CAHN-HILLIARD ON LATTICES: DYNAMIC TRANSITIONS AND PATTERN FORMATIONS

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Abstract. This article examines the dynamic phase transitions and pattern formations attributed to binary systems modeled by the Cahn-Hilliard equation. In particular, we consider a two-dimensional lattice structure and determine how different choices of the spanning vectors influence the resulting stability and pattern formations. As the trivial steady-state loses its linear stability, the binary system undergoes a dynamic transition which is shown to be characterized by both the geometry of the domain and the choice of parameters of the model. Unlike rectangular domains, we are able to observe the emergence of hexagonally–packed circles, as well as the familiar rolls and square structures. We begin with the decomposition of our function space into a stable and unstable eigenspace before calculating the center manifold that maps the former to the later. In analyzing the resulting reduced equations, we consider the different multiplicities that the critical eigenvalue can have, which is shown to be geometry-dependent. We briefly consider the long-range interaction model and determine that it produces similar results to the original model.

Contents

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1. Introduction

The Cahn-Hilliard model is a partial differential equation that describes the process of phase separation by which two components of a binary fluid spontaneously separate and form domains pure in each component; see, e.g., Cahn and Hilliard [1], Novick-Cohen and Segel [6], Reichl [8], and Pismen [7]. The Cahn-Hilliard model is also used in modeling sharp interfaces of materials such as in Liu and Shen [3], as well as as Shen and Yang [9], who developed a phase field model for the mixture of two incompressible fluids and its approximations by a Fourier-Spectral method. Many situations can be be modeled as a phase separation of binary systems, and the systematic study of solutions to the Cahn-Hilliard equation and their stabilities prove to be useful in the natural sciences.

The main objective of this paper is to initiate a study of dynamic transitions and pattern formations on a lattice periodic structure for the Cahn-Hilliard model without or with long-range interaction. The specific goal is then to explore how the geometry of the spatial domain, the physical parameters γ_2 and γ_3, and the control λ affect **1)** the type of phase transitions; **2)** the structure of the transition states; and **3)** the emergence of different patterns (rolls, squares, hexagons, etc.).

This article will examine the phase transition and pattern formation that occurs in a lattice domain system. The control parameter λ plays a critical role in determining the degeneracy of the basic solution $u = 0$ into patterns in the form of new solutions to the model. Some patterns found in the lattice domain include rolls, squares, hexagons, and rectangles in the far field.

It is classical that the Cahn-Hilliard model can be put in the perspective of an infinite dimensional dissipative dynamical system. The mathematical analysis of the model is carried using the dynamical transition theory developed by Ma and Wang [5]. The key ingredients of the analysis consist of the following. First, the solution on a lattice structure L with dual lattice $L^* = \{n_1k_1 + n_2k_2 | (n_1, n_2) \in \mathbb{Z}^2\}$ can be Fourier expanded; see Section 2 for details. The Fourier modes correspond to eigenfunctions of the linearized equation. This leads to a precise characterization of the critical thresholds, the principle of exchange of stabilities (PES), and the stable and unstable modes.

Second, we derive leading order approximations of the center manifold function, so that the stable modes are written as functions of the unstable/center modes. We then derive the leading order approximation of the reduced system of the original Cahn-Hilliard on the center manifold. The reduced system depends on the number of critical modes, the spatial geometry, and the physical parameters γ_2, γ_3, and λ. The reduced system captures the precise information on types of phase transitions, the structure of the transitions, and the related emerging patterns. For example, in the case where the dimension of the critical space is four, the type of transition is dictated by the sign of

$$b = \gamma_3 - A\gamma_2^2,$$

with

$$A = \max\left\{ (\frac{2}{27|k_1^c|^2} + \frac{4}{9}(\frac{1}{|k_1^c - k_2^c|^2 - |k_1^c|^2} + \frac{1}{|k_1^c + k_2^c|^2 - |k_1^c|^2})), \frac{2}{9|k_1^c|^2} \right\}.$$

If $b > 0$, the system undergoes a dynamical transition to a local attractor Σ_λ, homological to a $3D$ sphere S^3. Also, Σ_λ contains three circles of steady states and a two-dimensional torus of steady states. In addition, the solution on Σ_λ gives rise to square and roll patterns.

Reduction of our model into a system of reduced equations allows us to find equilibrium solutions and transition types at bifurcation. Possible transitions include a continuous (Type I), jump (Type II), and mixed (Type III) transition. In a continuous transition, emerging solutions stay withing a neighborhood of the basic solution during bifurcation. Conversely, a jump transition exhibits solutions quickly diverging from the basic solution and approaching the far field. Mixed transitions are those that exhibit behavior of both Type I and Type II.

In the multiplicity six case, hexagon packed patterns also appear, which are absent in the classical Cahn-Hilliard model on rectangular domains.

2. Cahn-Hilliard Model

Consider a binary system of components with concentrations u_A and u_B that experience long-range repulsive interaction. Assuming that the system is incompressible, we can work exclusively with u_A through the identity $u_A + u_B = 1$. Consider the free energy functional associated with the binary system:

$$\begin{aligned} F(u_A) = \int_U & (\frac{\mu}{2}|\nabla u_A|^2 + f(u_A) \\ & + \frac{\sigma_d}{2}(-\Delta)^{-\frac{1}{2}}(u_A - a) \cdot (-\Delta)^{-\frac{1}{2}}(u_A - a))dx + F_0. \end{aligned} \tag{2.1}$$

Here $U \subset \mathbb{R}^2$ is a bounded domain, $(-\Delta)^{-\frac{1}{2}}$ is a fractional power of the Laplacian, μ and σ_d are positive physical parameters, a is the concentration of component A in the disordered state, and F_0 is the free energy of the system in the disordered state. We assume that

$$f(u_A) = b_1(u_A - a)^2 + b_2(u_A - a)^3 + b_3(u_A - a)^4 \tag{2.2}$$

where b_1, b_2, and $b_3 > 0$ are arbitrary constants. The disordered state corresponds to a complete spread of component A and is written as

$$a = \frac{1}{|U|}\int_U u_A(x)dx \tag{2.3}$$

where $|U|$ is the area of the domain. In this paper, we will explore the case when U is a lattice domain, and the typical Neumann boundary conditions associated with the Cahn-Hilliard model will be replaced with a periodic condition over the lattice.

Let $l_1, l_2 \in \mathbb{R}^2$ be any set of linearly independent vectors. We consider a two-dimensional lattice L and its dual lattice L^* given by

$$\begin{aligned} L &= \{n_1 l_1 + n_2 l_2 | (n_1, n_2) \in \mathbb{Z}^2\}, \\ L^* &= \{n_1 k_1 + n_2 k_2 | (n_1, n_2) \in \mathbb{Z}^2\}, \end{aligned} \tag{2.4}$$

where $k_i \cdot l_j = 2\pi\delta_{ij}$ for $i, j \in \{1, 2\}$. Let U be the area enclosed by the parallelogram created by the vectors l_1 and l_2.

The non-dimensional form of the negative gradient flow of the free energy (2.1) is given by

$$\begin{aligned} &u_t = -\Delta^2 u - \lambda \Delta u + \Delta(\gamma_2 u^2 + \gamma_3 u^3), (x, t) \in \mathbb{R}^2 \times \mathbb{R}^+, \\ &u(x + l, t) = u(x, t), l \in L, \\ &u(x, 0) = \phi(x), \\ &\int_U u(x, t) dx = 0. \end{aligned} \tag{2.5}$$

The non-dimensional variables and parameters (suppressing the primes) are given by

$$\begin{aligned} x' &= \frac{x}{d}, & t' &= \frac{m\mu}{d^4} t, \\ u' &= u, & \lambda &= -\frac{d^2 b_1}{\mu}, \\ \gamma_2 &= \frac{d^2 b_2}{\mu}, & \gamma_3 &= \frac{d^2 b_3}{\mu}. \end{aligned} \tag{2.6}$$

These parameters were considered in [4]. Solutions of (2.5) take the form

$$u(x, t) = \sum_{k \in L^* \setminus \{0\}} (z_k(t) e^{ik \cdot x} + \overline{z_k(t)} e^{-ik \cdot x}), \tag{2.7}$$

where each point of the lattice can be written as $k = n_1 k_1 + n_2 k_2$ for some integers $(n_1, n_2) \neq (0, 0)$, as in Hoyle [2]. Observe that the solution is periodic in L as desired. From this, it can be calculated that

$$\Delta u(x, t) = \sum_{k \in L^* \setminus \{0\}} -|n_1 k_1 + n_2 k_2|^2 (z_k(t) e^{ik \cdot x} + \overline{z_k(t)} e^{-ik \cdot x}) \tag{2.8}$$

and

$$\Delta^2 u(x, t) = \sum_{k \in L^* \setminus \{0\}} |n_1 k_1 + n_2 k_2|^4 (z_k(t) e^{ik \cdot x} + \overline{z_k(t)} e^{-ik \cdot x}). \tag{2.9}$$

To put the model (2.5) in the perspective of nonlinear dissipative dynamical systems, we let

$$\begin{aligned} H &:= \{u \in L^2(U) \mid \int_U u dx = 0\}, \\ H_1 &:= \{u \in H^4(U) \cap H \mid u(x + l, t) = u(x, t), l \in L\}, \\ H_{1/2} &:= \{u \in H^2(U) \cap H \mid u(x + l, t) = u(x, t), l \in L\}. \end{aligned} \tag{2.10}$$

We shall split the linear component of (2.5) into two operators: one depending on the control parameter λ and the other not. Define $L_\lambda = -A + B_\lambda : H_1 \to H$ and $G : H_{1/2} \to H$ by

$$\begin{aligned} Au &= \Delta^2 u, \qquad\qquad B_\lambda u = -\lambda u, \\ G(u) &= \Delta(\gamma_2 u^2 + \gamma_3 u^3). \end{aligned} \tag{2.11}$$

Then, (2.5) can be written as

$$\begin{aligned} \frac{\partial u}{\partial t} &= L_\lambda u + G(u), \\ u(x,0) &= \phi(x). \end{aligned} \tag{2.12}$$

It is then classical to show that for any $\phi \in H$, (2.12) has a global in time solution

$$u \in L^2([0,T]; H_1) \cap L^\infty([0,T]; H), \text{ for all } T > 0. \tag{2.13}$$

In other words, (2.12) is a well-posed dynamical system.

3. Principle of Exchange of Stabilities

To study the dynamical transitions and pattern formations of (2.12), we first examine the linear instability, leading to the exchange of stabilities principle. The eigenvalues and eigenfunctions of L_λ subject to the periodicity in (2.5) are

$$\begin{aligned} \beta_{n_1 n_2}(\lambda) &= -|n_1 k_1 + n_2 k_2|^4 + \lambda |n_1 k_1 + n_2 k_2|^2 \\ &= -|n_1 k_1 + n_2 k_2|^2 (|n_1 k_1 + n_2 k_2|^2 - \lambda), \end{aligned} \tag{3.1}$$

$$e_{n_1 n_2} = e^{i(n_1 k_1 \cdot x + n_2 k_2 \cdot x)}, \tag{3.2}$$

as seen in (2.7) and (2.8). Note that both of these can be written equivalently as $\beta_k = -|k|^2(|k|^2 - \lambda)$ and $e_k = e^{ik\cdot x}$ and will be used interchangeably henceforth.

Let $S \subset \mathbb{Z}^2 \setminus \{(0,0)\}$ be the set of all integer weights (n_1, n_2) that minimize the magnitude of the vector $k = n_1 k_1 + n_2 k_2$. More explicitly, denote

$$\begin{aligned} S = \{ &(n_1, n_2) \in \mathbb{Z}^2 \setminus \{(0,0)\} \mid \\ &|n_1 k_1 + n_2 k_2|^2 = \min_{(p,q)\in \mathbb{Z}^2 \setminus \{(0,0)\}} |p k_1 + q k_2|^2 \}. \end{aligned} \tag{3.3}$$

It can be seen that the possible values of the cardinality of S are two, four, and six. For notation, when $\#S = 6$, the elements of S are (n_1^c, n_2^c), $(-n_1^c, -n_2^c)$, (n_3^c, n_4^c), $(-n_3^c, -n_4^c)$, (n_5^c, n_6^c), and $(-n_5^c, -n_6^c)$. Define the critical vectors of L^* by

$$\begin{aligned} k_1^c &= n_1^c k_1 + n_2^c k_2, \\ k_2^c &= n_3^c k_1 + n_4^c k_2, \\ k_3^c &= n_5^c k_1 + n_6^c k_2. \end{aligned} \tag{3.4}$$

Should $\#S = 2$ or $\#S = 4$, we will work only with one or two critical vectors, respectively.

We now give examples of three lattices in which the critical eigenvalue has multiplicity two, four, and six. When $\#S = 4$, the four vectors that have minimal magnitude are the two critical vectors and their opposites. Perhaps the simplest lattice to consider is the square lattice spanned by the vectors $k_1 = (1,0)$ and $k_2 = (0,1)$. In this case, the elements of the lattice that have minimal magnitude

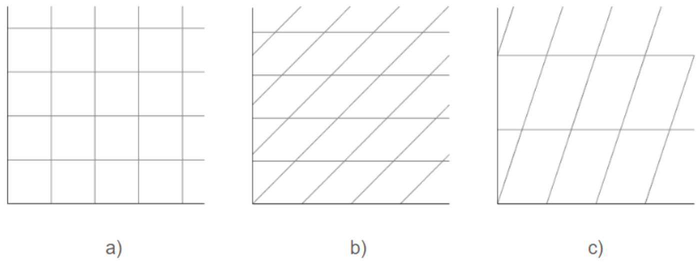

FIGURE 1. Examples of various lattice structures in which the critical eigenvalue has multiplicity; **(a)** four, **(b)** two, and **(c)** six.

are the two spanning vectors as well as their opposites.

When $\#S = 2$, only one vector and its additive inverse can achieve minimal magnitude, as is the case with the critical vector $k^c = (\frac{\sqrt{3}}{2} - 1, \frac{1}{2})$ of the lattice spanned by the vectors $k_1 = (1, 0)$ and $k_2 = (\frac{\sqrt{3}}{2}, \frac{1}{2})$. Notice that this critical vector is the difference of the two spanning vectors; that is $k^c = k_2 - k_1$.

When $\#S = 6$, we seek three vectors and their opposites with minimal magnitude. The lattice spanned by the vectors $k_1 = (1, 0)$ and $k_2 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ has critical vectors k_1, k_2, and $k_1 + k_2 = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$ as well as their additive inverses. FIGURE 1 shows graphs of these three lattices.

Define the spaces E_1^λ and E_2^λ by

$$\begin{aligned} E_1^\lambda &= span\{e^{\pm i(n_1^c k_1 + n_2^c k_2)\cdot x} \mid (n_1^c, n_2^c) \in S\}, \\ E_2^\lambda &= \overline{span\{u \mid \langle u, e_i \rangle = 0 \text{ for all } e_i \in E_1^\lambda\}}. \end{aligned} \tag{3.5}$$

We define the critical value of the control parameter by $\lambda_0 = |k_1^c|^2 = |k_2^c|^2 = |k_3^c|^2$. This critical value plays a central role in the stability of the basic solution $u = 0$. Specifically, as λ crosses the threshold λ_0, a finite number of the eigenvalues given by (3.1) become positive, and the basic solution becomes linearly unstable. This principle of exchange of stabilities is given mathematically as follows:

$$\begin{aligned} &\begin{cases} \beta_n(\lambda) < 0 & \text{if } \lambda < \lambda_0, \\ \beta_n(\lambda) = 0 & \text{if } \lambda = \lambda_0, \\ \beta_n(\lambda) > 0 & \text{if } \lambda > \lambda_0, \end{cases} \\ &\beta_{n_1 n_2}(\lambda_0) < 0 \text{ if } (n_1, n_2) \in \mathbb{Z} \times \mathbb{Z} \setminus (\{(0,0)\} \cup S), \end{aligned} \tag{3.6}$$

for all $n \in S$. Thus, with critical value λ_0, the eigenvalue $-|k_1^c|^2(|k_1^c|^2 - \lambda)$, has either multiplicity two, four, or six with basis vectors for the eigenspace

$$\begin{aligned} \{e_1 = e^{i(k_1^c\cdot x)}, e_2 = e^{-i(k_1^c\cdot x)}, e_3 = e^{i(k_2^c\cdot x)}, \\ e_4 = e^{-i(k_2^c\cdot x)}, e_5 = e^{i(k_3^c\cdot x)}, e_6 = e^{-i(k_3^c\cdot x)}\}, \end{aligned} \tag{3.7}$$

with the number of vectors equal to the cardinality of S. We now verify the existence of a dynamical phase transition of (2.11) as λ becomes larger than λ_0. Based on Theorem 2.1.3 in [5], we have the following dynamical transition theorem:

Theorem 3.1 (Existence of Transition). *The system (2.11) undergoes a dynamical transition from the basic state $u = 0$ as the control parameter λ crosses the critical threshold λ_0. The transition is one of the three types: continuous, catastrophic, or random, and the type is dictated by the nonlinear interaction.*

Remark: This theorem states that a transition occurs when $\lambda > \lambda_0$, and the transition will either be Type I, Type II, or Type III. The transition type is dependent on the geometry of the lattice and the system parameters λ, γ_2, and γ_3. It is determined by capturing the nonlinear interactions of stable and unstable modes, using dynamical transition theory and center manifold techniques. This will be the main focus of the remaining part of the paper.

