2 are adjacent. If e is not an interior edge, then it is a boundary edge. Note that if $f_1 \cap f_2 \neq e$ for any edge e, then f_1 and f_2 are not adjacent.*

Further, the set of faces $f \in F$ containing a vertex v can be arranged in a sequence $f_1, f_2, ... f_\delta$ such that f_i and f_δ are adjacent for $1 \leq i < \delta$.

Definition 2.3. (Interior Vertex) *If $\delta \neq 2$ and f_δ is adjacent to f_1 by an interior edge e, then the vertex v is said to be interior.*

Remark. *Should all edges of a surface be interior, then so are all vertices and we have a surface S without boundary. Boundary edges are constructed purely of boundary vertices.*

Definition 2.4. (Connectedness) *A cubulated surface $S = (V, E, F)$ is face-connected if for any $f, f' \in F$ there is a sequence of S-faces $S(f) = (f_1, f_2, ..., f_\delta)$ such that f_i is adjacent to f_{i+1} for $1 \leq i < \delta$ for a connected surface S.* [LMN21]

Remark. *This notion of combinatorial connectedness agrees with the traditional topological definition.*

Definition 2.5. (Automorphism Group) *A bijection $\alpha : V \to V$ is an automorphism of $S = (V, E, F)$ when $\{v_1, v_2, v_3, v_4\} \in F$ if and only if $\{\alpha(v_1), \alpha(v_2), \alpha(v_3), \alpha(v_4)\} \in F$ and α similarly respects edges. Under composition, the set of automorphisms form a group* $\mathrm{Aut}(S)$ [LMN21].

The automorphisms under our considerations may be assumed to be orientation-preserving.

Definition 2.6. (Maximally Symmetric) *A cubulated surface $S = (V, E, F)$ such that $|\mathrm{Aut}(S)| = 8|F|$ is maximally symmetric, as a fixed face of a cubulated surface S can be mapped to S' in at most 8 ways.*

3. The Combinatorial Fundamental Group

Just as we explore the discrete/combinatorial analogue of holonomy, we utilize the combinatorial analogue of the fundamental group in our methods.

3.1. Discrete Analogues of the Fundamental Group.

Definition 3.1. (Path) *A path along S is a sequence of faces $f_i \in F, 1 \leq i \leq n$ such that each $f_i$is adjacent to both faces $f_i - 1$ and $f_i + 1$.*

Definition 3.2. (Loop) *A loop is a path on S crossing a sequence of adjacent faces $f_i \in F$ for $1 \leq i \leq n$ such that $f_1 = f_n$ that starts and ends at the same face.*

Definition 3.3. (Loop Group) *We define the loop group $L_f(S)$ by classes of loops sharing backtracking equivalence under the operation of concatenation $\star$.*

Definition 3.4. (Backtracking Equivalence) *We consider loops $l \simeq l'$ if and only if one can be obtained from the other by replacing some portion (of l in this case) of form $(f_i, f_{i+1}, f_{i+2}, ... f_{i+k}, f_{(i+k)-1}, ... f_{i+2}, f_{i+1}, f_i)$ with f_i.*

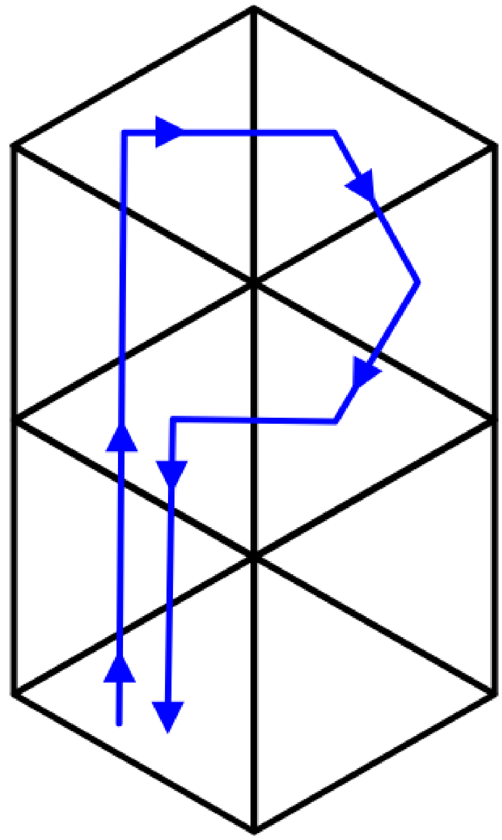


FIGURE 1. Lasso around an interior vertex

To illustrate, we would consider loops:

$$l = (f_1, f_2, f_3, f_4, f_3, f_5, f_1)$$

and

$$l' = (f_1, f_2, f_3, f_5, f_1)$$

equivalent (assuming adjacency of listed faces), where $f_i = f_3$ in this particular example. The group operation is well-defined on equivalence classes of loops.

Remark. *Backtracking allows for inverses, satisfying the requirement of inverse elements for the group $L_f(S)$.*

Theorem 3.1. *Assuming our surface S is path-connected, then $L_f(S)$ is independent of the base face up to isomorphism.*

Proof. This proof follows that of the prior work. [LMN21] □

Definition 3.5. (Lasso) *If h is a path from f_1 to f_n with $v_n \in f_n$ as an interior vertex, and l is the loop around v beginning and ending at f_n whose existence is guaranteed by Definition 2.3, then we have $h^{-1} \star l \star h$ as a lasso based at f_1.*

Definition 3.6. (Nullhomotopic Loop Group) *The subgroup group $C_f(S)$ of $L_f(S)$ generated by the set of all lassos based at f in $L_f(S)$ is the nullhomotopic loop group based at f under* *concatenation of loops. As $C(S)$ is preserved under the isomorphism of 3.1, we choose to omit the base face and reference $C(S)$ as our nullhomotopic loop group.*

Remark. *This definition of nullhomotopy agrees with the standard definition of nullhomotopic loops in a continuous sense; a combinatorial loop on S is nullhomotopic if and only if the same loop, considered instead as a continuous loop in the underlying topological space of S, is nullhomotopic. Similarly, we consider two loops homotopic if their continuous analogues in the underlying topological space are also homotopic.*

Theorem 3.2. *C(S) is a normal subgroup of L(S).*

Proof. Suppose $c \in C(S)$ and $l \in L(S)$. Then lcl^{-1} is homotopic to $l1l$ which is equal to the identity $1 \in C(S)$. Thus lcl^{-1} is nullhomotopic, so $lcl^{-1} \in C(S)$. [LMN21] ☐

Definition 3.7. (Combinatorial Fundamental Group) *We define the fundamental group of a combinatorial symmetric surface S as $\pi_1(S) := L(S)/C(S)$.*

Remark. *This definition agrees with the traditional notion of a fundamental group in that the fundamental group of a surface is isomorphic to the fundamental group of the surface's underlying topological space.*

For further details please reference the appendix.

4. The Combinatorial Holonomy Group

Rolling our cube $\mathcal{C}$ in loops on the surface S not only induces the holonomy of surface and realizes symmetries of $\mathcal{C}$, but also gives rise to the combinatorial holonomy group we define in this section. First we provide the definitions necessary to describe rolling the cube $\mathcal{C}$ on S.

Definition 4.1. (Orientation) *Given a face $f = \{v_1, v_2, v_3, v_4\} \in F$ of a cubulated surface S, an orientation of f is an ordering of the vertices of f (i.e. clockwise or counterclockwise). This ordering depends only on the relative positions of the vertices in the ordering, not on the specific order in which they are listed; that is, the orderings (v_1, v_2, v_3, v_4), (v_2, v_3, v_4, v_1), (v_3, v_4, v_1, v_2), (v_4, v_1, v_2, v_3) are all equivalent and can be denoted $[v_1, v_2, v_3, v_4]$. Similarly, an orientation of an edge $e = \{v_1, v_2\}$ of S is an ordering of the two vertices of e. For each edge, there are two possible orientations $[v_1, v_2]$ and $[v_2, v_1]$.*

Remark. *The orientation of $S = (V, E, F)$ is a choice of orientation for each face $f \in F$ such that whenever 2 faces share an edge, the two faces f and f' induce opposite orientations.*

All of surfaces described throughout this paper are be assumed to be orientable.

Definition 4.2. (Position) *Given a cube $\mathcal{C} = (V', E', F')$ and a cubulated surface $S = (V, E, F)$, we define a position*

$$p = \{(v'_i, v_1), (v'_{i+1}, v_2), (v'_{i+2}, v_3), (v'_{i+3}, v_4)\}.$$

as a map from some face f' of $\mathcal{C}$ to some face f of S that preserves the structure as cyclic graphs. The set of all possible positions given a pair of surfaces $\mathcal{C}, S$ is then $P(\mathcal{C}, S)$.

Definition 4.3. (Face Projections) *There exists natural face projections $\phi' : P(S', S) \to F$ and $\phi : P(S', S) \to F'$ which map a position to the face of S or S' (respectively) which positively aligns with the other surface.*

Definition 4.4. (Connected by single roll) *Consider the position*

$$p = \{(v_1', v_1), (v_2', v_2), (v_3', v_3), (v_4', v_4)\}$$

assuming $\{v_1, v_2\}$, $\{v_1', v_2'\}$ *are internal edges of* S, S' *(respectively). In other words, there exist faces* $f = \{v_1, v_2, v_5, v_6\} \in F$, $f' = \{v_1', v_2', v_5', v_6'\} \in F'$ *such that the position*

$$\tilde{p} = \{(v_1', v_1), (v_2', v_2), (v_5', v_5), (v_6', v_6)\}$$

is said to be connected to p *by a single roll.*

The choice of $\tilde{p}$ is determined by p and the internal edge $e = \{v_1, v_2\}$.

Theorem 4.1. *Let S = (V,E,F) be a connected, maximally symmetric triangulated surface without boundary, S'= (V',E',F') a connected symmetric surface, and* $f' \in F$ *a face of S'. Then each* $p \in P(S', S)_{f'}$ *gives a unique holonomy homomorphism* $h_p : L_f'(S') \to \mathrm{Aut}(S)$. *Moreover,* $h_{\beta(p)} = \beta \circ h_p \circ \beta^{-1}$ *for all* $\beta \in \mathrm{Aut}(S)$.

Proof. The inductive proof follows similarly to that of prior work. [LMN21] □

We now describe the structure of the combinatorial holonomy group and its usefulness in computation of holonomy.

Definition 4.5. (Combinatorial Holonomy Group) *For a maximally symmetric surface* S *without boundary and another symmetric surface* S' *without boundary, then let* $p \in P(S', S)_f'$ *be a position of* S' *over* S *based face* f *of* S. *We define the holonomy group of* S' *over* S *based at* p *as the image* $\mathrm{Hol}_p(S', S) := h_p(L_f(S)) \subset \mathrm{Aut}(S')$.

Definition 4.6. (Restricted Holonomy Group) *Then the restricted holonomy group is the image* $\mathrm{Hol}_{p0}(S, S') := h_p(L_f(S)) \trianglelefteq \mathrm{Hol}_p(S', S)$.

The initial position p will often be suppressed in the notation.

Theorem 4.2. *As* $\mathrm{Hol}_0(S', S) \trianglelefteq \mathrm{Hol}(S', S)$, *then for the quotient* $Q = \mathrm{Hol}(S'S)/\mathrm{Hol}_0(S', S)$, *there exists a unique surjective homorphism* $\psi : \pi_1(S') \to Q$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C(S') & \xrightarrow{i_1} & L(S') & \xrightarrow{q_1} & \pi_1(S') & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h|_{C(S')}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \psi} & & \\
0 & \longrightarrow & \mathrm{Hol}_0(S, S') & \xrightarrow{i_2} & \mathrm{Hol}(S, S') & \xrightarrow{q_2} & Q & \longrightarrow & 0
\end{array}$$

Proof. We see in the diagram above that the top and bottom rows are short exact sequences with i_1, i_2 as inclusion homomorphisms and q_1, q_2 as quotient homomoprhisms. The remainder of this proof is detailed in the prior work. [LMN21] □

Theorem 4.3. *Let S be a connected, maximally symmetric surface without boundary, and S' a connected, symmetric surface. If both S and S' are orientable, then* Hol*(S,S') contains only orientation-preserving automorphisms of S.*

Proof. This proof is detailed in the prior work. [LMN21] □

4.1. **Computing Holonomy.** After computing the restricted holonomy $\mathrm{Hol}_0(S', S)$ generated by the set of all lassos, we can pick a representative loop γ from each holonomy class $[\gamma] \in \pi_1(S)$ and compute the coset $h(\gamma)\mathrm{Hol}_0(S', S)$ such that we have decomposed the holonomy group into the restricted holonomy group $\mathrm{Hol}_0(S', S)$ and the cosets as created by nontrivial homotopy classes of the fundamental group. We can then study holonomy by the restricted holonomy and "fundamental group holonomy".

5. The Cube on Cubulated Surfaces

We are now prepared to direct our focus to computing holonomy of the cube on cubulated surfaces, beginning with a description of effective methods of computation.

Theorem 5.1. *As we assume S to be maximally symmetric, every vertex v of S has the same order. Now suppose every vertex v of S has degree n and let l be a lasso on S around a vertex v. Then $|h(l)| = n/gcd(n, \delta(v'))$.*

Proof. This proof is detailed in the prior work. [LMN21] ☐

The above result of the prior work effectively speeds up the process of computing holonomy, as does the new approach we developed as detailed below.