4. Multiplicity Four Case

4.1. **Center manifold reduction.** Consider the case where S has cardinality four. Let

$$E_1^\lambda = span\{e_1(\lambda), e_2(\lambda), e_3(\lambda), e_4(\lambda)\}, \tag{4.1}$$

denote the unstable and stable eigenspaces, respectively. Let P_i denote the canonical projection of H into E_i for $i = 1, 2$. Then *u(x,t)* belongs to the direct sum of these two spaces and can be written as

$$u(x,t) = y_1e_1 + y_2e_2 + y_3e_3 + y_4e_4 + z, \tag{4.2}$$

where $z \in E_2^\lambda$ is the stable component. Equation (2.12) can thus be written in the form

$$(y_t \cdot e + z_t) = L_\lambda(y \cdot e + z) + G(y \cdot e + z), \tag{4.3}$$

where $y = (y_1, y_2, y_3, y_4)$ and $e = (e_1, e_2, e_3, e_4)$. Let $e_i \in e$ be any of the four unstable eigenfunctions. Consider the projection of the Cahn-Hilliard equation that results from taking the inner product of both sides of (4.3) with e_i. Using the mutual orthogonality of the eigenfunctions, we have

$$\int_U \dot{y_i} e_i \bar{e}_i dx = \int_U L_\lambda(y_ie_i)\bar{e}_i dx + \int_U G(y \cdot e + z)\bar{e}_i dx. \tag{4.4}$$

Here integration is to be taken over the parallelogram U. Applying the linear operator to e_i and simplifying, we are left with

$$\dot{y_i} \int_U e_i\bar{e}_i dx = y_i\beta_{n_1^c n_2^c}(\lambda) \int_U e_i\bar{e}_i dx + \int_U G(y \cdot e + z)\bar{e}_i dx,$$

which implies that

$$\dot{y_i} = y_i\beta_{n_1^c n_2^c}(\lambda) + \frac{1}{|e_i|^2} \int_U G(y \cdot e + \Phi(y))\bar{e}_i dx, \tag{4.5}$$

where the norm of the eigenfunction is the area of the parallelogram $|l_1 \times l_2|$. Here we have written $z = \Phi(y)$ which is the center manifold function that maps the unstable eigenspace to the stable. Because the linear operator L is diagonal near

$\lambda = \lambda_0$, we can use the following formula for the center manifold (Theorem A.1.1 in [5]):

$$-L_\lambda \Phi(y) = P_2 G(y \cdot e) + h.o.t., \tag{4.6}$$

where P_2 denotes projection onto the stable eigenspace E_2^λ. Fourier decomposing the center manifold as $\Phi(y) = \sum_{k=5}^{\infty} \varphi_k(y) e_k$ allows us to write the left hand side of (4.6) as

$$-L_\lambda \Phi(y) = -\sum_{k=5}^{\infty} \varphi_k L_\lambda e_k = -\sum_{k=5}^{\infty} \varphi_k \beta_k(\lambda) e_k. \tag{4.7}$$

Let $e_k \in E_2^\lambda$ be any stable eigenfunction. Then taking the inner product of (4.6) with e_k yields

$$-\beta_k |e_k|^2 \varphi_k = \int_U G(y \cdot e) \bar{e_k} dx + h.o.t., \tag{4.8}$$

which implies that

$$\varphi_k(y) = -\frac{1}{\beta_k |e_k|^2} \int_U G(y \cdot e) \bar{e_k} dx + h.o.t., \tag{4.9}$$

where φ_k represents the Fourier coefficient of $\Phi(y)$ with basis function e_k. We shall use the following eight stable eigenfunctions for our approximation of the center manifold:

$$\begin{aligned} e_{5,6} &= e^{\pm 2ik_1^c \cdot x}, & e_{7,8} &= e^{\pm 2ik_2^c \cdot x}, \\ e_{9,10} &= e^{\pm i(k_1^c \cdot x + k_2^c \cdot x)}, & e_{11} &= e^{i(-k_1^c \cdot x + k_2^c \cdot x)}, \\ e_{12} &= e^{i(k_1^c \cdot x - k_2^c \cdot x)}. \end{aligned} \tag{4.10}$$

Associated with each of these eigenfunctions is a coefficient in y that is found by means of equation (4.9). Let $e_k \in \{e_i\}_{i=5}^{i=12}$ be any one of these stable eigenfunctions. Per equation (4.9), we have

$$\varphi_k(y) = -\frac{1}{\beta_k |e_k|^2} \int_U \Delta(\gamma_2 (y \cdot e)^2 + \gamma_3 (y \cdot e)^3) \bar{e_k} dx. \tag{4.11}$$

Using integration by parts and the fact that $\Delta \bar{e_k} = |k|^2 \bar{e_k}$, we can write

$$\varphi_k(y) = -\frac{|k|^2}{\beta_k |e_k|^2} \int_U (\gamma_2 (y \cdot e)^2 + \gamma_3 (y \cdot e)^3) \bar{e_k} dx. \tag{4.12}$$

The orthogonality of the eigenfunctions ensures that $\int e_i \bar{e_j} dx = |e_j|^2 \delta_{ij}$. When the nonlinear term of (4.12) is expanded, we see that the integral of every term vanishes save for those of $\gamma_2 (y \cdot e)^2 + \gamma_3 (y \cdot e)^3$ that have eigenfunction e_k. More specifically, take $e_5 = e^{2ik_1^c \cdot x}$. Dropping the cubic term, we have

$$\varphi_5(y) = -\frac{|k_{2n_1^c 2n_2^c}|^2}{\beta_{2n_1^c 2n_2^c} |e_5|^2} \int_U \gamma_2 (y_1 e_1 + y_2 e_2 + y_3 e_3 + y_4 e_4)^2 \bar{e_5} dx, \tag{4.13}$$

where the subscript on $k_{2n_1^c 2n_2^c}$ denotes the integers in the linear combination of the eigenfunction $e_5 = e^{i(2k_1^c \cdot x + 0k_2^c \cdot x)}$. Since $\bar{e_5} = e^{-2ik_1^c \cdot x}$, we are searching for terms of the square that contain the eigenfunction $e^{2ik_1^c \cdot x}$. Upon expanding, we observe that the only such term is $\gamma_2 y_1^2 e_1^2$, since $e_1^2 = e_5$. Therefore the integral of all other terms is zero and we are left with

$$\begin{aligned} \varphi_5(y) &= -\frac{|k_{2n_1^c 2n_2^c}|^2}{\beta_{2n_1^c 2n_2^c}|e_5|^2}\int_U \gamma_2 y_1^2 e_1^2 \bar{e_5} dx \\ &= -\frac{\gamma_2|k_{2n_1^c 2n_2^c}|^2}{\beta_{2n_1^c 2n_2^c}|e_5|^2} y_1^2 \int_U dx \\ &= -\frac{\gamma_2|k_{2n_1^c 2n_2^c}|^2}{\beta_{2n_1^c 2n_2^c}} y_1^2. \end{aligned} \tag{4.14}$$

where again $k_{2n_1^c 2n_2^c} = 2k_1^c + 0k_2^c$ and $\beta_{2n_1^c 2n_2^c} = -|k_{2n_1^c 2n_2^c}|^2(|k_{2n_1^c 2n_2^c}|^2 - \lambda)$ is the eigenvalue of e_5. We can obtain the other Fourier coefficients of the center manifold by proceeding in a similar manner for the rest of the stable eigenfunctions. Listed here, the coefficients are

$$\begin{aligned} \varphi_{5,6,7,8} &= -\frac{\gamma_2|k_{(2n_1^c 2n_2^c)}|^2}{\beta_{(2n_1^c 2n_2^c)}} y_{1,2,3,4}^2, \\ \varphi_9 &= -\frac{2\gamma_2|k_{(n_1^c+n_3^c, n_2^c+n_4^c)}|^2}{\beta_{(n_1^c+n_3^c, n_2^c+n_4^c)}} y_1 y_3, \\ \varphi_{10} &= -\frac{2\gamma_2|k_{(-n_1^c-n_3^c, -n_2^c-n_4^c)}|^2}{\beta_{(-n_1^c-n_3^c, -n_2^c-n_4^c)}} y_2 y_4, \\ \varphi_{11} &= -\frac{2\gamma_2|k_{(-n_1^c+n_3^c, -n_2^c+n_4^c)}|^2}{\beta_{(-n_1^c+n_3^c, -n_2^c+n_4^c)}} y_2 y_3, \\ \varphi_{12} &= -\frac{2\gamma_2|k_{(n_1^c-n_3^c, n_2^c-n_4^c)}|^2}{\beta_{(n_1^c-n_3^c, n_2^c-n_4^c)}} y_1 y_4. \end{aligned} \tag{4.15}$$

Writing the center manifold as $\Phi(y) = \sum_{k=5}^{12} \varphi_k(y) e_k$ and returning to equation (4.5), we can view the nonlinear term as

$$\gamma_2 P_1 \Delta(\sum_{j=1}^4 y_j e_j + \Phi)^2 + \gamma_3 P_1 \Delta(\sum_{j=1}^4 y_j e_j + \Phi)^3. \tag{4.16}$$

Letting $e_k \in E_1^\lambda$ and dropping higher-order terms, we have

(4.17)
$$\gamma_2 \int_U \Delta(\sum_{j=1}^4 y_j e_j)^2 \bar{e_k} dx + 2\gamma_2 \int_U \Delta((\sum_{j=1}^4 y_j e_j)\Phi)\bar{e_k} dx + \gamma_3 \int_U \Delta(\sum_{j=1}^4 y_j e_j)^3 \bar{e_k} dx.$$

Using integration by parts and the fact that $\Delta \bar{e_k} = |k|^2 \bar{e_k}$, we're left with

(4.18)
$$|k|^2(\gamma_2 \int_U (\sum_{j=1}^4 y_j e_j)^2 \bar{e_k} dx + 2\gamma_2 \int_U (\sum_{j=1}^4 y_j e_j)\Phi \bar{e_k} dx + \gamma_3 \int_U (\sum_{j=1}^4 y_j e_j)^3 \bar{e_k} dx).$$

Consider the differential equation for y_1 in the system (4.5). We have, along with (4.18), that

$$\dot{y_1} = y_1\beta_{n_1^c n_2^c}(\lambda) + \frac{|k_1^c|^2}{|e_1|^2}(\gamma_2 \int_U (\sum_{j=1}^4 y_j e_j)^2 \bar{e_1} dx$$

$$+ 2\gamma_2 \int_U (\sum_{j=1}^4 y_j e_j)\Phi \bar{e_1} dx + \gamma_3 \int_U (\sum_{j=1}^4 y_j e_j)^3 \bar{e_1} dx). \tag{4.19}$$

We search for cross-terms among $(\sum_{j=1}^4 y_j e_j)^2$ that contain the eigenfunction $e_1 = e^{ik_1^c \cdot x}$. However, observation shows that there are no such terms. This holds true, in fact, for each of the four differential equations in (4.5). Now looking at the term $(\sum_{j=1}^4 y_j e_j)\Phi$ and using our expansion of the center manifold, we see that the terms

$$y_2 e_2 \varphi_5 e_5,\ y_3 e_3 \varphi_{12} e_{12},\ y_4 e_4 \varphi_9 e_9,$$

all contain the eigenfunction e_1 once simplified. Thus, the only non-zero terms that result from the second integral in (4.19) are

$$2\gamma_2 |e_1|^2 (y_2\varphi_5 + y_3\varphi_{12} + y_4 + \varphi_9).$$

Finally, consider the term $(\sum_{j=1}^4 y_j e_j)^3$. We again seek cross-terms of this cube that contain the eigenfunction e_1 so that the integral may be non-trivial. The terms we find are

$$3y_1^2 e_1^2 y_2 e_2,\ 6y_1 e_1 y_3 e_3 y_4 e_4.$$

Thus the final integral in (4.19) becomes

$$\gamma_3 |e_1|^2 (y_1^2 y_2 + y_1 y_3 y_4).$$

Combining these results in the original differential equation (4.5) yields the following reduced equation

$$\dot{y_1} = y_1\beta_{n_1^c n_2^c}(\lambda) + |k_1^c|^2 (2\gamma_2(y_2\varphi_5 + y_3\varphi_{12} + y_4\varphi_9) + \gamma_3(3y_1^2 y_2 + 6y_1 y_3 y_4)). \tag{4.20}$$

Using the coefficients found in (4.15) and the expressions of the eigenvalues in (3.1), we can write $y_2\varphi_5 + y_3\varphi_{12} + y_4\varphi_9$ as

$$\gamma_2(-\frac{y_1^2 y_2}{|k_{(2n_1^c, 2n_2^c)}|^2 - \lambda} - \frac{2y_1 y_3 y_4}{|k_{(n_1^c - n_3^c, n_2^c - n_4^c)}|^2 - \lambda} - \frac{2y_1 y_3 y_4}{|k_{(n_1^c + n_3^c, n_2^c + n_4^c)}|^2 - \lambda})$$

$$= \gamma_2(-\frac{y_1^2 y_2}{4|k_1^c|^2 - \lambda} - \frac{2y_1 y_3 y_4}{|k_1^c - k_2^c|^2 - \lambda} - \frac{2y_1 y_3 y_4}{|k_1^c + k_2^c|^2 - \lambda}).$$

Using this, we can write (4.20) more conveniently as

$$\dot{y_1} = -|k_1^c|^2(|k_1^c|^2 - \lambda)y_1$$

$$- 2\gamma_2^2 |k_1^c|^2 (\frac{-2y_1 y_3 y_4}{|k_1^c - k_2^c|^2 - \lambda} - \frac{2y_1 y_3 y_4}{|k_1^c + k_2^c|^2 - \lambda} + \frac{y_1^2 y_2}{-4|k_1^c|^2 - \lambda})$$

$$+ \gamma_3 |k_1^c|^2 (3y_1^2 y_2 + 6y_1 y_3 y_4). \tag{4.21}$$

Following this procedure for the other three differential equations in (4.5) yields a system of reduced equations which will be used to determine the transition type and stability of solutions of the Cahn-Hilliard equation.

By omitting the higher order terms $o(3) := o(|y^3|) + O(|y^3|\beta_1(\lambda)|)$, the reduced system of (2.11) on the center manifold is given by the following:

$$y_{1t} = \beta_1(\lambda)y_1 - 2\gamma_2^2|k_1^c|^2\Big(\frac{-2y_1y_3y_4}{|k_1^c-k_2^c|^2-\lambda} - \frac{2y_1y_3y_4}{|k_1^c+k_2^c|^2-\lambda} + \frac{y_1^2y_2}{-4|k_1^c|^2+\lambda}\Big) - \gamma_3|k_1^c|^2(3y_1^2y_2 + 6y_1y_3y_4), \tag{4.22}$$

$$y_{2t} = \beta_1(\lambda)y_2 - 2\gamma_2^2|k_1^c|^2\Big(\frac{-2y_2y_3y_4}{|k_1^c-k_2^c|^2-\lambda} - \frac{2y_2y_3y_4}{|k_1^c+k_2^c|^2-\lambda} + \frac{y_1y_2^2}{-4|k_1^c|^2+\lambda}\Big) - \gamma_3|k_1^c|^2(3y_1y_2^2 + 6y_2y_3y_4), \tag{4.23}$$

$$y_{3t} = \beta_1(\lambda)y_3 - 2\gamma_2^2|k_1^c|^2\Big(\frac{-2y_1y_2y_3}{|k_1^c-k_2^c|^2-\lambda} - \frac{2y_1y_2y_3}{|k_1^c+k_2^c|^2-\lambda} + \frac{y_3^2y_4}{-4|k_1^c|^2+\lambda}\Big) - \gamma_3|k_1^c|^2(3y_3^2y_4 + 6y_1y_2y_3), \tag{4.24}$$

$$y_{4t} = \beta_1(\lambda)y_4 - 2\gamma_2^2|k_1^c|^2\Big(\frac{-2y_1y_2y_4}{|k_1^c-k_2^c|^2-\lambda} - \frac{2y_1y_2y_4}{|k_1^c+k_2^c|^2-\lambda} + \frac{y_3y_4^2}{-4|k_1^c|^2+\lambda}\Big) - \gamma_3|k_1^c|^2(3y_3y_4^2 + 6y_1y_2y_4). \tag{4.25}$$

Letting $\lambda = \lambda_0 = |k_1|^2$, $y_2 = \overline{y_1}$, and $y_4 = \overline{y_3}$ gives

$$y_{1t} = 2\gamma_2^2|k_1^c|^2\Big(\frac{2y_1|y_3|^2}{|k_1^c-k_2^c|^2-|k_1^c|^2} + \frac{2y_1|y_3|^2}{|k_1^c+k_2^c|^2-|k_1^c|^2} + \frac{y_1|y_1|^2}{3|k_1^c|^2}\Big) - \gamma_3|k_1^c|^2(3y_1|y_1|^2 + 6y_1|y_3|^2), \tag{4.26}$$

$$\overline{y_{1t}} = 2\gamma_2^2|k_1^c|^2\Big(\frac{2\overline{y_1}|y_3|^2}{|k_1^c-k_2^c|^2-|k_1^c|^2} + \frac{2\overline{y_1}|y_3|^2}{|k_1^c+k_2^c|^2-|k_1^c|^2} + \frac{\overline{y_1}|y_1|^2}{3|k_1^c|^2}\Big) - \gamma_3|k_1^c|^2(3\overline{y_1}|y_1|^2 + 6\overline{y_1}|y_3|^2), \tag{4.27}$$

$$y_{3t} = 2\gamma_2^2|k_1^c|^2\Big(\frac{2y_3|y_1|^2}{|k_1^c-k_2^c|^2-|k_1^c|^2} + \frac{2y_3|y_1|^2}{|k_1^c+k_2^c|^2-|k_1^c|^2} + \frac{y_3|y_3|^2}{3|k_1^c|^2}\Big) - \gamma_3|k_1^c|^2(3y_3|y_3|^2 + 6y_3|y_1|^2), \tag{4.28}$$

$$\overline{y_{3t}} = 2\gamma_2^2|k_1^c|^2\Big(\frac{2\overline{y_3}|y_1|^2}{|k_1^c-k_2^c|^2-|k_1^c|^2} + \frac{2\overline{y_3}|y_1|^2}{|k_1^c+k_2^c|^2-|k_1|^2} + \frac{\overline{y_3}|y_3|^2}{3|k_1^c|^2}\Big) - \gamma_3|k_1^c|^2(3\overline{y_3}|y_3|^2 + 6\overline{y_3}|y_1|^2). \tag{4.29}$$

Now break up the variables into real and imaginary components, so that $y_1 = a_1 + a_2 i$ and $y_3 = a_3 + a_4 i$. By combining real and imaginary parts, this gives the leading order approximation of reduced system at the critical threshold λ_0:

$$\begin{aligned} a_{1t} &= a_1(\xi(a_1^2+a_2^2) + \eta(a_3^2+a_4^2)),\\ a_{2t} &= a_2(\xi(a_1^2+a_2^2) + \eta(a_3^2+a_4^2)),\\ a_{3t} &= a_3(\eta(a_1^2+a_2^2) + \xi(a_3^2+a_4^2)),\\ a_{4t} &= a_4(\eta(a_1^2+a_2^2) + \xi(a_3^2+a_4^2)), \end{aligned} \tag{4.30}$$

where

$$\begin{aligned} \xi &= \frac{2\gamma_2^2 - 9|k_1^c|^2\gamma_3}{3},\\ \eta &= 2|k_1^c|^2\Big(\frac{2\gamma_2^2}{|k_1^c-k_2^c|^2-|k_1^c|^2} + \frac{2\gamma_2^2}{|k_1^c+k_2^c|^2-|k_1^c|^2} - 3\gamma_3\Big). \end{aligned} \tag{4.31}$$

It can be seen that for $i, j, k, l \in [1, 4] \cap \mathbb{Z}$ where $i \neq j \neq k \neq l$, the 48 straight line orbits come from the 24 straight lines given by

$$a_i = a_j = a_k = 0, \tag{4.32}$$

$$a_1^2 = a_2^2 = a_3^2 = a_4^2, \tag{4.33}$$

$$a_i = a_j = 0 \text{ and } a_k^2 = a_l^2. \tag{4.34}$$

Finally, the reduced system

$$\begin{aligned} a_{1t} &= \beta a_1 + a_1(\xi(a_1^2 + a_2^2) + \eta(a_3^2 + a_4^2)) + o(3), \\ a_{2t} &= \beta a_2 + a_2(\xi(a_1^2 + a_2^2) + \eta(a_3^2 + a_4^2)) + o(3), \\ a_{3t} &= \beta a_3 + a_3(\eta(a_1^2 + a_2^2) + \xi(a_3^2 + a_4^2)) + o(3), \\ a_{4t} &= \beta a_4 + a_4(\eta(a_1^2 + a_2^2) + \xi(a_3^2 + a_4^2)) + o(3), \end{aligned} \tag{4.35}$$

where $\beta = \beta_{10}(\lambda)$ and $o(3) = o(|a|^3) + O(|a|^3 * |\beta|)$. By combining equations as well as letting $a_1^2 + a_2^2 = r_1^2$ and $a_3^2 + a_4^2 = r_2^2$ gives the system $(*)$

$$\begin{aligned} r_{1t} &= \beta r_1 + r_1(\xi r_1^2 + \eta r_2^2) + o(3), \\ r_{2t} &= \beta r_2 + r_2(\eta r_1^2 + \xi r_2^2) + o(3). \end{aligned} \tag{4.36}$$

4.2. **Dynamical transition theorem.**

Theorem 4.1 (Transition Types of Multiplicity Four Case). *Consider system (2.12). Let the multiplicity of $\beta_1(\lambda_0)$ be four. Let*

$$A = \max\{(\frac{2}{27|k_1^c|^2} + \frac{4}{9}(\frac{1}{|k_1^c - k_2^c|^2 - |k_1^c|^2} + \frac{1}{|k_1^c + k_2^c|^2 - |k_1^c|^2})), \frac{2}{9|k_1^c|^2}\}. \tag{4.37}$$

Then the following assertions hold true:
1. If

$$\gamma_3 > A\gamma_2^2, \tag{4.38}$$

then as λ crosses λ_0, the system (2.11) undergoes a continuous (Type I) dynamic transition to a local attractor Σ_λ, homological to S^3.
2. If

$$\gamma_3 < A\gamma_2^2, \tag{4.39}$$

then the system undergoes a jump (Type II) dynamic transition as λ crosses λ_0.

Remark: If the system undergoes a Type II transition, then the following are true:
1. Let $\lambda < \lambda_0$ and λ be near λ_0, then the system undergoes a sub-critical bifurcation.
2. There exists $\lambda^* < \lambda_0$ at which a saddle node bifurcation occurs.
These statements are true regarding all Type II transitions in future theorems.

Proof. The transition type at the critical point $\lambda_0 = |k_1|^2$ is given by the system

$$\begin{aligned} r_{1t} &= r_1(\xi r_1^2 + \eta r_2^2), \\ r_{2t} &= r_2(\eta r_1^2 + \xi r_2^2), \end{aligned} \tag{4.40}$$

where ξ and η are defined in (4.31). It can be calculated that

(4.41)
$$\xi+\eta>0 \iff \gamma_3<\left(\frac{2}{27|k_1^c|^2}+\frac{4}{9}\left(\frac{1}{|k_1^c-k_2^c|^2-|k_1^c|^2}+\frac{1}{|k_1^c+k_2^c|^2-|k_1|^2}\right)\right)\gamma_2^2,$$

(4.42)
$$\xi+\eta<0 \iff \gamma_3>\left(\frac{2}{27|k_1^c|^2}+\frac{4}{9}\left(\frac{1}{|k_1^c-k_2^c|^2-|k_1^c|^2}+\frac{1}{|k_1^c+k_2^c|^2-|k_1^c|^2}\right)\right)\gamma_2^2,$$

(4.43)
$$\xi>0 \iff \gamma_3<\frac{2}{9|k_1^c|^2}\gamma_2^2,$$

(4.44)
$$\xi<0 \iff \gamma_3>\frac{2}{9|k_1^c|^2}\gamma_2^2,$$

(4.45)
$$\eta>0 \iff \gamma_3<\frac{4|k_1^c|^2}{3}\left(\frac{1}{|k_1^c-k_2^c|^2-|k_1^c|^2}+\frac{1}{|k_1^c+k_2^c|^2-|k_1^c|^2}\right)\gamma_2^2,$$

(4.46)
$$\eta<0 \iff \gamma_3>\frac{4|k_1^c|^2}{3}\left(\frac{1}{|k_1^c-k_2^c|^2-|k_1^c|^2}+\frac{1}{|k_1^c+k_2^c|^2-|k_1^c|^2}\right)\gamma_2^2.$$

Note that there are no elliptic orbits as the system (2.12) is a gradient system (see Lemma A.2.7 in [5]). Observe that on the straight line $r_1=r_2$, the system satisfies $\frac{dr_2}{dr_1}=\frac{r_2}{r_1}$ where $\xi+\eta\neq 0$ and $r_1,r_2>0$. It can be seen that $r_1=0$ and $r_2=0$ as well as $r_1=r_2$ are the straight lines corresponding to the three straight line orbits of this system.

Note that on the straight line $r_1=r_2$, the system reduces to

(4.47)
$$\begin{aligned} r_{1t}&=r_1^3(\xi+\eta),\\ r_{2t}&=r_2^3(\eta+\xi). \end{aligned}$$

Thus, when $\xi+\eta>0$, the solutions tend away from the origin, and when $\xi+\eta<0$, the solutions tend towards the origin. In other words, when $\gamma_3<(\frac{2}{27|k_1^c|^2}+\frac{4}{9}(\frac{1}{|k_1^c-k_2^c|^2-|k_1^c|^2}+\frac{1}{|k_1^c+k_2^c|^2-|k_1^c|^2}))\gamma_2^2$, the solutions tend away from the origin, and when $\gamma_3>(\frac{2}{27|k_1^c|^2}+\frac{4}{9}(\frac{1}{|k_1^c-k_2^c|^2-|k_1^c|^2}+\frac{1}{|k_1^c+k_2^c|^2-|k_1^c|^2}))\gamma_2^2$, the solutions tend towards the origin.

Note that on the straight line $r_1=0$, the system reduces to

(4.48)
$$\begin{aligned} r_{1t}&=0,\\ r_{2t}&=\xi r_2^3. \end{aligned}$$

Thus, when $\xi>0$, the solutions tend away from the origin, and when $\xi<0$, the solutions tend towards the origin. In other words, when $\gamma_3<\frac{2}{9|k_1^c|^2}\gamma_2^2$, the solutions tend away from the origin, and when $\gamma_3>\frac{2}{9|k_1^c|^2}\gamma_2^2$, the solutions tend towards the origin.

Note that on the straight line $r_2=0$, the system reduces to

(4.49)
$$\begin{aligned} r_{1t}&=\xi r_1^3,\\ r_{2t}&=0. \end{aligned}$$

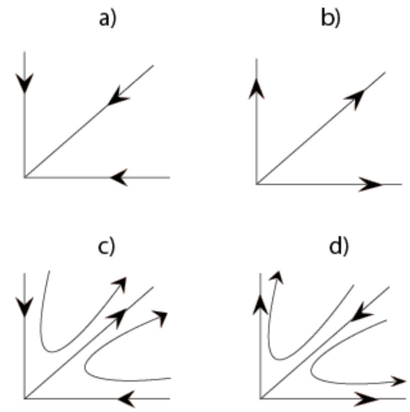

FIGURE 2. Straight line orbits for: (a) $\xi + \eta < 0$ and $\xi < 0$; (b) $\xi + \eta > 0$ and $\xi > 0$; (c) $\xi + \eta > 0$ and $\xi < 0$; (d) $\xi + \eta < 0$ and $\xi > 0$.

Thus, when $\xi > 0$, the solutions tend away from the origin, and when $\xi < 0$, the solutions tend towards the origin. In other words, when $\gamma_3 < \frac{2}{9|k_1^c|^2}\gamma_2^2$, the solutions tend away from the origin, and when $\gamma_3 > \frac{2}{9|k_1^c|^2}\gamma_2^2$, the solutions tend towards the origin.

By putting this together, when $\xi + \eta < 0$ and $\xi < 0$, or equivalently $\gamma_3 > \max\{(\frac{2}{27|k_1^c|^2} + \frac{4}{9}(\frac{1}{|k_1^c - k_2^c|^2 - |k_1^c|^2} + \frac{1}{|k_1^c + k_2^c|^2 - |k_1^c|^2})), \frac{2}{9|k_1^c|^2}\}\gamma_2^2$, solutions along all three of the straight lines mentioned above approach the origin, so the transition is Type I.

When $\xi + \eta > 0$ and $\xi > 0$, or equivalently $\gamma_3 < \min\{(\frac{2}{27|k_1^c|^2} + \frac{4}{9}(\frac{1}{|k_1^c - k_2^c|^2 - |k_1^c|^2} + \frac{1}{|k_1^c + k_2^c|^2 - |k_1^c|^2})), \frac{2}{9|k_1^c|^2}\}\gamma_2^2$, solutions along all three of the straight lines mentioned above tend away from the origin, so the transition is Type II.

If $\frac{2}{9|k_1^c|^2}\gamma_2^2 < (\frac{2}{27|k_1^c|^2} + \frac{4}{9}(\frac{1}{|k_1^c - k_2^c|^2 - |k_1^c|^2} + \frac{1}{|k_1^c + k_2^c|^2 - |k_1^c|^2}))\gamma_2^2$ and $\frac{2}{9|k_1^c|^2}\gamma_2^2 < \gamma_3 < (\frac{2}{27|k_1^c|^2} + \frac{4}{9}(\frac{1}{|k_1^c - k_2^c|^2 - |k_1^c|^2} + \frac{1}{|k_1^c + k_2^c|^2 - |k_1^c|^2}))\gamma_2^2$, then $\xi < 0$ and $\xi + \eta > 0$. This means that solutions along the line $r_1 = r_2$ tend away from the origin, but solutions along the lines $r_1 = 0$ and $r_2 = 0$ tend towards the origin, so the transition is Type II.

If $\frac{2}{9|k_1^c|^2}\gamma_2^2 > (\frac{2}{27|k_1^c|^2} + \frac{4}{9}(\frac{1}{|k_1^c - k_2^c|^2 - |k_1^c|^2} + \frac{1}{|k_1^c + k_2^c|^2 - |k_1^c|^2}))\gamma_2^2$ and $(\frac{2}{27|k_1^c|^2} + \frac{4}{9}(\frac{1}{|k_1^c - k_2^c|^2 - |k_1^c|^2} + \frac{1}{|k_1^c + k_2^c|^2 - |k_1^c|^2}))\gamma_2^2 < \gamma_3 < \frac{2}{9|k_1^c|^2}\gamma_2^2$, then $\xi > 0$ and $\xi + \eta < 0$. This means that solutions along the line $r_1 = r_2$ tend towards the origin, but solutions along the lines $r_1 = 0$ and $r_2 = 0$ tend away from the origin, so the transition is Type II. □

FIGURE 1 show graphs of the straight line orbits in r_1, r_2 space. Notice that only FIGURE 1(A) features solutions that approach the origin, hence why it is Type I.

4.3. **Structure of the set of transition states.** To study the detailed structure of the local attractor Σ_λ in Theorem 4.1, representing all transition states for

$\lambda > \lambda_0$, we examine the nontrivial fixed points of the reduced system by solving

$$\begin{aligned} \beta r_1 &= -r_1(\xi r_1^2 + \eta r_2^2), \\ \beta r_2 &= -r_2(\eta r_1^2 + \xi r_2^2). \end{aligned} \tag{4.50}$$

The solutions are

$$p_1 =(r_1, r_2) = (0, \sqrt{\frac{-\beta}{\xi}}), \tag{4.51}$$

$$p_2 =(r_1, r_2) = (\sqrt{\frac{-\beta}{\xi}}, 0), \tag{4.52}$$

$$p_3 =(r_1, r_2) = (\sqrt{\frac{-\beta}{\xi+\eta}}, \sqrt{\frac{-\beta}{\xi+\eta}}), \quad \text{if } \xi^2 \neq \eta^2. \tag{4.53}$$

Note that the general solution u in Σ_λ is given by

$$u = y_1 e_1 + y_2 e_2 + y_3 e_3 + y_4 e_4 + \Phi(y, \lambda), \tag{4.54}$$

with $\Phi = o(|y|)$. Therefore, the eigenfunctions $\sum_{k=1}^4 y_k e_k$ dictate the typical patterns of solutions, represented by Σ_λ, for λ beyond the critical threshold λ_0. The Jacobian of the system at a fixed point (r_1, r_2) is given by

$$J = \begin{pmatrix} \beta + 3\xi r_1^2 + \eta r_2^2 & 2\eta r_1 r_2 \\ 2\eta r_1 r_2 & \beta + \eta r_1^2 + 3\xi r_2^2 \end{pmatrix}. \tag{4.55}$$

Consider the solution $(r_1, r_2) = (0, \sqrt{\frac{-\beta}{\xi}})$. It can be seen that the Jacobian calculated at this solution is $diag(\beta + \eta|\frac{\beta}{\xi}|, \beta + 3\xi|\frac{\beta}{\xi}|)$. Thus, if $\beta > 0$, then $-|\xi| < \eta$ implies that this solution is a saddle and $-|\xi| > \eta$ implies this solution is stable.