Definition 5.1. (Face-Adjacent Group) *We define a face-adjacent group $F'(S')$ of a symmetric surface S' as the set of adjacent faces $f' \in F'$ of S' with face projection $\phi : P(S', S) \to F$ such that $p = \{f'_p, f_p\}$ with $\phi'(p) = f'_p \in F'$ and $\phi(p) = f_p \in F$. The action of rolling S' on S then induces a chain of adjacent S'-faces $f'_{p1}, f'_{p2}, ...f'_{p\delta} \in F'$ of the polyhedra S' rolled on the surface S with which we can "trace" the face positions $f_p \in F(S')$ on S in order to realize the symmetry induced.*

Lemma 5.2. *There is a canonical isomorphism $F' \to \mathbb{Z}/\mathbb{Z}n'$ such that rolling S' around a vertex v on S acts as an equivalence class $[n]$.*

Proof. Let the holonomy of S' on S be a subgroup $F' \to \mathbb{Z}/n'\mathbb{Z}$ for $n' = |v'|$ be a canonical isomorphism sending a face $f'_p \in F'(S')$ to the position of another adjacent face $f_{p\delta} \in F'$ such that a $\mathbb{Z}/n'\mathbb{Z}$ symmetry is induced. Starting with initial position p matching v' to v, then $\mathbb{Z}$ as generated by lassos acts on S' such that $f'_{p1}, f'_{p2}, ...f'_{p\delta} \in F'$ of S' is a chain of relevant adjacent faces rolled on S in a preferred clockwise lasso c once around v and across n' edges as the preferred positive generator of isometry-preserving rotational symmetries.

Then we have a single lasso around a fixed interior vertex v on S such that $f_1 = f_n$ in S and f'_{p1} is sent to $f'_{p\delta}$ in $F'(S')$ for a $\mathbb{Z}/n'\mathbb{Z}$ symmetry of equivalence class $[n']$.

☐

Remark. *For every lasso c, we roll across n' edges to return to the identity element and only for $n' \mid n$ where $n' = |v'|$ and $n = |v|$ does a trivial holonomy exist by loops on S. Note that if $n' \nmid n$ only nontrivial holonomy exists for $\mathbb{Z}$ acting on $F'(S')$.*

Combinatorial holonomy, while previously studied via the lens of the tetrahedron, has potential to be generalized to the cube and further polyhedra. Let us consider the case then of rolling any polyhedra on itself.

Theorem 5.3. *Any polyhedra rolled in a lasso on itself always results in trivial restricted holonomy.*

Proof. For any polyhedra S' rolled on itself S around any vertex v of S, note $n' = n$ and hence $n'|n$ with v' fixed to v such that a single lasso c in the preferred clockwise direction always rolls around n' edges sending $f'_{p1} \in F'(S')$ back to $f'_{p1} \in F'(S')$ by the isomorphism of 5.2 such that the full holonomy induced by lassos c is trivial. By 4.2, note that the fundamental group of the surface is trivial, so this completes the proof. ☐

Remark. *This fact is clearly true for the cube $\mathcal{C}$ rolled on itself S, where S is also a cubulated surface.*

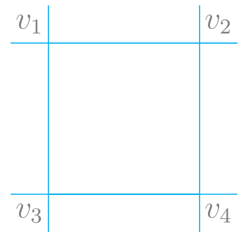


FIGURE 2. Infinite grid

Fixed v	Cycles for $h(c)$	Cycles for $h(c^{-1})$
v_1	(2,3,4)(1)	(2,4,3)(1)
v_2	(3,4,1)(2)	(3,1,4)(2)
v_3	(4,1,2)(3)	(4,2,1)(3)
v_4	(1,2,3)(4)	(1,3,2)(4)

FIGURE 3. Lassos on a cubulated surface S

Cycles for $h(l)$	Lasso Concatenation
(1,2)(3,4), (1,3)(2,4), (1,4)(2,3)	$h(c) * h(c'), h(c^{-1}) * h(c'^{-1})$
$e = (1)(2)(3)(4)$	$h(c) * h(c') * h(c'')$, $h(c^{-1}) * h(c'^{-1}) * h(c''^{-1})$

FIGURE 4. Loops on a cubulated surface S

Theorem 5.4. *The holonomy group of the cube $\mathcal{C}$ on the infinite square grid S is isomorphic to A_4.*

Proof. Recall that any plane in $\mathbb{R}^2$ has a trivial fundamental group with a single homotopy class. Then the class of nullhomotopic loops on the infinite square grid S generated by lassos c consists of all loops on the surface, with all loops having a possible decomposition as lassos. Let the holonomy induced by a lasso c around a fixed vertex v of the cube $\mathcal{C}$ on the infinite grid S be $h(c)$ and that of the inverse lasso path be $h(c^{-1})$. Observe then that the holonomy induced by lassos on S yield 3-cycles by 5.2 all of which are contained in A_4. Additionally, Table 3 shows that all 3-cycles are actually achieved. As any A_n is generated by 3-cycles and A_4 is contained in S_4 as the group of even permutations, then it is evident that the holonomy of the cube on the infinite square grid is closed under and isomorphic to A_4 such that the holonomy $\text{Hol}(\mathcal{C}, S) \leq A_4$ for the cube $\mathcal{C}$ on any infinitely gridded surface S'. Please see tables 3 and 4 for all elements of A_4 as generated by lassos on S. □

Remark. *Assume $h(c)$ and $h(c^{-1})$ are lasso paths of minimum length such that they each go around 1 vertex and cover 4 faces. For any nontrivial $h(l)$ of minimum length, note that its two distinct consecutive lassos c and c' must be adjacent to one another, share the same directional path, and loop initial/ending faces such that $l(f_1) = c(f_1) = c(f_n) = c'(f_1) = c'(f_n) = l(f_n)$.*

This observation of the shortest possible loop on the infinite grid S is useful in considering the below theorem on the smallest grid needed to realize $A_4 \leq Hol(\mathcal{C}, S)$.

Theorem 5.5. *A cubulated surface S has $A_4 \leq Hol(\mathcal{C}, S)$ if it contains at least a 2×3 gridded subsurface.*

Proof. The proof follows similarly to that of 5.4. □

Propositon 5.6. *The restricted holonomy $Hol_0(\mathcal{C}, S) \leq A_4$ for the cube $\mathcal{C}$ on any cubulated surface S.*

Proof. The proof follows similarly to that of 5.4. □

As the above examples attempt to illustrate, it is relatively easy to find $A_4 \leq \mathrm{Hol}(\mathcal{C}, S)$. As A_4 is a maximal normal subgroup of S_4, our challenge then is to find subgroups outside of A_4 yet within S_4. We now turn our attention to annuli in our attempt to realize parameters of surfaces for the holonomy subgroups of interest.