Consider the solution $(r_1, r_2) = (\sqrt{\frac{-\beta}{\xi}}, 0)$. It can be seen that the Jacobian calculated at this solution is $diag(\beta + 3\xi|\frac{\beta}{\xi}|, \beta + \eta|\frac{\beta}{\xi}|)$. Thus, if $\beta > 0$, then $-|\xi| < \eta$ implies that this solution is a saddle and $-|\xi| > \eta$ implies this solution is stable.

Consider the solution $(r_1, r_2) = (\sqrt{\frac{-\beta}{\xi+\eta}}, \sqrt{\frac{-\beta}{\xi+\eta}})$ with the condition that $\xi^2 \neq \eta^2$. The Jacobian calculated at this solution is

$$\begin{pmatrix} \beta + (3\xi + \eta)|\frac{\beta}{\xi+\eta}| & 2\eta|\frac{\beta}{\xi+\eta}| \\ 2\eta|\frac{\beta}{\xi+\eta}| & \beta + (3\xi + \eta)|\frac{\beta}{\xi+\eta}| \end{pmatrix}. \tag{4.56}$$

It can be calculated that the eigenvalues of this matrix are $\beta - (\eta - 3\xi)|\frac{\beta}{\xi+\eta}|$ and $\beta + 3\beta(\frac{\eta+\xi}{\beta} * |\frac{\beta}{\xi+\eta}|)$. If $\beta > 0$, then $\eta > \xi$ implies this solution is stable and $\eta < \xi$ implies this solution is a saddle. In conclusion, we have the following theorem:

Theorem 4.2 (Structure of Σ_λ)**.** *In the collapsed phase space (r_1, r_2), the bifurcated attractor from case one in Theorem 4.1, $\Sigma_\lambda \approx S^3$, collapses to an arc in the first quadrant, Σ_λ^r, which contains three fixed points: p_1, p_2, p_3, given in (4.83-4.85). For $j \in \{1, 2, 3\}$, p_j generates the circles (p_1 and p_2) or torus (p_3) of steady states $(r_1^{(j)} e^{i\theta}, r_2^{(j)} e^{i\theta})$, which are all contained in Σ_λ.*

4.4. **Example: square lattice.** Let $l_1 = (\frac{2\pi}{50}, 0)$ and $l_2 = (0, \frac{2\pi}{50})$. The scaling factor 50 is chosen so that the patterns are easier to be visualized. Then, the dual lattice is spanned by the vectors $k_1 = (50, 0)$ and $k_2 = (0, 50)$. The critical vectors in this case are $k_1^c = k_1$ and $k_2^c = k_2$. Thus $|k_1^c|^2 = |k_2^c|^2 = 50$ and $|k_1^c + k_2^c|^2 = |k_1^c - k_2^c|^2 = 5000$. We also have that $\beta = 50\lambda - 2500$, $\xi = \frac{2}{3}\gamma_2^2 - 150\gamma_3$, and $\eta = \frac{8}{99}\gamma_2^2 - 300\gamma_3$. From the theorem in section 6, when $\gamma_3 > \max\{\frac{37}{22275}, \frac{2}{450}\}\gamma_2^2 = \frac{2}{450}\gamma_2^2$, all straight line orbits tend towards the origin and the transition is Type I. When $\frac{37}{22275}\gamma_2^2 < \gamma_3 < \frac{2}{450}\gamma_2^2$, solutions along the straight line $r_1 = r_2$ tend away from zero, but solutions along $r_1 = 0$ and $r_2 = 0$ tend towards the origin and the transition is Type II. When $\gamma_3 < \frac{37}{22275}\gamma_2^2$, all straight line orbits tend away from the origin and the transition is Type II.

The three stationary solutions are

$$\begin{aligned}
p_1 &= (r_1, r_2) = (0, \sqrt{\frac{7500 - 150\lambda}{2\gamma_2^2 - 450\gamma_3}}),\\
p_2 &= (r_1, r_2) = (\sqrt{\frac{7500 - 150\lambda}{2\gamma_2^2 - 450\gamma_3}}, 0),\\
p_3 &= (r_1, r_2) = (\sqrt{\frac{247500 - 4950\lambda}{74\gamma_2^2 - 44550\gamma_3}}, \sqrt{\frac{247500 - 4950\lambda}{74\gamma_2^2 - 44550\gamma_3}}).
\end{aligned} \tag{4.57}$$

The trivial solution is unstable when the control parameter exceeds the critical threshold, i.e. when $\lambda > 50$. Now let $\lambda > 50$. For the first and second solutions, $\frac{26}{27}\gamma_2^2 < \gamma_3 < \frac{22}{9}\gamma_2^2$ implies the solutions are stable, and else are saddles. For the third solution, $\frac{26}{27}\gamma_2^2 < \gamma_3 < \frac{22}{9}\gamma_2^2$ implies the solution is stable, and else is a saddle.

Consider the solution $(r_1, r_2) = (0, \sqrt{\frac{7500-150\lambda}{2\gamma_2^2-450\gamma_3}})$. Recalling that $a_1^2 + a_2^2 = r_1^2$, $a_3^2 + a_4^2 = r_2^2$, and further that $y_1 = a_1 + a_2 i$, $y_3 = a_3 + a_4 i$, we see that our stationary solutions to the reduced system are radial. From (4.2), we can write the solutions as $u(x,t) = y_1 e_1 + \bar{y_1}\bar{e_1} + y_3 e_3 + \bar{y_3}\bar{e_3}$ where the stable component is of little significance anymore and can be dropped. In this case, $r_1 = 0$ so the solution becomes $u = y_3 e_3 + \bar{y_3}\bar{e_3}$. Expanding and noting that $k_2^c = (0, 50)$ and $e_3 = e^{ik_2^c x}$, we have

$$\begin{aligned}
u(x,t) &= (a_3 + ia_4)(\cos(k_2^c \cdot x)\\
&\quad + i\sin(k_2^c \cdot x)) + (a_3 - ia_4)(\cos(k_2^c \cdot x) - i\sin(k_2^c \cdot x))\\
&= 2(a_3\cos(50x_2) - a_4\sin(50x_2)),
\end{aligned} \tag{4.58}$$

where $x = (x_1, x_2)$. As (a_3, a_4) run along the circle $a_3^2 + a_4^2 = r_2^2$, a set of stationary solutions is generated in (x,t)-space. The principle exchange of stability guarantees that patterns in the form of solutions to (2.5) emerge as λ crosses the critical threshold $\lambda_0 = 50$. FIGURE 2 shows a graph of the stationary solution $(r_1, r_2) = (0, \sqrt{\frac{7500-150\lambda}{2\gamma_2^2-450\gamma_3}})$ when $\lambda = 50.1$, $\gamma_2 = 1$, and $\gamma_3 = \frac{17}{450}$. In this case, $(r_1, r_2) = (0, 1)$ and $\gamma_3 < \frac{26}{27}\gamma_2^2$ so that the solution is a saddle and we have chosen the point $(a_3, a_4) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ on the circle of solutions. The characteristic patterns for this stationary solution are horizontal rolls. Graphically, we see that the size of the domain is responsible for the amount of rolls within the square, and the

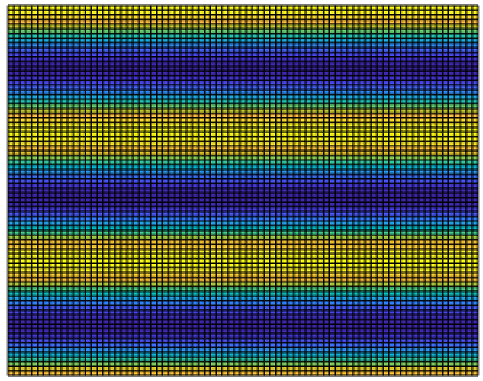

FIGURE 3. Horizontal rolls exhibited by the stationary solution $u(x,t) = \sqrt{2}\cos(50x_2) - \sqrt{2}\sin(50x_2)$.

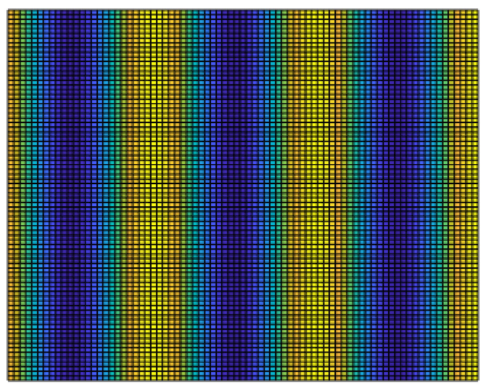

FIGURE 4. Vertical rolls exhibited by the stationary solution $u(x,t) = \sqrt{2}\cos(50x_1) - \sqrt{2}\sin(50x_1)$.

patterns shift vertically as the parameters (a_1, a_2) run along the unit circle.

Likewise, the solution for $(r_1, r_2) = (\sqrt{\frac{7500-150\lambda}{2\gamma_2^2-450\gamma_3}}, 0)$ can be written as

$$\begin{aligned} u(x,t) &= (a_1 + ia_2)(\cos(k_1^c \cdot x) \\ &\quad + i\sin(k_1^c \cdot x)) + (a_1 - ia_2)(\cos(k_1^c \cdot x) - i\sin(k_1^c \cdot x)) \\ &= 2(a_1\cos(50x_1) - a_2\sin(50x_1)), \end{aligned} \tag{4.59}$$

which also produces rolls, however this time horizontal. FIGURE 3 shows a graph of this solution for the same choice of constants and parameters used in FIGURE 2.

Finally, the solution for $(r_1, r_2) = (\sqrt{\frac{247500-4950\lambda}{74\gamma_2^2-44550\gamma_3}}, \sqrt{\frac{247500-4950\lambda}{74\gamma_2^2-44550\gamma_3}})$ can be written as the sum of the previous solutions

$$u(x,t) = 2(a_1\cos(x_1) - a_2\sin(x_1) + a_3\cos(x_2) - a_4\sin(x_2)), \tag{4.60}$$

where both (a_1, a_2) and (a_3, a_4) run along different circles centered at the origin with radii $r_1 = r_2$. FIGURE 4 shows the graph of this solution for $\lambda = 50.1$, $\gamma_2 = 1$,

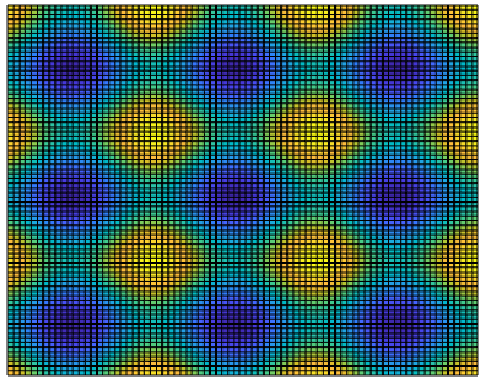

FIGURE 5. Square-packed circles exhibited by stationary solution $u(x,t) = \sqrt{2}\cos(50x_1) - \sqrt{2}\sin(50x_1) + \sqrt{2}\cos(50x_2) - \sqrt{2}$ $\sin(50x_2)$.

$\gamma_3 = \frac{569}{445500}$. In this case, $(r_1, r_2) = (1, 1)$, and $\gamma_3 > \frac{26}{27}\gamma_2^2$ so that the solution is stable. Here, we have chosen $(a_1, a_2) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $(a_3, a_4) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ as the two points on the unit circle. Notice here that the characteristic patterns are circles and that they are arranged in a square-like fashion throughout the lattice. We will encounter another pattern that exhibits circles packed in a different manner, namely hexagonally-packed circles.

5. Multiplicity Two Case

5.1. **Dynamical transition theorem.** Consider the same situation as above but assume that the pairs of (n_1, n_2) that minimize $|n_1 k_1 + n_2 k_2|^2$ are (n_1^c, n_2^c) and $(-n_1^c, -n_2^c)$. Equivalently, the cardinality of S is two. In this case, $\beta_{n_1 n_2}(\lambda_0) < 0$ if $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0), (n_1^c, n_2^c), (-n_1^c, -n_2^c)\}$. Thus, with critical value λ_0, the eigenvalue $-|n_1^c k_1 + n_2^c k_2|^2(|n_1^c k_1 + n_2^c k_2|^2 - \lambda) = -|k_c|^2(|k_c|^2 - \lambda)$, has multiplicity two:

$$
\begin{aligned}
E_1^\lambda &= span\{e_1 = e^{i(k_c \cdot x)}, e_2 = e^{-i(k_c \cdot x)}\}, \\
E_2^\lambda &= \overline{span\{e_3(\lambda), \ldots\}}.
\end{aligned} \tag{5.1}
$$

An example of this is when the lattice is spanned by the vectors $l_1 = (\frac{\pi}{25}, -\frac{\sqrt{3}\pi}{25})$ and $l_2 = (0, \frac{2\pi}{25})$, which will be discussed in section 5.2. The solution can thus be written as

$$
u(x,t) = y_1 e_1 + y_2 e_2 + z, \tag{5.2}
$$

where $z \in E_2^\lambda$ is the stable component. By similar computation, the center manifold function up to higher order terms is given by

$$
\phi(x) = \frac{-\gamma_2 y_1^2}{4|k_c|^2 - \lambda} e^{2ik_c \cdot x} - \frac{\gamma_2 y_2^2}{4|k_c|^2 - \lambda} e^{-2ik_c \cdot x}. \tag{5.3}
$$

Using this function, it can be calculated that the reduced equations for this system are

$$\begin{aligned} y_{1t} &= -|k_c|^2(|k_c|^2-\lambda)y_1 + \frac{2|k_c|^2\gamma_2^2 y_1^2 y_2}{4|k_c|^2-\lambda} - 3|k_c|^2 y_1^2 y_2\gamma_3 + o(3), \\ y_{2t} &= -|k_c|^2(|k_c|^2-\lambda)y_2 + \frac{2|k_c|^2\gamma_2^2 y_1 y_2^2}{4|k_c|^2-\lambda} - 3|k_c|^2 y_1 y_2^2\gamma_3 + o(3). \end{aligned} \tag{5.4}$$

Theorem 5.1 (Transition Types for Multiplicity Two)**.** *Assume the multiplicity of β_1 is two at $\lambda = \lambda_0 = |k_c|^2$. The following are true:*
1. If $\gamma_3 > \frac{2}{9|k_c|^2}\gamma_2^2$ the system undergoes a continuous dynamical transition (Type I) to $\Sigma_\lambda \approx S^1$ consisting of a circle of steady-states as λ crosses λ_0.
2. If $\gamma_3 < \frac{2}{9|k_c|^2}\gamma_2^2$ the system undergoes a jump dynamical transition (Type II) as λ crosses λ_0.

Proof. By letting

$$\begin{aligned} \lambda = \lambda_0 = |k_c|^2, \quad y_1 = a_1 + a_2 i, \\ y_2 = a_1 - a_2 i, \qquad \eta = \frac{2}{3}\gamma_2^2 - 3|k_c|^2\gamma_3, \end{aligned} \tag{5.5}$$

the system given by (5.4) can be rewritten as

$$\begin{aligned} a_{1t} &= \eta a_1(a_1^2 + a_2^2), \\ a_{2t} &= \eta a_2(a_1^2 + a_2^2). \end{aligned} \tag{5.6}$$

By analyzing this system, it can be seen that all solutions tend towards the origin when $\eta < 0$ and tend away from the origin when $\eta > 0$. Thus, the transition is Type I when $\eta < 0$ and Type II when $\eta > 0$. □

By using the approximative system

$$a_{1t} = \beta a_1 + \eta a_1(a_1^2 + a_2^2), \tag{5.7}$$

$$a_{2t} = \beta a_2 + \eta a_2(a_1^2 + a_2^2), \tag{5.8}$$

and letting $a_1^2 + a_2^2 = r^2$, this system can be rewritten as

$$r_t = \beta r + \eta r^3. \tag{5.9}$$

The nontrivial equilibrium of this system is $r = \sqrt{\frac{-\beta}{\eta}}$. The Jacobian of this system at a fixed point r is

$$J = \left(\beta + 3\eta r^2\right). \tag{5.10}$$

When $r = \sqrt{\frac{-\beta}{\eta}}$, the eigenvalue of the Jacobian is $\beta + 3\eta|\frac{\beta}{\eta}|$. If $\beta > 0$, then $\eta < 0$ must be true, which implies that this solution will be stable. Stationary solutions in this case are given by

$$\begin{aligned} u(x,t) &= y_1 e_1 + \bar{y_1}\bar{e_1} \\ &= (a_1 + ia_2)(\cos(k_c\cdot x) + i\sin(k_c\cdot x)) \\ &+ (a_1 - ia_2)(\cos(k_c\cdot x) - i\sin(k_c\cdot x)) \\ &= 2(a_1\cos(k_c\cdot x) - a_2\sin(k_c\cdot x)), \end{aligned} \tag{5.11}$$

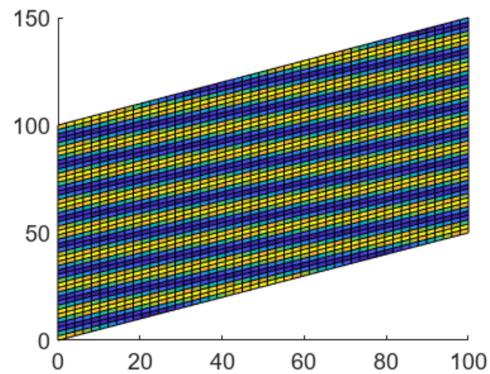

FIGURE 6. Stationary solution r=1

where (a_1, a_2) run along the circle $a_1^2 + a_2^2 = r^2$. Note that solutions depend solely on the two critical vectors of the lattice in which the magnitudes are least, and that the spanning vectors play no direct role besides specifying the domain.

5.2. **Example: roll patterns on parallelogram.** Let $l_1 = (\frac{\pi}{25}, -\frac{\sqrt{3}\pi}{25})$ and $l_2 = (0, \frac{2\pi}{25})$. Then the dual lattice is spanned by the vectors where $k_1 = (50, 0)$ and $k_2 = (25\sqrt{3}, 25)$. It can be shown that $|k_1|^2 = |k_2|^2 = 2500$, $|k_1 + k_2|^2 = 3125 + 1250\sqrt{3}$, and $|k_1 - k_2|^2 = 5000 - 2500\sqrt{3}$. The critical points of the lattice are $k_2 - k_1$ and $k_1 - k_2$ and so we will use the analysis outlined in the section dealing lwith multiplicity two. In this case, it can be shown that $\beta = -(5000 - 2500\sqrt{3})(5000 - 2500\sqrt{3} - \lambda)$ and $\eta = \frac{2}{3}\gamma_2^2 - 3(5000 - 2500\sqrt{3})\gamma_3$. From section 5.1, when $\gamma_3 > \frac{2}{9(5000-2500\sqrt{3})}\gamma_2^2$, all straight line orbits tend towards the origin and the transition is Type I. When $\gamma_3 < \frac{2}{9(5000-2500\sqrt{3})}\gamma_2^2$ all straight line orbits tend away from the origin and the transition is Type II.

The nontrivial stationary solution is $r = \sqrt{\frac{(2-\sqrt{3})(2-\sqrt{3}-\lambda)}{\frac{2}{3}\gamma_2^2 - 3(2-\sqrt{3})\gamma_3}}$. The non-trivial solution is always stable for $\lambda > \lambda_0$. By section 5.1, the solution can be written as

$$u(x, t) = 2(a_1 \cos(k_c \cdot x) - a_2 \sin(k_c \cdot x)), \tag{5.12}$$

where $k_c = k_2 - k_1 = (\frac{\sqrt{3}}{2} - 1, \frac{1}{2})$ and (a_1, a_2) run along the circle $a_1^2 + a_2^2 = r^2$. FIGURE 5 shows a graph of the solution for $\lambda = 2$, $\gamma_2 = 1$, and $\gamma_3 = -\frac{7-6\sqrt{3}}{18-9\sqrt{3}}$, in which case $r = 1$. The parameters (a_1, a_2) are evaluated at the point $(\frac{\sqrt{3}}{2}, \frac{1}{2})$. Notice the characteristic patterns are horizontal rolls similar to the square case.

6. Multiplicity Six Case

6.1. **Dynamical transition theorem.** Consider the same situation as above but assume that $\#S = 6$. Then,

$$S = \{(n_1^c, n_2^c), (-n_1^c, -n_2^c), (n_3^c, n_4^c), (-n_3^c, -n_4^c), (n_5^c, n_6^c), (-n_5^c, -n_6^c)\}. \tag{6.1}$$

Thus, we have $\beta_{n_1n_2}(\lambda_0) < 0$ for all $(n_1, n_2) \in \mathbb{Z}^2 \setminus (S \cup \{(0,0)\})$. Thus, with critical value λ_0, the eigenvalue

$$\begin{aligned} \beta_{n_1^c n_2^c}(\lambda) &= -|n_1^c k_1 + n_2^c k_2|^2 (|n_1^c k_1 + n_2^c k_2|^2 - \lambda) \\ &= -|k_c|^2(|k_c|^2 - \lambda), \end{aligned} \tag{6.2}$$

has multiplicity six with

$$\begin{aligned} e_1 &= e^{i(k_1^c \cdot x)}, & e_2 &= e^{-i(k_1^c \cdot x)}, & e_3 &= e^{i(k_2^c \cdot x)}, \\ e_4 &= e^{-i(k_2^c \cdot x)}, & e_5 &= e^{i(k_3^c \cdot x)}, & e_6 &= e^{-i(k_3^c \cdot x)}, \\ E_1^\lambda &= span\{e_1, ..., e_6\}, \\ E_2^\lambda &= \overline{span\{e_7, e_8, ...\}}. \end{aligned} \tag{6.3}$$

The solution can thus be written as

$$u(x,t) = \sum_{i=1}^{6} y_i e_i + z \in E_1^\lambda \bigoplus E_2^\lambda, \tag{6.4}$$

where $z \in E_2^\lambda$ is the stable component. By similar computation, the center manifold function up to higher order terms is given by $\phi(x) = \sum_{i=7}^{24} \phi_i e_i$. Using the notation that

$$\begin{aligned} e_7 &= e^{2ik_1^c \cdot x}, & e_8 &= e^{-2ik_1^c \cdot x}, & e_9 &= e^{2ik_2^c \cdot x}, \\ e_{10} &= e^{-2ik_2^c \cdot x}, & e_{11} &= e^{2ik_3^c \cdot x}, & e_{12} &= e^{-2ik_3^c \cdot x}, \\ e_{13} &= e^{i(k_1^c + k_2^c) \cdot x}, & e_{14} &= e^{-i(k_1^c + k_2^c) \cdot x}, & e_{15} &= e^{i(k_1^c + k_3^c) \cdot x}, \\ e_{16} &= e^{-i(k_1^c + k_3^c) \cdot x}, & e_{17} &= e^{i(k_2^c + k_3^c) \cdot x}, & e_{18} &= e^{-i(k_2^c + k_3^c) \cdot x}, \\ e_{19} &= e^{i(k_1^c - k_2^c) \cdot x}, & e_{20} &= e^{-i(k_1^c - k_2^c) \cdot x}, & e_{21} &= e^{i(k_1^c - k_3^c) \cdot x}, \\ e_{22} &= e^{-i(k_1^c - k_3^c) \cdot x}, & e_{23} &= e^{i(k_2^c - k_3^c) \cdot x}, & e_{24} &= e^{-i(k_2^c - k_3^c) \cdot x}, \end{aligned} \tag{6.5}$$

it can be calculated that the coefficients of the manifold are

$$
\begin{aligned}
\phi_7 &= \frac{-\gamma_2 y_1^2}{4|k_1^c|^2-\lambda}, & \phi_8 &= \frac{-\gamma_2 y_2^2}{4|k_1^c|^2-\lambda},\\
\phi_9 &= \frac{-\gamma_2 y_3^2}{4|k_1^c|^2-\lambda}, & \phi_{10} &= \frac{-\gamma_2 y_4^2}{4|k_1^c|^2-\lambda},\\
\phi_{11} &= \frac{-\gamma_2 y_5^2}{4|k_1^c|^2-\lambda}, & \phi_{12} &= \frac{-\gamma_2 y_6^2}{4|k_1^c|^2-\lambda},\\
\phi_{13} &= \frac{-2\gamma_2 y_1 y_3}{|k_1^c+k_2^c|^2-\lambda}, & \phi_{14} &= \frac{-2\gamma_2 y_2 y_4}{|k_1^c+k_2^c|^2-\lambda},\\
\phi_{15} &= \frac{-2\gamma_2 y_1 y_5}{|k_1^c+k_3^c|^2-\lambda}, & \phi_{16} &= \frac{-2\gamma_2 y_2 y_6}{|k_1^c+k_3^c|^2-\lambda},\\
\phi_{17} &= \frac{-2\gamma_2 y_3 y_5}{|k_2^c+k_3^c|^2-\lambda}, & \phi_{18} &= \frac{-2\gamma_2 y_4 y_6}{|k_2^c+k_3^c|^2-\lambda},\\
\phi_{19} &= \frac{-2\gamma_2 y_1 y_4}{|k_1^c-k_2^c|^2-\lambda}, & \phi_{20} &= \frac{-2\gamma_2 y_2 y_3}{|k_1^c-k_2^c|^2-\lambda},\\
\phi_{21} &= \frac{-2\gamma_2 y_1 y_6}{|k_1^c-k_3^c|^2-\lambda}, & \phi_{22} &= \frac{-2\gamma_2 y_2 y_5}{|k_1^c-k_3^c|^2-\lambda},\\
\phi_{23} &= \frac{-2\gamma_2 y_3 y_6}{|k_2^c-k_3^c|^2-\lambda}, & \phi_{24} &= \frac{-2\gamma_2 y_4 y_5}{|k_2^c-k_3^c|^2-\lambda}.
\end{aligned} \tag{6.6}
$$

Let

$$
\begin{aligned}
\lambda &= \lambda_0 = |k_1^c|^2\\
y_1 &= a_1 + a_2 i,\\
y_2 &= a_1 - a_2 i,\\
y_3 &= a_3 + a_4 i,\\
y_4 &= a_3 - a_4 i,\\
y_5 &= a_5 + a_6 i,\\
y_6 &= a_5 - a_6 i,\\
D_{12}^{\pm} &= \frac{1}{|k_1^c \pm k_2^c|^2 - |k_1^c|^2},\\
D_{13}^{\pm} &= \frac{1}{|k_1^c \pm k_3^c|^2 - |k_1^c|^2},\\
D_{23}^{\pm} &= \frac{1}{|k_2^c \pm k_3^c|^2 - |k_1^c|^2},\\
\xi &= -3|k_1^c|^2\gamma_3 + \frac{2}{3}\gamma_2^2,\\
\eta &= -6|k_1^c|^2\gamma_3 + |k_1^c|^2(4D_{12}^- + 4D_{12}^+)\gamma_2^2,\\
\chi &= -6|k_1^c|^2\gamma_3 + |k_1^c|^2(4D_{13}^- + 4D_{13}^+)\gamma_2^2,\\
\omega &= -6|k_1^c|^2\gamma_3 + |k_1^c|^2(4D_{23}^- + 4D_{23}^+)\gamma_2^2.
\end{aligned} \tag{6.7}
$$

Then, the reduced system on the center manifold can be rewritten as

$$
\begin{aligned}
a_{1t} &= \beta a_1 + a_1(\xi(a_1^2+a_2^2) + \eta(a_3^2+a_4^2) + \chi(a_5^2+a_6^2)) + o(3),\\
a_{2t} &= \beta a_2 + a_2(\xi(a_1^2+a_2^2) + \eta(a_3^2+a_4^2) + \chi(a_5^2+a_6^2)) + o(3),\\
a_{3t} &= \beta a_3 + a_3(\eta(a_1^2+a_2^2) + \xi(a_3^2+a_4^2) + \omega(a_5^2+a_6^2)) + o(3),\\
a_{4t} &= \beta a_4 + a_4(\eta(a_1^2+a_2^2) + \xi(a_3^2+a_4^2) + \omega(a_5^2+a_6^2)) + o(3),\\
a_{5t} &= \beta a_5 + a_5(\chi(a_1^2+a_2^2) + \omega(a_3^2+a_4^2) + \xi(a_5^2+a_6^2)) + o(3),\\
a_{6t} &= \beta a_6 + a_6(\chi(a_1^2+a_2^2) + \omega(a_3^2+a_4^2) + \xi(a_5^2+a_6^2)) + o(3).
\end{aligned}
\tag{6.8}
$$

It can be calculated that

(6.9)

$$
\begin{aligned}
\xi > 0 &\iff \gamma_3 < \frac{2}{9|k_1^c|^2}\gamma_2^2,\\
\xi < 0 &\iff \gamma_3 > \frac{2}{9|k_1^c|^2}\gamma_2^2,\\
\xi + \omega > 0 &\iff \gamma_3 < (\frac{2}{27|k_1^c|^2} + \frac{4}{9}D_{23}^- + \frac{4}{9}D_{23}^+)\gamma_2^2,\\
\xi + \omega < 0 &\iff \gamma_3 > (\frac{2}{27|k_1^c|^2} + \frac{4}{9}D_{23}^- + \frac{4}{9}D_{23}^+)\gamma_2^2,\\
\xi + \chi > 0 &\iff \gamma_3 < (\frac{2}{27|k_1^c|^2} + \frac{4}{9}D_{13}^- + \frac{4}{9}D_{13}^+)\gamma_2^2,\\
\xi + \chi < 0 &\iff \gamma_3 > (\frac{2}{27|k_1^c|^2} + \frac{4}{9}D_{13}^- + \frac{4}{9}D_{13}^+)\gamma_2^2,\\
\xi + \eta > 0 &\iff \gamma_3 < (\frac{2}{27|k_1^c|^2} + \frac{4}{9}D_{12}^- + \frac{4}{9}D_{12}^+)\gamma_2^2,\\
\xi + \eta < 0 &\iff \gamma_3 > (\frac{2}{27|k_1^c|^2} + \frac{4}{9}D_{12}^- + \frac{4}{9}D_{12}^- D_{12}^+)\gamma_2^2,\\
\xi + \eta + \chi > 0 &\iff \gamma_3 < (\frac{2}{45|k_1^c|^2} + \frac{4}{15}D_{12}^- + \frac{4}{15}D_{12}^+ + \frac{4}{15}D_{13}^- + \frac{4}{15}D_{13}^+)\gamma_2^2,\\
\xi + \eta + \chi < 0 &\iff \gamma_3 > (\frac{2}{45|k_1^c|^2} + \frac{4}{15}D_{12}^- + \frac{4}{15}D_{12}^+ + \frac{4}{15}D_{13}^- + \frac{4}{15}D_{13}^+)\gamma_2^2,\\
\eta + \xi + \omega > 0 &\iff \gamma_3 < (\frac{2}{45|k_1^c|^2} + \frac{4}{15}D_{12}^- + \frac{4}{15}D_{12}^+ + \frac{4}{15}D_{23}^- + \frac{4}{15}D_{23}^+)\gamma_2^2,\\
\eta + \xi + \omega < 0 &\iff \gamma_3 > (\frac{2}{45|k_1^c|^2} + \frac{4}{15}D_{12}^- + \frac{4}{15}D_{12}^+ + \frac{4}{15}D_{23}^- + \frac{4}{15}D_{23}^+)\gamma_2^2,\\
\chi + \omega + \xi > 0 &\iff \gamma_3 < (\frac{2}{45|k_1^c|^2} + \frac{4}{15}D_{13}^- + \frac{4}{15}D_{13}^+ + \frac{4}{15}D_{23}^- + \frac{4}{15}D_{23}^+)\gamma_2^2,\\
\chi + \omega + \xi < 0 &\iff \gamma_3 > (\frac{2}{45|k_1^c|^2} + \frac{4}{15}D_{13}^- + \frac{4}{15}D_{13}^+ + \frac{4}{15}D_{23}^- + \frac{4}{15}D_{23}^+)\gamma_2^2.
\end{aligned}
$$

By letting $r_1^2 = a_1^2 + a_2^2$, $r_2^2 = a_3^2 + a_4^2$, and $r_3^2 = a_5^2 + a_6^2$, the system becomes

$$r_{1t} = r_1(\xi r_1^2 + \eta r_2^2 + \chi r_3^2), \tag{6.10}$$

$$r_{2t} = r_2(\eta r_1^2 + \xi r_2^2 + \omega r_3^2), \tag{6.11}$$

$$r_{3t} = r_3(\chi r_1^2 + \omega r_2^2 + \xi r_3^2). \tag{6.12}$$

The straight lines corresponding to this system are

$$r_1 = r_2 = 0; \tag{6.13}$$
$$r_1 = r_3 = 0; \tag{6.14}$$
$$r_2 = r_3 = 0; \tag{6.15}$$
$$r_1 = 0 \text{ and } r_2 = r_3; \tag{6.16}$$
$$r_2 = 0 \text{ and } r_1 = r_3; \tag{6.17}$$
$$r_3 = 0 \text{ and } r_1 = r_2; \tag{6.18}$$
$$r_1 = r_2 = r_3. \tag{6.19}$$