6. Holonomy on Annuli

Definition 6.1. (Annular Strip). *Let a cubulated annular strip A be an $m \times 1$ annulus where $|m| = |f| \in F$ for a surface $A = (V, E, F)$ be cut such that the cut ends are now identified with each other without boundary. Then we have a connected surface S for which we can only roll in two possible directions (i.e. clockwise or counterclockwise) to form loops under backtracking equivalence.*

Propositon 6.1. *For the cube $\mathcal{C}$ rolled on any $m \times 1$ annular strip A; there is a trivial, disjoint 2-cycle, or 4-cycle holonomy isomorphic to A.*

Proof. Let $\mathcal{C}$ be a cube on an annular strip A of length m and let f_1' denote the face of $\mathcal{C}$ initially touching the surface of A. A single roll on this surface fixes two faces and yields a symmetry about the corresponding four-fold axis of symmetry (this subgroup is generated by a 4-cycle, which we can choose to be a single turn in the direction of rolling). As every fourth roll results in f_1' touching the surface, the symmetry that results after one loop will be this 4-cycle to the power of $m \pmod 4$. For $m \in [0]$ we get the identity and $m \in [2]$ a pair of disjoint 2-cycles. If $m \in [1]$ or $[3]$, we get at 4-cycle . The holonomy of $\mathcal{C}$ on an annular strip A can be summarized as in figure 6.

□

Definition 6.2. (L-shaped annular strip). *Let a cubulated L-shaped annular strip be a surface A with a width of n faces and a height of m faces with the extreme faces identified (as shown in figure 5).*

Theorem 6.2. *The symmetry resulting from rolling a cube on an L-shaped annulus is even if the parity of m and $n - 1$ match.*

Proof. Consider an L-shaped annulus A with a width n and height m. The clockwise loop realized on this surface consists first of $n - 1$ rolls to the left, then m rolls upward. The holonomy of our surface A is equivalent to that of two perpendicular annuli, as shown in figure 5, so we find the resultant permutation (or symmetry) of a loop on the (horizontal) annular strip of length $n - 1$ and the same for the (vertical) annular strip of length m. Traveling along the horizontal annular strip yields the same permutation as the one you get after one loop along $n - 1 \pmod 4$. The same applies for traveling on the vertical annular strip, for an annulus of length $m \pmod 4$. Composing these permutations provides the parity of the permutation on the L-shaped annulus. Since an even permutation results only from

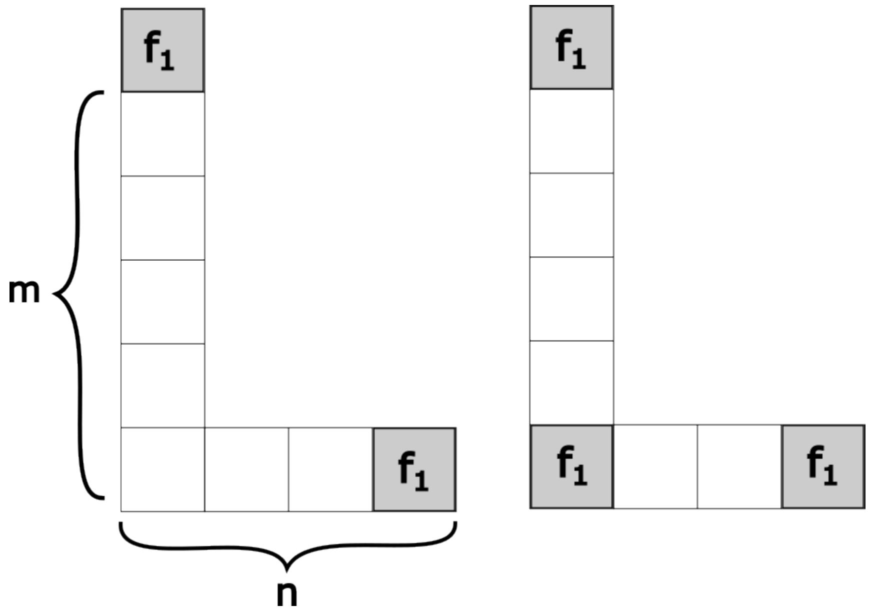


FIGURE 5. L-shaped annulus on the left, alongside an annulus with equivalent holonomy.

Number of rows (mod 4)	Holonomy
0	Trivial
1 or 3	$\mathbb{Z}/4\mathbb{Z}$
2	$\mathbb{Z}/2\mathbb{Z}$

FIGURE 6. Summary of holonomy groups for families of straight annuli

the composition of two even permutations or two odd permutations, the parity of $n-1$ and m must match in order to produce an even permutation for the L-shaped annulus. □

7. Holonomy on Tori

Motivated to study the torus like our previous counterparts, we sought to explore if given a certain $m \times n$ torus with turning number, we could predict a certain holonomy. Beyond computing holonomy groups for particular surfaces, we also generalize the holonomy for families of cubulated T^2 tori as will follow.

7.1. Parity of the $m \times n$ Torus.

We begin with the example of the 3×3 cubulated torus.

Propositon 7.1. *The holonomy of the cube $\mathcal{C}$ on the 3 x 3 cubulated torus T is isomorphic to S_4.*

Proof. Deconstructing or "ungluing/unfolding" our torus under homotopy, we have a 3 x 3 cubulated square plane with opposite pairs of edges identified with each other. Referencing 5.5 which proves the minimum grid needed to realize the restricted holonomy, then the holonomy of our cubulated torus with 9 faces has $\mathrm{Hol}_0(S', S) \leq A_4$. As a straight-pathed

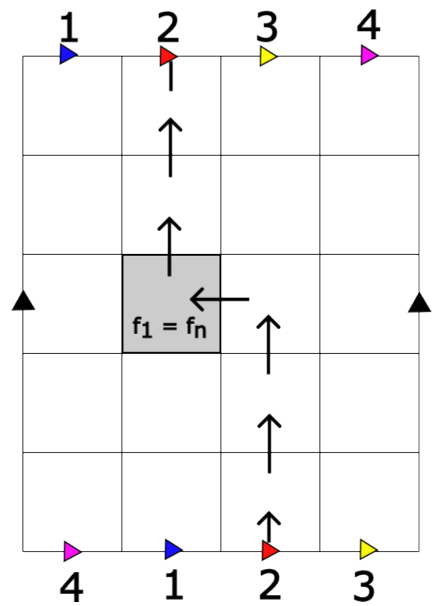


FIGURE 7. Flat torus with turning number $\lambda = 1$. The loop depicted is a longitude.

loop around the torus results in $\mathbb{Z}/4\mathbb{Z}$ holonomy outside of A_4, we know by 5.4 that the holonomy of the cube S' on our 3×3 torus T must be isomorphic to S_4. □

Remark. *We have an $m \times n$ cubulated torus isomorphic to S_4 where m and n are both odd.*

We will soon understand the general reasoning behind this, as detailed in 7.2.