On these straight lines, the systems that emerge are respectively

$$\begin{cases} r_{3t} &= \xi r_3^3; \end{cases} \tag{6.20}$$
$$\begin{cases} r_{2t} &= \xi r_2^3; \end{cases} \tag{6.21}$$
$$\begin{cases} r_{1t} &= \xi r_1^3; \end{cases} \tag{6.22}$$
$$\begin{cases} r_{2t} &= r_2^3(\xi + \omega), \\ r_{3t} &= r_3^3(\xi + \omega); \end{cases} \tag{6.23}$$
$$\begin{cases} r_{1t} &= r_1^3(\xi + \chi), \\ r_{3t} &= r_3^3(\xi + \chi); \end{cases} \tag{6.24}$$
$$\begin{cases} r_{1t} &= r_1^3(\xi + \eta), \\ r_{2t} &= r_2^3(\xi + \eta); \end{cases} \tag{6.25}$$
$$\begin{cases} r_{1t} &= r_1^3(\xi + \eta + \chi), \\ r_{2t} &= r_2^3(\xi + \eta + \omega), \\ r_{3t} &= r_3^3(\xi + \chi + \omega). \end{cases} \tag{6.26}$$

Thus, the only way for all solutions to go to zero along these straight lines is for all of the coefficients to be negative. Let

$$A = \{\frac{\xi}{\gamma_2^2}, \frac{\xi + \omega}{\gamma_2^2}, \frac{\xi + \chi}{\gamma_2^2}, \frac{\xi + \eta}{\gamma_2^2}, \frac{\xi + \eta + \chi}{\gamma_2^2}, \frac{\xi + \eta + \omega}{\gamma_2^2}, \frac{\xi + \chi + \omega}{\gamma_2^2}\}, \tag{6.27}$$

or equivalently,

$$\begin{aligned} A = \{&\frac{2}{9|k_1^c|^2}, (\frac{2}{27|k_1^c|^2} + \frac{4}{9}D_{23}^- + \frac{4}{9}D_{23}^+), (\frac{2}{27|k_1^c|^2} + \frac{4}{9}D_{13}^- + \frac{4}{9}D_{13}^+), \\ &(\frac{2}{27|k_1^c|^2} + \frac{4}{9}D_{12}^- + \frac{4}{9}D_{12}^+), \\ &(\frac{2}{45|k_1^c|^2} + \frac{4}{15}D_{12}^- + \frac{4}{15}D_{12}^+ + \frac{4}{15}D_{13}^- + \frac{4}{15}D_{13}^+), \\ &(\frac{2}{45|k_1^c|^2} + \frac{4}{15}D_{12}^- + \frac{4}{15}D_{12}^+ + \frac{4}{15}D_{23}^- + \frac{4}{15}D_{23}^+), \\ &(\frac{2}{45|k_1^c|^2} + \frac{4}{15}D_{13}^- + \frac{4}{15}D_{13}^+ + \frac{4}{15}D_{23}^- + \frac{4}{15}D_{23}^+)\} \end{aligned} \tag{6.28}$$

From this, it can be seen that the transition is Type I when $\gamma_3 > (\max A)\gamma_2^2$ and Type II when $\gamma_3 < (\max A)\gamma_2^2$. This can be stated in the following theorem:

Theorem 6.1 (Transition Types for Multiplicity Six Case)**.** *Suppose k_1^c, k_2^c, and k_3^c are defined as previously and $k_3^c = ak_1^c + bk_2^c$ such that $a, b \neq 0$ and $(a, b) \in \mathbb{Z}^2 \setminus \{(1,1), (-1,1), (-1,-1), (1,-1)\}$. Then, the system undergoes a continuous (Type I) transition to Σ_λ homological to S^3 when $\gamma_3 > (\max A)\gamma_2^2$ (implying that $\xi < 0$) and undergoes a jump (Type II) transition when $\gamma_3 < (\max A)\gamma_2^2$.*

6.2. **Structure of the set of transition states.** By using the approximative system

$$r_{1t} = \beta r_1 + r_1(\xi r_1^2 + \eta r_2^2 + \chi r_3^2) + o(3), \tag{6.29}$$

$$r_{2t} = \beta r_2 + r_2(\eta r_1^2 + \xi r_2^2 + \omega r_3^2) + o(3), \tag{6.30}$$

$$r_{3t} = \beta r_3 + r_3(\chi r_1^2 + \omega r_2^2 + \xi r_3^2) + o(3), \tag{6.31}$$

the nontrivial equilibria of this system can be calculated to be

$$\begin{aligned}
p_1: &\quad (r_1, r_2, r_3) = (\sqrt{\frac{-\beta}{\xi}}, 0, 0),\\
p_2: &\quad (r_1, r_2, r_3) = (0, \sqrt{\frac{-\beta}{\xi}}, 0),\\
p_3: &\quad (r_1, r_2, r_3) = (0, 0, \sqrt{\frac{-\beta}{\xi}}),\\
p_4: &\quad (r_1, r_2, r_3) = (0, \sqrt{\frac{-\beta}{\omega+\xi}}, \sqrt{\frac{-\beta}{\omega+\xi}}),\\
p_5: &\quad (r_1, r_2, r_3) = (\sqrt{\frac{-\beta}{\chi+\xi}}, 0, \sqrt{\frac{-\beta}{\chi+\xi}}),\\
p_6: &\quad (r_1, r_2, r_3) = (\sqrt{\frac{-\beta}{\eta+\xi}}, \sqrt{\frac{-\beta}{\eta+\xi}}, 0),\\
p_7: &\quad (r_1, r_2, r_3) = (\sqrt{\frac{\beta(\xi-\omega)(-\eta+\xi-\chi+\omega)}{\eta^2\xi - 2\eta\chi\omega + \xi(-\xi^2+\chi^2+\omega^2)}},\\
&\qquad \sqrt{\frac{\beta(\xi-\chi)(-\eta+\xi+\chi-\omega)}{\eta^2\xi - 2\eta\chi\omega + \xi(-\xi^2+\chi^2+\omega^2)}},\\
&\qquad \sqrt{\frac{\beta(\xi-\eta)(\eta+\xi-\chi-\omega)}{\eta^2\xi - 2\eta\chi\omega + \xi(-\xi^2+\chi^2+\omega^2)}}).
\end{aligned} \tag{6.32}$$

The Jacobian of the reduced system at a fixed point (r_1, r_2, r_3) is

$$J = \begin{pmatrix} \beta + 3\xi r_1^2 + \eta r_2^2 + \chi r_3^2 & 2\eta r_1 r_2 & 2\chi r_1 r_3 \\ 2\eta r_1 r_2 & \beta + \eta r_1^2 + 3\xi r_2^2 + \omega r_3^2 & 2\omega r_2 r_3 \\ 2\chi r_1 r_3 & 2\omega r_2 r_3 & \beta + \chi r_1^2 + \omega r_2^2 + 3\xi r_3^2 \end{pmatrix}. \tag{6.33}$$

The stability of each solution can be determined from calculating the eigenvalues of this matrix at each equilibria, such as in the example in the next section.

Theorem 6.2 (Structure of Σ_λ)**.** *Under a Type I transition, the bifurcated attractor Σ_λ from Theorem 6.1 is homological to S^5, with p_1, p_2, and p_3 correspond to circles of solutions; p_4, p_5, and p_6 correspond to torii of solutions; and p_7 corresponding to an $S^1 \times S^1 \times S^1$ surface.*

6.3. **Example: roll patterns.** Let $l_1 = (2\pi, \frac{7\sqrt{15}\pi}{15})$ and $l_2 = (0, \frac{8\sqrt{15}\pi}{15})$. Then the dual lattice is spanned by $k_1 = (1,0)$ and $k_2 = (-\frac{7}{8}, \frac{\sqrt{15}}{4})$. Note that $|k_1 - k_2|^2 = \frac{285}{64}$, $|k_1 + k_2|^2 = \frac{61}{64}$. The critical points of the lattice are thus k_1, $-k_1$, k_2, $-k_2$, $2k_1 + 2k_2$, and $-2k_1 - 2k_2$, so we will use the analysis outlined in the previous sections dealing with multiplicity six with higher coefficient linear dependence. Let $\gamma_2 = 1$ and $\gamma_3 = 2$. Observe that $\beta = \lambda - 1$, $\xi = -\frac{16}{3}$, $\eta = -\frac{63764}{663}$, $\chi = -\frac{36076}{3657}$, $\omega = -\frac{314956}{38577}$. Let $\lambda = \lambda_0 = 1$ and consider the straight line orbits of the system. From THEOREM 6.1, we see that $max(A) = \xi = -\frac{16}{3}$, and thus the transition is Type I because $\gamma_3 > (\max A)\gamma_2^2$. Now let $\lambda = 1.1$ so that we may consider the pattern formation that results from the dynamic transition as λ crossed the critical threshold. The trivial solution $(r_1, r_2, r_3) = (0,0,0)$ obviously becomes unstable as $\lambda > \lambda_0 = 1$. Next consider the solution $(r_1, r_2, r_2) = (\sqrt{\frac{-\beta}{\xi}}, 0, 0) = (\sqrt{\frac{3}{160}}, 0, 0)$. The Jacobian evaluated at this solution is

$$J = \begin{pmatrix} -0.2 & 0 & 0 \\ 0 & -1.703 & 0 \\ 0 & 0 & -0.085 \end{pmatrix}, \tag{6.34}$$

and so the solution is stable. Observe that because β, ξ, η, and χ are all negative, the solutions $(r_1, r_2, r_3) = (0, \sqrt{\frac{-\beta}{\xi}}, 0)$ and $(r_1, r_2, r_3) = (0, 0, \sqrt{\frac{-\beta}{\xi}})$ are also both stable as their Jacobians are diagonal matrices with negative entries. The solution $(r_1, r_2, r_2) = (\sqrt{\frac{-\beta}{\xi}}, 0, 0)$ can be written as

$$u(x,t) = 2(a_1 \cos(k_1^c \cdot x) - a_2 \sin(k_1^c \cdot x)), \tag{6.35}$$

where $a_1^2 + a_2^2 = r_1^2$. Graphs of this solution (and the previous two) are similar to those of the multiplicity two case. Now consider the solution $(r_1, r_2, r_3) = (0, \sqrt{\frac{-\beta}{\omega+\xi}}, \sqrt{\frac{-\beta}{\omega+\xi}}) = (0, \frac{38577}{5207000}, \frac{38577}{5207000})$. The Jacobian evaluated at this solution is

$$J = \begin{pmatrix} 0.094 & 0 & 0 \\ 0 & 0.0987 & -0.000896 \\ 0 & -0.000896 & 0.987 \end{pmatrix}, \tag{6.36}$$

with eigenvalues $(\frac{24899}{250000}, \frac{24451}{250000}, \frac{47}{500})$, from which we see that the solution is unstable in all directions. This solution can be written as

$$u(x,t) = 2(a_3 \cos(k_2^c \cdot x) - a_4 \sin(k_2^c \cdot x) + a_5 \cos(k_3^c \cdot x) - a_6 \sin(k_3^c \cdot x)), \tag{6.37}$$

where $a_3^2 + a_4^2 = r_2^2$ and $a_5^2 + a_6^2 = r_3^2$. Graphs of this solution are similar to those of the multiplicity four case. Consider the solution p_7. Substituting values for β, ξ, ω, η, and χ produce undefined values for r_1 and r_2. Subsequently, this solution does not exist for the values of the parameters chosen.

7. Multiplicity Six with $k_3^c = k_1^c + k_2^c$

7.1. **Center manifold reduction.** Consider the same situation as above but assume that $k_3^c = k_1^c + k_2^c$. This scenario is important because different coefficients would have zero in the denominator in the previous computations. If $k_3^c = k_1^c - k_2^c$, then the lattice can be redefined with $k_2^c = -k_2^c$ and the rest of these computations follow. In this case,

$$S = \{(n_1^c, n_2^c), (-n_1^c, -n_2^c), (n_3^c, n_4^c), (-n_3^c, -n_4^c), (n_5^c, n_6^c), (-n_5^c, -n_6^c)\}. \tag{7.1}$$

Thus, we have $\beta_{n_1 n_2}(\lambda_0) < 0$ for all $(n_1, n_2) \in \mathbb{Z}^2 \setminus (S \cup \{(0,0)\})$. Thus, with critical value λ_0, the eigenvalue

$$\begin{aligned} \beta_{n_1^c n_2^c}(\lambda) &= -|n_1^c k_1 + n_2^c k_2|^2 (|n_1^c k_1 + n_2^c k_2|^2 - \lambda) \\ &= -|k_c|^2 (|k_c|^2 - \lambda), \end{aligned} \tag{7.2}$$

has multiplicity six with

$$\begin{aligned} e_1 &= e^{i(k_1^c \cdot x)}, & e_2 &= e^{-i(k_1^c \cdot x)}, \quad e_3 = e^{i(k_2^c \cdot x)}, \\ e_4 &= e^{-i(k_2^c \cdot x)}, & e_5 &= e^{i(k_3^c \cdot x)}, \quad e_6 = e^{-i(k_3^c \cdot x)}, \\ E_1^\lambda &= span\{e_1, ..., e_6\}, \\ E_2^\lambda &= \overline{span\{e_7, e_8, ...\}}. \end{aligned} \tag{7.3}$$

The solution can thus be written as

$$u(x,t) = \sum_{i=1}^{6} y_i e_i + z, \tag{7.4}$$

where $z \in span\{e_7, e_8, ...\}$ is the stable component. By similar computation, the center manifold function up to higher order terms is given by $\phi(x) = \sum_{i=7}^{24} \phi_i e_i$. Using the notation that

$$\begin{aligned} e_7 &= e^{2ik_1^c \cdot x}, & e_8 &= e^{-2ik_1^c \cdot x}, & e_9 &= e^{2ik_2^c \cdot x}, \\ e_{10} &= e^{-2ik_2^c \cdot x}, & e_{11} &= e^{2ik_3^c \cdot x}, & e_{12} &= e^{-2ik_3^c \cdot x}, \\ e_{13} &= e^{i(k_1^c - k_2^c) \cdot x}, & e_{14} &= e^{-i(k_1^c - k_2^c) \cdot x}, & e_{15} &= e^{i(2k_1^c + k_2^c) \cdot x}, \\ e_{16} &= e^{-i(2k_1^c + k_2^c) \cdot x}, & e_{17} &= e^{i(k_2^c + 2k_2^c) \cdot x}, & e_{18} &= e^{-i(k_2^c + 2k_2^c) \cdot x}, \end{aligned} \tag{7.5}$$

it can be calculated that the coefficients of the manifold are

$$\begin{aligned} \phi_7 &= \frac{-\gamma_2 y_1^2}{4|k_1^c|^2 - \lambda}, & \phi_8 &= \frac{-\gamma_2 y_2^2}{4|k_1^c|^2 - \lambda}, \\ \phi_9 &= \frac{-\gamma_2 y_3^2}{4|k_1^c|^2 - \lambda}, & \phi_{10} &= \frac{-\gamma_2 y_4^2}{4|k_1^c|^2 - \lambda}, \\ \phi_{11} &= \frac{-\gamma_2 y_5^2}{4|k_1^c|^2 - \lambda}, & \phi_{12} &= \frac{-\gamma_2 y_6^2}{4|k_1^c|^2 - \lambda}, \\ \phi_{13} &= \frac{-2\gamma_2 y_1 y_4}{|k_1^c - k_2^c|^2 - \lambda}, & \phi_{14} &= \frac{-2\gamma_2 y_2 y_3}{|k_1^c - k_2^c|^2 - \lambda}, \\ \phi_{15} &= \frac{-2\gamma_2 y_1 y_5}{|2k_1^c + k_2^c|^2 - \lambda}, & \phi_{16} &= \frac{-2\gamma_2 y_2 y_6}{|2k_1^c + k_2^c|^2 - \lambda}, \\ \phi_{17} &= \frac{-2\gamma_2 y_3 y_5}{|k_1^c + 2k_2^c|^2 - \lambda}, & \phi_{18} &= \frac{-2\gamma_2 y_4 y_6}{|k_1^c + 2k_2^c|^2 - \lambda}. \end{aligned} \tag{7.6}$$

Using this function and by letting

$$
\begin{aligned}
\lambda &= \lambda_0 = |k_1^c|^2 \\
y_1 &= a_1 + a_2 i, \\
y_2 &= a_1 - a_2 i, \\
y_3 &= a_3 + a_4 i, \\
y_4 &= a_3 - a_4 i, \\
y_5 &= a_5 + a_6 i, \\
y_6 &= a_5 - a_6 i, \\
\xi &= 2|k_1^c|^2\left(\frac{2\gamma_2^2}{|k_1^c - k_2^c|^2 - |k_1^c|^2} - 3\gamma_3\right), \\
\eta &= -3|k_1^c|^2\gamma_3 + \frac{2}{3}\gamma_2^2 \\
\chi &= 2|k_1^c|^2\left(\frac{2\gamma_2^2}{|2k_1^c + k_2^c|^2 - |k_1^c|^2} - 3\gamma_3\right), \\
\omega &= 2|k_1^c|^2\left(\frac{2\gamma_2^2}{|k_1^c + 2k_2^c|^2 - |k_1^c|^2} - 3\gamma_3\right), \\
\tau &= 2|k_1^c|^2\gamma_2,
\end{aligned} \tag{7.7}
$$

the reduced system can be rewritten as

$$
\begin{aligned}
a_{1t} &= \beta a_1 + a_1(\eta(a_1^2 + a_2^2) + \xi(a_3^2 + a_4^2) + \chi(a_5^2 + a_6^2)) - \tau a_3 a_5 - \tau a_4 a_6 + o(3), \\
a_{2t} &= \beta a_2 + a_2(\eta(a_1^2 + a_2^2) + \xi(a_3^2 + a_4^2) + \chi(a_5^2 + a_6^2)) - \tau a_3 a_6 + \tau a_4 a_5 + o(3), \\
a_{3t} &= \beta a_3 + a_3(\xi(a_1^2 + a_2^2) + \eta(a_3^2 + a_4^2) + \omega(a_5^2 + a_6^2)) - \tau a_1 a_5 - \tau a_2 a_6 + o(3), \\
a_{4t} &= \beta a_4 + a_4(\xi(a_1^2 + a_2^2) + \eta(a_3^2 + a_4^2) + \omega(a_5^2 + a_6^2)) - \tau a_1 a_6 + \tau a_2 a_5 + o(3), \\
a_{5t} &= \beta a_5 + a_5(\chi(a_1^2 + a_2^2) + \omega(a_3^2 + a_4^2) + \eta(a_5^2 + a_6^2)) - \tau a_1 a_3 + \tau a_2 a_4 + o(3), \\
a_{6t} &= \beta a_6 + a_6(\chi(a_1^2 + a_2^2) + \omega(a_3^2 + a_4^2) + \eta(a_5^2 + a_6^2)) - \tau a_2 a_3 - \tau a_1 a_4 + o(3).
\end{aligned} \tag{7.8}
$$

From this point, the transition dynamics can be calculated using the different straight lines in a six dimensional space. In order to make the calculations simpler, we will impose another condition on the original solution of the equation: u must be even in x.