For any cubulated torus T, we denote the *longitudes* (traveling along columns) as vertical chains of adjacent cube faces along T and *meridians* (traveling along rows) as chains of horizontal adjacent cube faces perpendicular to any longitude on T. To illustrate that we can use longitudes and meridians in describing any $m \times n$ cubulated tori T, note then that we would have 5 longitudes and 4 meridians for the 5×4 torus T (for example).

A quick computation using representative loops from these homotopy classes allows us to generalize beyond the holonomy of this initial example.

Theorem 7.2. *Let T be a cubulated $m \times n$ torus with at least a 2×3 subsurface. Then T has holonomy isomorphic to A_4 if and only if* m *and* n *are of even parity and holonomy isomorphic to S_4 if either m or n are odd.*

Proof. First, assume m and n are even. As we know by 5.5 that the restricted holonomy is at least A_4, we now pick a representative loop from the homotopy classes of the longitude as one of the generators of the fundamental group in computing full holonomy. (The argument for the classes of the meridian is analogous, so we only consider longitude henceforth.) We roll in a straight line along the column until it reaches the initial face such that the representative loop is equivalent to following an annular strip of length n. As n is even, the resulting symmetry is either a pair of disjoint 2-cycles or the identity and is (clearly) contained within A_4. As a generator for the fundamental group, this representative implies that all other homotopy classes will be also be contained within A_4. If the cube takes loops

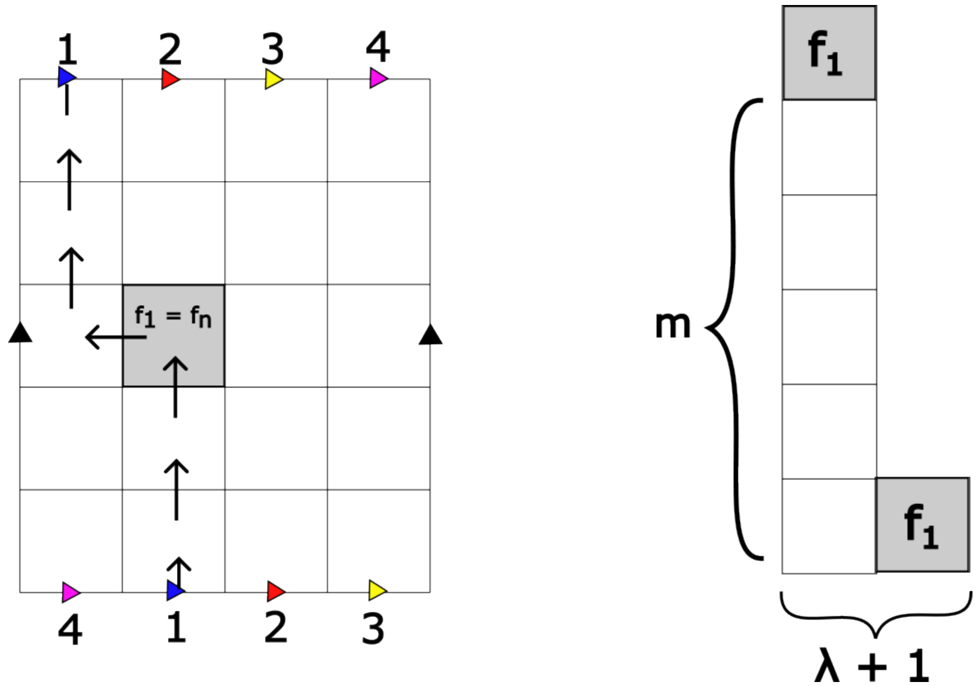


FIGURE 8. Flat torus with turning number $\lambda = 1$, alongside the L-shaped annulus corresponding with the generating loop of the fundamental group shown.

going perpendicular to the rows (to compute the other homotopy classes), this reasoning holds as well since the number of rows is also even.

Now assume T has holonomy isomorphic to A_4. Assume also (toward a contradiction) that either m or n is odd. Then take a loop equivalent to an annular strip perpendicular to the rows (if n is odd) or along the row (if m is odd). Since this path is odd, the resulting symmetry is a 4-cycle, which means the torus has S_4 holonomy, a contradiction. This also shows that if m or n is odd, then T has S_4 holonomy. □

7.2. **Parity and Turning Numbers.** We discovered that the parity of both the number of rows and the length of the rows determines the holonomy for any torus. The it was natural to explore the effect of a turning number on holonomy. In this way, we developed predictions for holonomy on $m \times n$ cubulated tori.

We prove that the parity of turning number, along with the parity of the row and row-length, provides a general theory for the holonomy of cubulated tori.

Definition 7.1. (Turning Number) *Let the turning number of an $m \times n$ surface S be a shift in the identification of the sides of the torus. The turning number μ denotes how many times the edges on the right side of the torus are shifted downward, while λ denotes how many times the edges on the bottom of the torus are shifted to the right. Additionally, μ and λ are non-negative integers.*

Theorem 7.3. *Let T be an $m \times n$ cubulated torus with turning numbers μ and λ. Then T has A_4 holonomy if the number of rows is of the same parity as λ and the number of columns is of the same parity as μ.*

Proof. As was the case in the proof for Theorem 7.2, we know the restricted holonomy is A_4, so we must find the permutation resulting from traveling on loops which generate the fundamental group of T. We choose one such generator to be the loop l_1 which begins at face $f_1 \in T$, rolls to the left λ times (to counteract the turning effect), and then rolls upward m times. The permutation π resulting from this loop is equivalent to that resulting from an L-shaped annulus with height m and width $\lambda + 1$. According to the proof of Theorem 6.2, π is even if the parity of m and $(\lambda + 1) - 1 = \lambda$ match. Thus, if the parity of m and λ match, then the permutation resulting from l is even. The same reasoning holds for the generator of the fundamental group traveling along the meridians - the chosen loop l_2 travels downward μ times and then to the right n times (this is the other generator of the fundamental group). As the parity of μ and n match, the permutation resulting from l_2 is also even. As both l_1 and l_2 result in even permutations, the holonomy of the cube on T is A_4. □

References

[Goo15] Frederick M. Goodman, *Algebra: Abstract and Concrete*, SemiSimple Press, 2015.

[LMN21] Aram Lindroth, Alanna Manfredini, and Nathan Nguyen, *Holonomy of combinatorial surfaces*, Duke DOmath Project Notes, July 2021.

Note: This appendix was written by Dr. Seppo Niemi-Colvin as a previous contribution to the research conducted at Duke University, but is included here as it has not been published elsewhere. As the Seifert-Van-Kampen theorem is often used to describe the fundamental group of a space X via two open path-connected subspaces which cover X, here we apply the theorem to describe the structure of our combinatorial fundamental group. Hence this proves the compatibility of the traditional and combinatorial formulations of the fundamental group.

8. Appendix

Let $S = (V, F)$ be a combinatorial surface without boundary with vertices V and faces F. We will now describe the dual surface to S which we will represent as $\tilde{S} = (\tilde{V}, \tilde{F})$. While it is not necessarily a triangulated surface it can be described as a polygonal surface, whereby each face is a polygon. The vertices of $\tilde{S}$ are made up of the faces of S, so $\tilde{V} = F$. Two vertices in $\tilde{V}$ are connected by an edge, if those faces shared an edge in S. The faces in $\tilde{S}$ correspond to the vertices in S, and thus each face will have the number of sides equal to the degree of the corresponding veretex in S.

8.1. **Realizing the dual surface as a triangulation.** While $\tilde{S}$ is not a priori a triangulated surface (and the decomposition above will be most useful for our work), it can be naturally triangulated in a way which illustrates that it's geometric realization $|\tilde{S}|$ is homeomorphic to $|S|$. This triangulation will give the barrycentric subdivision $\operatorname{sd} S$ for S which we will discuss now before associating each face of $\tilde{S}$ with a collection of faces in $\operatorname{sd} S$. This is not necessary to understand the fundamental group calculation beyond confirming that $\pi_1(|\tilde{S}|, *) \cong \pi_1(|S|)$.

The vertices $\operatorname{sd} V$ of $\operatorname{sd} S$ correspond to $V \cup E \cup F$ where E is the set of edges in S. One can think of the vertex coming from an edge as representing the midpoint of that edge, and the vertex coming from a face to represent the center of that face. We call the new vertices associated to edges and faces, their barrycenters. The faces $\operatorname{sd} F$ of $\operatorname{sd} S$ are chains $v \subset e \subset f$ where $v \in E, e \in E, f \in F$. Taking the barrycentric subdivision does not change the homeomorphism type of $|S|$.

For each face of $\tilde{S}$ we have a corresponding vertex u in S, which is also a vertex in $\operatorname{sd} S$. One can then take the star $\operatorname{st} u$ in $\operatorname{sd} S$, which is the union of all the faces (and edges) of $\operatorname{sd} S$, which contain u. We will consider the star closed and thus it contains any edges and vertices that are in faces of $\operatorname{st} u$ even if those edges and vertices do not contain u themselves.

For an edge e of $\tilde{S}$ is determined by an edge e' of S connecting two faces f_1 and f_2, which share two vertices v_1 and v_2 in S. We can then associate to e the intersection of $\operatorname{st} v_1$ and $\operatorname{st} v_2$. This is an edge from the barryenter of f_1 to the barrycenter of e' to the barrycenter of f_2.

Finally we can associate to a vertex $v \in \tilde{S}$ a face f in S and thus three vertices v_1, v_2, v_3 in S, and in $\operatorname{sd} S$ this becomes $\operatorname{st} v_1 \cap \operatorname{st} v_2 \cap \operatorname{st} v_3$, which will be just the barrycenter of f.

8.2. **The graph of the dual surface.** Before we compute the fundamental group of the entire surface, let us compute the fundamental group of the graph of $\tilde{S}$, by which I mean the union of the vertices and edges of $\tilde{S}$ but no faces attached. I will denote the graph as K_1.

Theorem 8.1. *The group $\pi_1(|K_1|, v_0)$ is isomorphic to the group L given by considering combinatorial loops in K_1 up to adding and subtracting backtracking.*

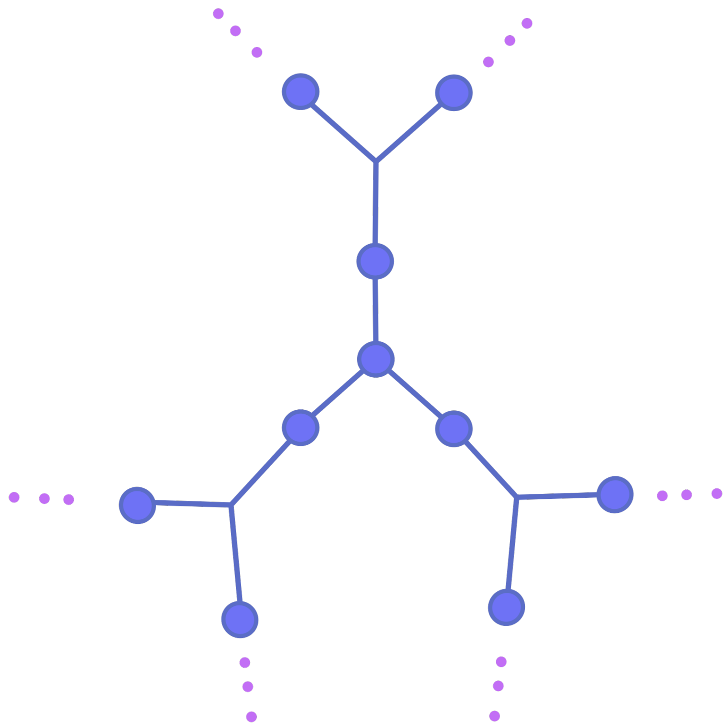


FIGURE 9. The start of the infinite trivalent graph. These branches continue indifinitely and once branched no two vertices get identified. It is a covering space for every connected trivalent graph

Proof. The elements of L are lists of vertices of $\tilde{S}$ (and thus faces of S)

$$v_0, v_1, v_2, \ldots, v_{n-1}, v_n = v_0,$$

where consecutive vertices must be related by an edge in $|K_1|$ Multiplication is given by concatenation where the duplicate vertex is given by concatenation with the doubled vertex v_0 in the middle reduced to a singleton. Concatination is naturally associative and the identity is given by the singleton v_0.. Equivalence under backtracking means that given a configuration $\ldots, v_{i-1}, v_i, v_{i+1}, v_i, v_{i+2}, \ldots$ we can replace it with a configuration $\ldots, v_{i-1}, v_i, v_{i+2}, \ldots$ and vice versa. Under this equivalence, the inverse of a loop $v_0, v_1, \ldots, v_{n-1}, v_n, v_0$ is $v_0, v_n, v_{n-1}, \ldots, v_1, v_0$. Let $\varphi : L \to \pi_1(|K_1|, v_0)$ take a combinatorial loop $v_0, v_1, \ldots, v_n = v_0$ to the class $[\gamma_l]$ where $\gamma_l : [0,1] \to |K_1|$ is defined to be $[\frac{i-1}{n}, \frac{i}{n}]$ to the edge connecting v_{i-1} and v_i. Note that the backtracking equivalence can be given by a homotopy so this map is well defined.

Meanwhile given some loop γ based at v_0, homotopy can cut out partial excursions onto edges and reparamaterize the rest to achieve a path γ' of the form above.