7.2. u even in x.

Assume that $u(x,t) = u(-x,t)$. For this to be true, then

$$
\begin{aligned}
&\sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}^+\setminus\{(0,0)\}} (z_{n_1n_2}(t)e^{i(n_1(k_1\cdot x)+n_2(k_2\cdot x))} + \overline{z_{n_1n_2}(t)}e^{-i(n_1(k_1\cdot x)+n_2(k_2\cdot x))}) \\
= &\sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}^+\setminus\{(0,0)\}} (z_{n_1n_2}(t)e^{-i(n_1(k_1\cdot x)+n_2(k_2\cdot x))} + \overline{z_{n_1n_2}(t)}e^{i(n_1(k_1\cdot x)+n_2(k_2\cdot x))}).
\end{aligned} \tag{7.9}
$$

This condition implies that $z(t) = \overline{z(t)}$, so $z(t) \in \mathbb{R}$ for all t. This also means that

$$\begin{aligned} u(x,t) &= \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}^+\setminus\{(0,0)\}} z_{n_1n_2}(t)(e^{i(n_1(k_1\cdot x)+n_2(k_2\cdot x))} + e^{-i(n_1(k_1\cdot x)+n_2(k_2\cdot x))}) \\ &= \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}^+\setminus\{(0,0)\}} (2z_{n_1n_2}(t)\cos(n_1(k_1\cdot x)+n_2(k_2\cdot x)) \\ &= \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}^+\setminus\{(0,0)\}} (\tilde{z}_{n_1n_2}(t)\cos(n_1(k_1\cdot x)+n_2(k_2\cdot x)). \end{aligned}$$

For the remaining part of this paper, we will be suppressing the tilde.

Now assume that $\#S = 3$ (this can occur since u is even so the negative of a mode is the same as the mode itself) and $k_3^c = k_1^c + k_2^c$. In this case,

$$S = \{(n_1^c, n_2^c), (n_3^c, n_4^c), (n_5^c, n_6^c)\}. \tag{7.10}$$

Thus, we have $\beta_{n_1n_2}(\lambda_0) < 0$ for all $(n_1, n_2) \in \mathbb{Z}\times\mathbb{Z}^+ \setminus (S \cup \{(0,0)\})$. Thus, with critical value λ_0, the eigenvalue

$$\begin{aligned} \beta_{n_1^cn_2^c}(\lambda) &= -|n_1^ck_1 + n_2^ck_2|^2(|n_1^ck_1 + n_2^ck_2|^2 - \lambda) \\ &= -|k_c|^2(|k_c|^2 - \lambda), \end{aligned} \tag{7.11}$$

has multiplicity six with

$$\begin{aligned} &e_1 = \cos(k_1^c\cdot x), \quad e_2 = \cos(k_2^c\cdot x), \quad e_3 = \cos(k_3^c\cdot x), \\ &E_1^\lambda = span\{e_1, e_2, e_3\}, \\ &E_2^\lambda = \overline{span\{e_4, e_5, ...\}}. \end{aligned} \tag{7.12}$$

The solution can thus be written as

$$u(x,t) = \sum_{i=1}^{3} y_ie_i + z, \tag{7.13}$$

where $z \in E_2^\lambda$ is the stable component. By similar computation, the center manifold function up to higher order terms is given by $\phi(x) = \sum_{i=4}^{24}\phi_ie_i$. Using the notation that

$$\begin{aligned} e_4 &= \cos(2k_1^c\cdot x), & e_5 &= \cos(2k_2^c\cdot x), \\ e_6 &= \cos(2k_3^c\cdot x), & e_7 &= \cos((k_1^c - k_2^c)\cdot x), \\ e_8 &= \cos((2k_1^c + k_2^c)\cdot x), & e_9 &= \cos((k_1^c + 2k_2^c)\cdot x), \end{aligned} \tag{7.14}$$

it can be calculated that the coefficients of the manifold are

$$\begin{aligned} \phi_4 &= \frac{-\gamma_2y_1^2}{8|k_1^c|^2 - 2\lambda}, & \phi_5 &= \frac{-\gamma_2y_2^2}{8|k_1^c|^2 - 2\lambda}, \\ \phi_6 &= \frac{-\gamma_2y_3^2}{8|k_1^c|^2 - 2\lambda}, & \phi_7 &= \frac{-\gamma_2y_1y_2}{|k_1^c - k_2^c|^2 - \lambda}, \\ \phi_8 &= \frac{-\gamma_2y_1y_3}{|2k_1^c + k_2^c|^2 - \lambda}, & \phi_9 &= \frac{-\gamma_2y_2y_3}{|k_1^c + 2k_2^c|^2 - \lambda}. \end{aligned} \tag{7.15}$$

Using this function and by letting

$$\begin{aligned}
&\lambda = \lambda_0 = |k_1^c|^2, \\
&\xi = -\frac{3}{4}|k_1^c|^2\gamma_3 + \frac{1}{6}\gamma_2^2, \\
&\eta = -\frac{3}{2}|k_1^c|^2\gamma_3 + \frac{|k_1^c|^2}{|k_1^c - k_2^c|^2 - |k_1^c|^2}\gamma_2^2, \\
&\chi = -\frac{3}{2}|k_1^c|^2\gamma_3 + \frac{|k_1^c|^2}{|2k_1^c + k_2^c|^2 - |k_1^c|^2}\gamma_2^2, \\
&\omega = -\frac{3}{2}|k_1^c|^2\gamma_3 + \frac{|k_1^c|^2}{|k_1^c + 2k_2^c|^2 - |k_1^c|^2}\gamma_2^2, \\
&\tau = -|k_1^c|^2\gamma_2,
\end{aligned} \tag{7.16}$$

the reduced system can be rewritten as

$$\begin{aligned}
y_{1t} &= \beta y_1 + y_1(\xi y_1^2 + \eta y_2^2 + \chi y_3^2) + \tau y_2 y_3 + o(3), \\
y_{2t} &= \beta y_2 + y_2(\eta y_1^2 + \xi y_2^2 + \omega y_3^2) + \tau y_1 y_3 + o(3), \\
y_{3t} &= \beta y_3 + y_3(\chi y_1^2 + \omega y_2^2 + \xi y_3^2) + \tau y_1 y_2 + o(3).
\end{aligned} \tag{7.17}$$

By algebraic calculations, it can be shown that if $|k_1^c|^2 = |k_2^c|^2 = |k_3^c|^2$ and $k_3^c = k_1^c + k_2^c$, then $|k_1^c - k_2^c|^2 = |2k_1^c + k_2^c|^2 = |k_1^c + 2k_2^c|^2$. This implies, that

$$\begin{aligned}
\xi &= -\frac{3}{4}|k_1^c|^2\gamma_3 + \frac{1}{6}\gamma_2^2, \\
\eta = \chi = \omega &= -\frac{3}{2}|k_1^c|^2\gamma_3 + \frac{|k_1^c|^2}{|k_1^c - k_2^c|^2 - |k_1^c|^2}\gamma_2^2, \\
\tau &= -|k_1^c|^2\gamma_2,
\end{aligned} \tag{7.18}$$

and the reduced system can be rewritten as

$$\begin{aligned}
y_{1t} &= \beta y_1 + \xi y_1^3 + \eta y_1 y_2^2 + \eta y_1 y_3^2 + \tau y_2 y_3 + o(3), \\
y_{2t} &= \beta y_2 + \xi y_2^3 + \eta y_2 y_3^2 + \eta y_1 y_2^2 + \tau y_1 y_3 + o(3), \\
y_{3t} &= \beta y_3 + \xi y_3^3 + \eta y_1^2 y_3 + \eta y_2^2 y_3 + \tau y_1 y_2 + o(3).
\end{aligned} \tag{7.19}$$

7.3. **Dynamical transition theorem.**

Theorem 7.1 (Transition Types with $k_3^c = k_1^c + k_2^c$)**.** *Consider the system defined in (7.52-7.54):*
Case 1: If $\gamma_2 = 0$, then the system undergoes a continuous (Type I) transition to $\Sigma_\lambda \approx S^2$ if $\gamma_3 > 0$, and undergoes a jump (Type II) transition if $\gamma_3 < 0$.
Case 2: If $\gamma_2 > 0$, the system undergoes a jump (Type II) transition if $\gamma_3 < \frac{2}{9|k_1^c|^2}\gamma_2^2$, and a continuous (Type I) transition to $\Sigma_\lambda \approx S^2$ if $\gamma_3 > \frac{2}{9|k_1^c|^2}\gamma_2^2$.
Case 3: If $\gamma_2 < 0$, the system undergoes a jump (Type II) transition.

Proof. Case 1: If $\gamma_2 = 0$, then

$$\xi = -\frac{3}{4}|k_1^c|^2\gamma_3, \quad \eta = -\frac{3}{2}|k_1^c|^2\gamma_3, \quad \tau = 0. \tag{7.20}$$

The system then becomes

$$y_{1t} = \beta y_1 + \xi y_1(y_1^2 + 2y_2^2 + 2y_3^2) + o(3), \tag{7.21}$$

$$y_{2t} = \beta y_2 + \xi y_2(2y_1^2 + y_2^2 + 2y_3^2) + o(3), \tag{7.22}$$

$$y_{3t} = \beta y_3 + \xi y_3(2y_1^2 + 2y_2^2 + y_3^2) + o(3). \tag{7.23}$$

If $\xi > 0$, the all solutions will tend away from the origin, and if $\xi < 0$ all solutions will tend towards the origin. Equivalently, if $\gamma_3 < 0$, the all solutions will tend away from the origin, and if $\gamma_3 > 0$ all solutions will tend towards the origin. Therefore, if $\gamma_3 < 0$, the transition is Type II and if $\gamma_3 > 0$, the transition is Type I.
Case 2: If $\gamma_2 > 0$ then $\tau < 0$. The straight lines corresponding to this system are

$$\begin{aligned} y_1 &= y_2 = 0, \\ y_1 &= y_3 = 0, \\ y_2 &= y_3 = 0, \\ y_1 = 0 &\text{ and } y_2^2 = y_3^2, \\ y_2 = 0 &\text{ and } y_1^2 = y_3^2, \\ y_3 = 0 &\text{ and } y_1^2 = y_2^2, \\ y_1^2 &= y_2^2 = y_3^2. \end{aligned} \tag{7.24}$$

Let $i, j, k \in [1, 3] \cap \mathbb{Z}$ such that $i \neq j$, $i \neq k$, and $j \neq k$. Along the lines of the form $y_i = y_j = 0$, the system reduces to

$$y_{kt} = \xi y_k^3. \tag{7.25}$$

Observe that since $\xi = -\frac{3}{4}|k_1^c|^2\gamma_3 + \frac{1}{6}\gamma_2$,

$$\xi > 0 \iff \gamma_3 < \frac{2}{9|k_1^c|^2}\gamma_2^2. \tag{7.26}$$

It can be seen that if $\xi < 0$, solutions along these lines tend towards the origin and if $\xi > 0$, solutions along these lines tend away from the origin. Equivalently, if $\gamma_3 > \frac{2}{9|k_1^c|^2}\gamma_2^2$, solutions along these lines tend towards the origin and if $\gamma_3 < \frac{2}{9|k_1^c|^2}\gamma_2^2$, solutions along these lines tend away from the origin. It can also be seen that along the lines of the form $y_k = 0$ and $y_i^2 = y_j^2$, there are no straight line solutions because at least one of y_i and y_j must be zero.

Along the line $y_1 = y_2 = y_3$ the system reduces to

$$\begin{aligned} y_{1t} &= \beta y_1 + y_1^3(\xi + 2\eta) + \tau y_1^2 + o(3), \\ y_{2t} &= \beta y_2 + y_2^3(\xi + 2\eta) + \tau y_2^2 + o(3), \\ y_{3t} &= \beta y_3 + y_3^3(\xi + 2\eta) + \tau y_3^2 + o(3). \end{aligned} \tag{7.27}$$

By truncating this system to second order (which can be done since we are considering small perturbations near the origin), the system becomes

$$\begin{aligned} y_{1t} &= \tau y_1^2, \\ y_{2t} &= \tau y_2^2, \\ y_{3t} &= \tau y_3^2. \end{aligned} \tag{7.28}$$

Since $\tau < 0$, all solutions along this straight line that start near zero tend towards zero.
Case 3: If $\gamma_2 < 0$ then $\tau > 0$. The straight lines corresponding to this system are

$$\begin{aligned} y_1 &= y_2 = 0, \\ y_1 &= y_3 = 0, \\ y_2 &= y_3 = 0, \\ y_1 = 0 &\text{ and } y_2^2 = y_3^2, \\ y_2 = 0 &\text{ and } y_1^2 = y_3^2, \\ y_3 = 0 &\text{ and } y_1^2 = y_2^2, \\ y_1^2 &= y_2^2 = y_3^2. \end{aligned} \tag{7.29}$$

Let $i, j, k \in [1, 3] \cap \mathbb{Z}$ such that $i \neq j$, $i \neq k$, and $j \neq k$. Along the lines of the form $y_i = y_j = 0$, the system reduces to

$$y_{kt} = \xi y_k^3. \tag{7.30}$$

Observe that since $\xi = -\frac{3}{4}|k_1^c|^2\gamma_3 + \frac{1}{6}\gamma_2$,

$$\xi > 0 \iff \gamma_3 < \frac{2}{9|k_1^c|^2}\gamma_2^2. \tag{7.31}$$

It can be seen that if $\xi < 0$, solutions along these lines tend towards the origin and if $\xi > 0$, solutions along these lines tend away from the origin. Equivalently, if $\gamma_3 > \frac{2}{9|k_1^c|^2}\gamma_2^2$, solutions along these lines tend towards the origin and if $\gamma_3 < \frac{2}{9|k_1^c|^2}\gamma_2^2$, solutions along these lines tend away from the origin. It can also be seen that along the lines of the form $y_k = 0$ and $y_i^2 = y_j^2$, there are no straight line solutions because at least one of y_i and y_j must be zero.

Along the line $y_1 = y_2 = y_3$ the system reduces to

$$y_{1t} = \beta y_1 + y_1^3(\xi + 2\eta) + \tau y_1^2 + o(3), \tag{7.32}$$

$$y_{2t} = \beta y_2 + y_2^3(\xi + 2\eta) + \tau y_2^2 + o(3), \tag{7.33}$$

$$y_{3t} = \beta y_3 + y_3^3(\xi + 2\eta) + \tau y_3^2 + o(3). \tag{7.34}$$

By truncating this system to second order (which can be done since we are considering small perturbations near the origin), the system becomes

$$\begin{aligned} y_{1t} &= \tau y_1^2, \\ y_{2t} &= \tau y_2^2, \\ y_{3t} &= \tau y_3^2. \end{aligned} \tag{7.35}$$

Since $\tau > 0$, all solutions along this straight line that start near zero tend away from zero.