Now assume that there is some loop l such that $\varphi(l) = e$, and suppose that we can write l as $v_0, v_1, \ldots, v_n = v_0$. The universal covering space of $|K_1|$ will be an infinite trivalent tree T (see Figure 9), and γ_l lifts to a loop in T based at a particular lift v_0' of v_0. However, note that T is contractible via a contraction that, providing a metric on T, has every points distance to the base point decreasing. This provides a specific homotopy of γ_l that can be acheived via repeated uses of backtracking.. □

Note that a rooted trivalent tree (T, v_0) can be thought of as three rooted binary trees, whose roots are all connected to the base root v_0. This coincides with our discussion that once we decide an initial direction to move, a path is subsequently determined by a series of choices of left and right. This provides a proof that our intuition about rolling up to backtracking coincides with the fundamental group. We will now compute the fundamental group of $|K_1|$. The

Theorem 8.2. *Let $n := |\tilde{E}| - |\tilde{V}| + 1$, where $\tilde{V}$ and $\tilde{E}$ are resectively the set of verties and edges for $\tilde{S}$. Then, $\pi_1(|K_1|, v_0) \cong *_n\mathbb{Z}$*

Proof. Let T (not the same as in the proof of 8.1) be a spanning tree for K_0, i.e. a subgraph that contains all the vertices and no cycles. Note that the number of edges in T must be $|V| - 1$. As such the number of edges not in T is n, and denote the set of such edges as E'. The rest of the proof follows similarly to example 1.22 in Hatcher. □

8.3. **The fundamental group of the dual surface.** We now show that the subgroup generated by lassos in L is the subgroup of contractible loops, i.e. the kernel of the map $L \to \pi_1(|\tilde{S}|, v_0)$.

Definition 8.1. *A* lasso *l is a loop in L that consists of a path γ from v_0 to a particular vertex $v \in \tilde{V}$, then doing a loop λ around one of the faces f incident to v, before returning along γ^{-1} to v_0. We will write this as $\gamma^{-1} \cdot \lambda \cdot \gamma$. Here $\cdot$ is concatenation on paths written in the order of function composition. The loop above is specifically* a lasso around f.

The following two lemmas will be helpful in identifying the subgroup generated by lassos as the kernel of the map $i : \pi_1(|K_1|, v_0) \to \pi_1(\tilde{S}, v_0)$, since the Seifert-van Kampen theorem will present it to us in a particular way. As an intermediate step we will need similar results about the subgroup generated by lassos around a particular face.

Lemma 8.3. *Let l be a lasso around a face f and let N_l be the smallest normal subgroup of L containing l. Then N_l is generated by all the lassos going around f. Furthermore the smallest normal subgroup N containing all lassos is the subgroup generated by those lassos.*

Proof. Let H_f be the subgroup generated by all of lassos going around f. First, we will show that H_f is normal and therefore $N_l \subseteq H_l$. First note that for an individual lasso h around f, and an element $g \in L$, we have that ghg^{-1} is a lasso around f since we can let $\gamma' = \gamma \cdot g$ be the path from v_0 to v. As such $ghg^{-1} \in H_f$.

Now, given a general element $h \in H_f$ we know that $h = h_1 \cdot h_2 \cdots h_n$ where each h_i is a lasso around f. Given $g \in L$ we have that

$$\begin{aligned} ghg^{-1} &= g\,(h_1 \cdot h_2 \cdots h_n)\,g^{-1} \\ &= (gh_1g^{-1}) \cdot (gh_2g^{-1}) \cdots (gh_ng^{-1}) \in H_f, \end{aligned}$$

as needed to show that H_f is normal.

Now let h_1 and h_2 be loops around the face f. We will show that h_2 is conjugate to h_1 and thus is in N_{h_1} Then each h_i can be constructed using a path γ_i to some vertex v_i incident to f and a path λ_i traveling once around f starting and ending at v_i.

Because we are ultimately trying to show that $h_2 \in N_{h_1}$ can assume without loss of generality that both λ_i induce the same orientation on f, and consider λ' to be the path connecting v_1 to v_2 in this direction. If these were oriented in opposite directions, we can

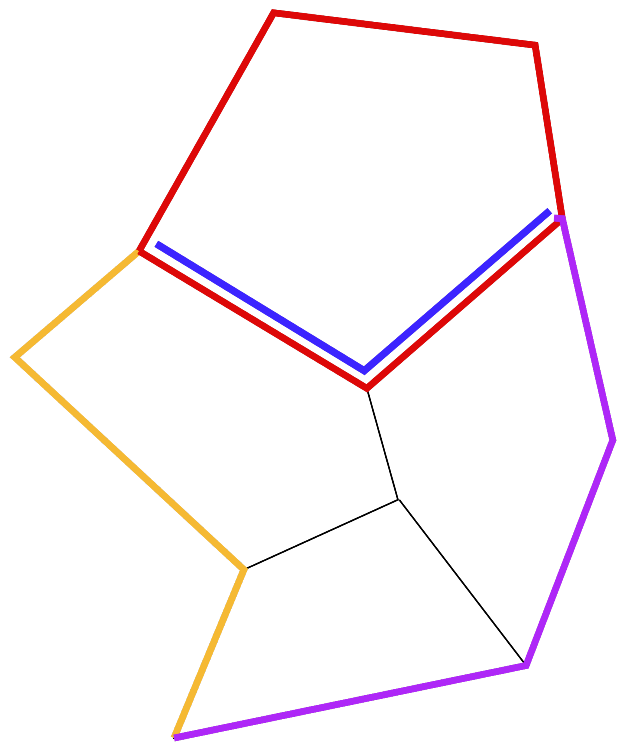


FIGURE 10. A figure representing some of the paths that occur in 8.3. Here orange is γ_1 and purple is γ_2. The red can represent λ_1 or λ_2 depending on which vertex is considered the starting and ending point. The blue represents λ'.

show that h_2^{-1}, which will induce the same orientation on f is conjugate to h_1, which will force $h_2^{-1} \in N_{h_1}$ and thus $h_2 \in N_{h_1}$. Note that $\lambda_2 = \lambda' \cdot \lambda_2 \cdot \lambda'^{-1}$. Let $g := \gamma_2^{-1} \cdot \lambda' \cdot \gamma_1$. Then,

$$\begin{aligned} gh_1g^{-1} =& (\gamma_2^{-1} \cdot \lambda' \cdot \gamma_1) \cdot (\gamma_1^{-1} \cdot \lambda_1 \cdot \gamma_1) \cdot (\gamma_1^{-1} \cdot \lambda'^{-1} \cdot \gamma_2) \\ &= \gamma_2^{-1} \cdot \lambda' \cdot \lambda_1 \cdot \lambda'^{-1} \cdot \gamma_2 \\ &= \gamma_2^{-1} \cdot \lambda_2 \cdot \gamma_2 \\ &= h_2, \end{aligned}$$

as needed.

To show that N is generated by the lassos, we merely need to show that the subgroup H generated by all the lassos is normal. The proof follows similarly to above in that the conjugate of a lasso is a lasso. □

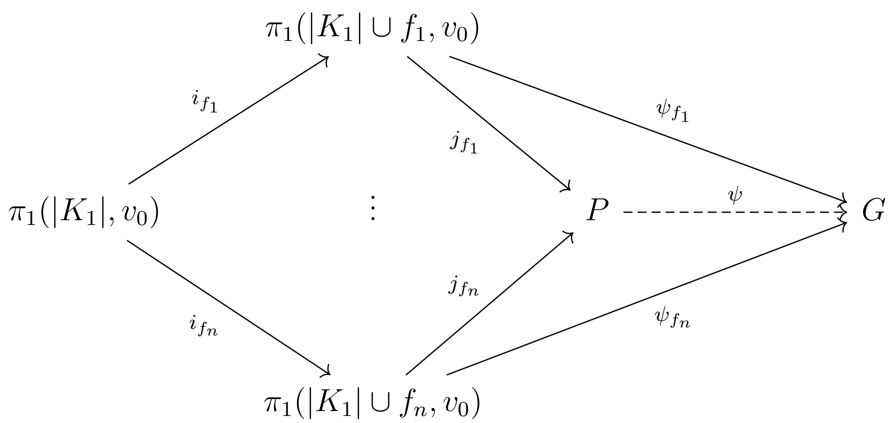
Theorem 8.4. *Let S represent a polyhedral surface and $\tilde{S}$ it's dual with 1-skeleton K_1. Then, the kernel of the map induced by inclusion $i : \pi_1(|K_1|, v_0) \to \pi_1(|\tilde{S}|, v_0)$ is the subgroup N generated by loops around the faces of $\tilde{S}$, i.e. around the vertices of S. Furthermore this map is surjective.*

Proof. The proof will proceed in two parts. For a given face $f \in \tilde{S}$, we first show that $\pi_1(|K_1| \cup f, v_0) \cong \pi_1(|K_1|, v_0)/N_f$ with the inclusion from $|K_1|$ to $|K_1| \cup f$ inducing the canonical projection. To see this let U_1 be $|K_1|$ union the face f sans a single point on the interior, and let U_2 be an open neighborhood of the new face f union a path γ from f to v_0.

We can set it up so that U_1 retracts onto $|K_1|$, U_2 retracts onto f with the path to v_0 and $U_1 \cap U_2$ tracts onto the image of the lasso around f defined using γ. As such, combining the Seifert-van Kampen theorem and Lemma 8.3 gets the desired result.

Now for the second part, for each face $f \in \tilde{S}$ let U_f be a neighborhood of $|K_1| \cup f$ in $\tilde{S}$. In particular, letting p_f be a point in face f, we can set $U_f = \tilde{S} - \cup_{f' \neq f} p_{f'}$. In this way we have the U_f deformation retracts onto $|K_1| \cup f$. Furthermore for any collection of faces A containing more than one eleemnt, we have that $\cap_{f \in A} U_f = \tilde{S} - \cup_{f \in \tilde{F}} p_f$ which deformation retracts onto $|K_1|$. So in particular double and tripple intersections are path connected.

This allows us to use the Seifert-van Kampen theorem to write $\pi_1(\tilde{S}, v_0)$ as the free amalgamation of the $\pi_1(|K_1| \cup f, v_0)$ along $\pi_1(|K_1|, v_0)$. Note that the free amalgamation has the universal property of the pushout. In particular, what this means is that given a group P and maps $j_f : \pi_1(|K_1| \cup f, v_0) \to P$, where given any collection of homomorphisms $\psi_f : \pi_1(|K_1| \cup f, v_0) \to G$ such that the diagram below commutes there exists a unique $\psi : P \to G$ commuting as below, then there exists a canonical isomoprhism between P and $\pi_1(\tilde{S}, v_0)$. Note that the diagram only contains the maps for two faces to prevent it from getting too crowded.



Now let $P := \pi_1(|K_1|, v_0)/N$, which is possible because by 8.3, N is normal. Because $N_f \trianglelefteq N$, there exists canonical projections $j_f : \pi_1(|K_1| \cup f, v_0) \to \pi_1(|K_1|, v_0)/N$. These commute with the inclusion maps from $i_f : \pi_1(|K_1|, v_0) \to \pi_1(|K_1| \cup f, v_0)$, since those maps canonically identify $\pi_1(|K_1| \cup f, v_0)$ with $\pi_1(|K_1|, v_0)/N_f$. Furthermore, assume we have some collection of maps $\psi_f : \pi_1(|K_1| \cup f, v_0) \to G$ for some group G that commute as above.

The fact that the diagram commutes as above means that there is a single homomorphism $\tilde{\psi} : \pi_1(|K_1|, v_0) \to G$ such that $\psi_f \circ i_f = \tilde{\psi}$ for all faces . Furthermore, $\tilde{\psi}$ must have N in its kernel. We can see this because each ι_f has N_f in its kernel thus forcing N_f to be in the kernel of $\tilde{\psi} = \psi_f \circ \iota_f$ Therefore, the smallest normal subgroup containing the N_f, which is N, must be in the kernel of $\tilde{\psi}$.

As such, by the universal property of the quotient we have that $\tilde{\psi} = \psi \circ j$ where j is the projection from $\pi_1(|K_1|, v_0) \to P$ and a unique map $\psi : P \to G$. That this commutes with the rest of the diagram, comes down to j factoring as $j_f \circ i_f$ for any face f.

As such, we have shown that P satisfies the same universal property as $\pi_1(\tilde{S}, v_0)$ and thus they must be canonically isomorphic. □

From this theorem we get the following corollary.

Corollary 8.5. *The combinatorial fundamental group given by L/N is isomorphic to $\pi_1(\tilde{S}, v_0) \cong \pi_1(S, v_0)$*

Proof. Combine Theorem 8.1 with Theorem 8.4. The isomorphism in Theorem 8.1 between L and $\pi_1(|K_1|, v_0)$ identifies combinatorial lassos with continuous lassos as needed. The identification of $\pi_1(\tilde{S}, v_0)$ with $\pi_1(S, v_0)$ comes from them representing homeormorphic spaces and in fact are equivalent after subdivision. □

I didn't end up needing this lemma but I'm keeping it here for your reference.

Lemma 8.6. *Let G be a group. For a collection of subsets $\{A_1, A_2, \ldots, A_n\} \subset G$, let N_i be the smallest normal subgroup of G containing A_i and let N be the smallest normal subgroup containing $\cup_i A_i$. Then N is the smallest normal subgroup containing all the N_i.*

Proof. Note that the smallest normal subgroup N_A containing a collection A is the intersection of all the normal subgroups containing A. The collection of normal subgroups containing A is the same as the collection of normal subgroups containing N_A. As such exchanging each A_i for N_i dose not change set of normal subgroups we are intersecting when we compute N. □

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