□

7.4. **Structure of the set of transition states.** Consider the approximative system

$$y_{1t} = \beta y_1 + y_1(\xi y_1^2 + \eta y_2^2 + \eta y_3^2) + \tau y_2 y_3 + o(3), \tag{7.36}$$

$$y_{2t} = \beta y_2 + y_2(\eta y_1^2 + \xi y_2^2 + \eta y_3^2) + \tau y_1 y_3 + o(3), \tag{7.37}$$

$$y_{3t} = \beta y_3 + y_3(\eta y_1^2 + \eta y_2^2 + \xi y_3^2) + \tau y_1 y_2 + o(3). \tag{7.38}$$

Assume that $\gamma_2 < 0$ so that $\tau > 0$ (if $\gamma_2 > 0$, the same equilibria and stability will persist). The nontrivial equilibria of this system can be calculated as in Hoyle [2] to be

- Rolls:
  - $(y_1, y_2, y_3) = (\pm\sqrt{\frac{-\beta}{\xi}}, 0, 0)$,
  - $(y_1, y_2, y_3) = (0, \pm\sqrt{\frac{-\beta}{\xi}}, 0)$,
  - $(y_1, y_2, y_3) = (0, 0, \pm\sqrt{\frac{-\beta}{\xi}})$;
- Hexagons:
  - $y_1 = y_2 = y_3 = \frac{-\tau\pm\sqrt{\tau^2-4\beta\xi-8\beta\eta}}{2\xi+4\eta}$ if $\tau^2 - 4\beta\xi - 8\beta\eta \geq 0$;
- Rectangles:
  - $(y_1, y_2, y_3) = (\frac{\tau}{\xi-\eta}, \pm\sqrt{\frac{-1}{\xi+\eta}(\beta + \frac{\tau^2\xi}{(\xi-\eta)^2})}, \pm\sqrt{\frac{-1}{\xi+\eta}(\beta + \frac{\tau^2\xi}{(\xi-\eta)^2})})$,
  - $(y_1, y_2, y_3) = (\pm\sqrt{\frac{-1}{\xi+\eta}(\beta + \frac{\tau^2\xi}{(\xi-\eta)^2})}, \frac{\tau}{\xi-\eta}, \pm\sqrt{\frac{-1}{\xi+\eta}(\beta + \frac{\tau^2\xi}{(\xi-\eta)^2})})$,
  - $(y_1, y_2, y_3) = (\pm\sqrt{\frac{-1}{\xi+\eta}(\beta + \frac{\tau^2\xi}{(\xi-\eta)^2})}, \pm\sqrt{\frac{-1}{\xi+\eta}(\beta + \frac{\tau^2\xi}{(\xi-\eta)^2})}, \frac{\tau}{\xi-\eta})$.

Some of these solutions are in far-fields, so we will not consider those. The solutions that are not in far fields are

- Rolls:
  - $\pm p_1 = (\pm\sqrt{\frac{-\beta}{\xi}}, 0, 0)$,
  - $\pm p_2 = (0, \pm\sqrt{\frac{-\beta}{\xi}}, 0)$,
  - $\pm p_3 = (0, 0, \pm\sqrt{\frac{-\beta}{\xi}})$;
- Hexagons (if $\tau^2 - 4\beta\xi - 8\beta\eta \geq 0$):
  - $p_4 = (\frac{-\tau+\sqrt{\tau^2-4\beta\xi-8\beta\eta}}{2\xi+4\eta}, \frac{-\tau+\sqrt{\tau^2-4\beta\xi-8\beta\eta}}{2\xi+4\eta}, \frac{-\tau+\sqrt{\tau^2-4\beta\xi-8\beta\eta}}{2\xi+4\eta})$.

The Jacobian of this system at a fixed point (y_1, y_2, y_3) is

$$J = \begin{pmatrix} \beta + 3\xi y_1^2 + \eta y_2^2 + \eta y_3^2 & 2\eta y_1 y_2 + \tau y_3 & 2\eta y_1 y_3 + \tau y_2 \\ 2\eta y_1 y_2 + \tau y_3 & \beta + \eta y_1^2 + 3\xi y_2^2 + \eta y_3^2 & 2\eta y_2 y_3 + \tau y_1 \\ 2\eta y_1 y_3 + \tau y_2 & 2\eta y_2 y_3 + \tau y_1 & \beta + \eta y_1^2 + \eta y_2^2 + 3\xi y_3^2 \end{pmatrix}. \tag{7.39}$$

The stability of each solution can be determined from calculating the eigenvalues of this matrix at each equilibrium.

Theorem 7.2 (Stability of Roll Solutions)**.** *Consider the roll solutions $\pm p_1$, $\pm p_2$, and $\pm p_3$ and assume $\beta > 0$ (or equivalently $\lambda > \lambda_0$):*
Case 1: If $\eta < \xi - \tau\sqrt{\frac{-\xi}{\beta}}$, then the roll solutions all have three stable eigenvalues.
Case 2: If $\xi - \tau\sqrt{\frac{-\xi}{\beta}} < \eta < \xi + \tau\sqrt{\frac{-\xi}{\beta}}$, then the roll solutions all have two stable

eigenvalues and one unstable eigenvalue. Moreover, the unstable directions for $\pm p_1$ *are* $(0, \pm 1, 1)$*, for* $\pm p_2$ *are* $(\pm 1, 0, 1)$*, and for* $\pm p_3$ *are* $(\pm 1, 1, 0)$*.*
Case 3: If $\eta > \xi + \tau\sqrt{\frac{-\xi}{\beta}}$*, then the roll solutions all have one stable eigenvalue and two unstable eigenvalue. Moreover, the stable direction will be towards the origin and the two unstable directions are orthogonal to the stable direction.*

Proof. Consider the roll solutions $\pm p_1$. It can be seen that

$$J(\pm p_1) = \begin{pmatrix} \beta + 3\xi|\frac{-\beta}{\xi}| & 0 & 0 \\ 0 & \beta + \eta|\frac{-\beta}{\xi}| & \pm\tau\sqrt{\frac{-\beta}{\xi}} \\ 0 & \pm\tau\sqrt{\frac{-\beta}{\xi}} & \beta + \eta|\frac{-\beta}{\xi}| \end{pmatrix}, \tag{7.40}$$

$$J(\pm p_2) = \begin{pmatrix} \beta + \eta|\frac{-\beta}{\xi}| & 0 & \pm\tau\sqrt{\frac{-\beta}{\xi}} \\ 0 & \beta + 3\xi|\frac{-\beta}{\xi}| & 0 \\ \pm\tau\sqrt{\frac{-\beta}{\xi}} & 0 & \beta + \eta|\frac{-\beta}{\xi}| \end{pmatrix}, \tag{7.41}$$

$$J(\pm p_3) = \begin{pmatrix} \beta + \eta|\frac{-\beta}{\xi}| & \pm\tau\sqrt{\frac{-\beta}{\xi}} & 0 \\ \pm\tau\sqrt{\frac{-\beta}{\xi}} & \beta + \eta|\frac{-\beta}{\xi}| & 0 \\ 0 & 0 & \beta + 3\xi|\frac{-\beta}{\xi}| \end{pmatrix}. \tag{7.42}$$

By computation, the eigenvalues and eigenvectors of $J(\pm p_1)$ are

$$\begin{aligned} \lambda_1 &= \beta + 3\xi|\frac{-\beta}{\xi}|,\ v_1 = (1, 0, 0), \\ \lambda_2 &= \beta + \eta|\frac{-\beta}{\xi}| + \tau\sqrt{\frac{-\beta}{\xi}},\ v_2 = (0, \pm 1, 1), \\ \lambda_3 &= \beta + \eta|\frac{-\beta}{\xi}| - \tau\sqrt{\frac{-\beta}{\xi}},\ v_3 = (0, \mp 1, 1), \end{aligned} \tag{7.43}$$

the eigenvalues and eigenvectors of $J(\pm p_2)$ are

$$\begin{aligned} \lambda_1 &= \beta + 3\xi|\frac{-\beta}{\xi}|,\ v_1 = (1, 0, 0), \\ \lambda_2 &= \beta + \eta|\frac{-\beta}{\xi}| + \tau\sqrt{\frac{-\beta}{\xi}},\ v_2 = (\pm 1, 0, 1), \\ \lambda_3 &= \beta + \eta|\frac{-\beta}{\xi}| - \tau\sqrt{\frac{-\beta}{\xi}},\ v_3 = (\mp 1, 0, 1), \end{aligned} \tag{7.44}$$

and the eigenvalues and eigenvectors of $J(\pm p_3)$ are

$$\begin{aligned}
&\lambda_1 = \beta + 3\xi|\frac{-\beta}{\xi}|, \; v_1 = (1,0,0),\\
&\lambda_2 = \beta + \eta|\frac{-\beta}{\xi}| + \tau\sqrt{\frac{-\beta}{\xi}}, \; v_2 = (\pm 1, 1, 0),\\
&\lambda_3 = \beta + \eta|\frac{-\beta}{\xi}| - \tau\sqrt{\frac{-\beta}{\xi}}, \; v_3 = (\mp 1, 1, 0).
\end{aligned} \tag{7.45}$$

Now assume that $\beta > 0$, so $\xi < 0$ must be true. In this case, the eigenvalues reduce to

$$\begin{aligned}
&\lambda_1 = -2\beta,\\
&\lambda_2 = \beta + \eta\frac{-\beta}{\xi} + \tau\sqrt{\frac{-\beta}{\xi}},\\
&\lambda_3 = \beta + \eta\frac{-\beta}{\xi} - \tau\sqrt{\frac{-\beta}{\xi}}.
\end{aligned} \tag{7.46}$$

From this, the following statements emerge:

$$\begin{aligned}
&\lambda_1 < 0,\\
&\lambda_2 > 0 \iff \eta > \xi - \tau\sqrt{\frac{-\xi}{\beta}},\\
&\lambda_3 > 0 \iff \eta > \xi + \tau\sqrt{\frac{-\xi}{\beta}}.
\end{aligned} \tag{7.47}$$

Since $\xi < 0$, if $\eta < \xi - \tau\sqrt{\frac{-\xi}{\beta}}$, these solutions will have three stable eigenvalues, if $\xi - \tau\sqrt{\frac{-\xi}{\beta}} < \eta < \xi + \tau\sqrt{\frac{-\xi}{\beta}}$, the solution will have two stable eigenvalues and one unstable eigenvalue, and if $\eta > \xi + \tau\sqrt{\frac{-\xi}{\beta}}$, these solutions will have two unstable eigenvalues and one stable eigenvalue. □

Consider the hexagon solution p_4. It can be seen that

$$J(p_4) = \begin{pmatrix} \beta + 3\xi y_1^2 + 2\eta y_1^2 & 2\eta y_1^2 + \tau y_1 & 2\eta y_1^2 + \tau y_1 \\ 2\eta y_1^2 + \tau y_1 & \beta + 2\eta y_1^2 + 3\xi y_1^2 & 2\eta y_1^2 + \tau y_1 \\ 2\eta y_1^2 y_1^2 + \tau y_1 & 2\eta y_1^2 + \tau y_1 & \beta + 2\eta y_1^2 + 3\xi y_1^2 \end{pmatrix}. \tag{7.48}$$

This solution will be further explored in the example below.

7.5. **Example: roll and hexagonal patterns.** Let $l_1 = (\frac{\pi}{25}, -\frac{\sqrt{3}\pi}{75})$ and $l_2 = (0, -\frac{2\sqrt{3}\pi}{75})$. The dual lattice is spanned by the vectors $k_1 = (50, 0)$ and $k_2 = (-25, -25\sqrt{3})$. Note that $k_1 + k_2 = (25, -25\sqrt{3})$ and $|k_1|^2 = |k_2|^2 = |k_1 + k_2|^2 = 2500$. The critical points of the lattice are thus k_1, $-k_1$, k_2, $-k_2$, $k_1 + k_2$, and $-k_1 - k_2$, so we will use the analysis outlined in the previous sections dealing with multiplicity six where $k_1^c = k_1$, $k_2^c = k_2$, and $k_3^c = k_1 + k_2$. Observe that $\beta = 2500\lambda - 6250000$, $\xi = -1875\gamma_3 + \frac{1}{6}\gamma_2^2$, $\eta = -3750\gamma_3 + \frac{1}{2}\gamma_2^2$, and $\tau = -2500\gamma_2$.

Let $\lambda = \lambda_0 = 1$ and consider the straight line orbits of the system. Assume $\gamma_2 = 0$. From THEOREM 7.1, we see that the transition is Type I if $\gamma_3 > 0$ and Type II if $\gamma_3 < 0$. For $\gamma_2 > 0$, the transition is Type I if $\gamma_3 > \frac{2}{9}\gamma_2^2$ and Type II if $\gamma_3 < \frac{2}{9}\gamma_2^2$. For $\gamma_2 < 0$, the transition is always Type II and solutions along these orbits will tend away from the origin.

Now let $\lambda = 2501$ so that we may consider the pattern formation that results from the dynamic transition at $\lambda = \lambda_0 = 2500$. Consider the trivial solution $y_1 = y_2 = y_3 = 0$. This solution loses its stability as the control parameter exceeds the critical threshold, i.e. when $\lambda > \lambda_0 = 2500$. Now consider the rolls solutions

$$y_i = \pm\sqrt{\frac{37500000 - 15000\lambda}{-11250\gamma_3 + \gamma_2^2}},\ y_j = 0,\ y_k = 0,\ i \neq j \neq k, \tag{7.49}$$

for $i \in \{1, 2, 3\}$. THEOREM 7.1 determines the stability of these six solutions in terms of γ_2 and γ_3. Let $\gamma_2 = 1$, $\gamma_3 = 2$, and $\lambda = 2501$. Then $\xi = -\frac{22499}{6}$, $\eta = -\frac{14999}{2}$, $\tau = -2500$, and $\beta = 1$. From this, we can calculate that $\xi + \tau\sqrt{\frac{-\xi}{\beta}} = -156839$ and $\xi - \tau\sqrt{\frac{-\xi}{\beta}} = 149340$. It then follows that

$$\xi + \tau\sqrt{\frac{-\xi}{\beta}} < \eta < \xi - \tau\sqrt{\frac{-\xi}{\beta}}, \tag{7.50}$$

and Case 2 of the theorem applies. We see that the rolls each have two stable eigenvalues and one unstable eigenvalue. The unstable directions for p_1 are $(0, \pm 1, 1)$, for p_2 are $(\pm 1, 0, 1)$, and for p_3 are $(\pm 1, 1, 0)$. From (7.13), by ignoring higher order terms, we can write the rolls solutions as

$$u(x, t) = \sum_{i=1}^{3} y_i e_i, \tag{7.51}$$

where the coefficients y_i are found by means of equation (16.1); in this case our nontrivial coefficient is ± 0.817 and the others are zero . Thus, our six solutions are

$$\begin{aligned} u_{1,2}(x,t) &= \pm 0.817\cos(50x_1), \\ u_{3,4}(x,t) &= \pm 0.817\cos(-25x_1 - 25\sqrt{3}x_2), \\ u_{5,6}(x,t) &= \pm 0.817\cos(25x_1 - 25\sqrt{3}x_2). \end{aligned} \tag{7.52}$$

FIGURE 5 shows a graph of the solution u_1. Notice that the rolls are vertical, a result of the x_1 term inside the cosine. In contrast, FIGURE 6 shows a graph of the solution u_3 where the rolls are oriented at a different angle.

Consider the hexagonal solution p_4 given by

$$y_1 = y_2 = y_3 = \frac{-\tau + \sqrt{\tau^2 - 4\beta\xi - 8\beta\eta}}{2\xi + 4\eta}. \tag{7.53}$$

Using our previous values of γ_2, γ_3, and λ, the solution becomes

$$y_1 = y_2 = y_3 = -0.134. \tag{7.54}$$

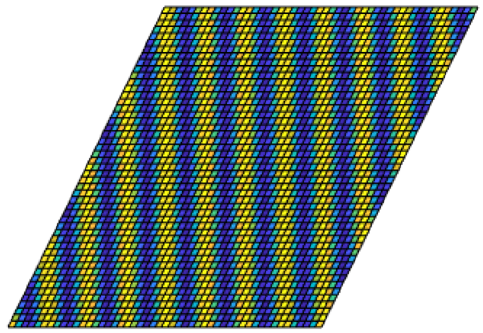

FIGURE 7. Rolls exhibited by the stationary solution $u(x,t) = 0.817\cos(50x_1)$

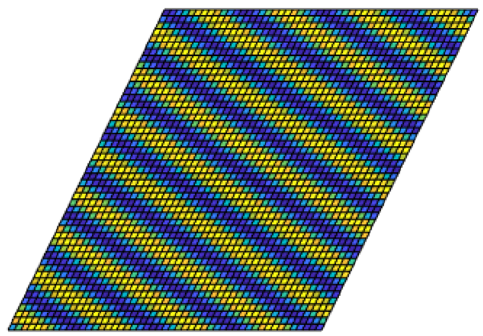

FIGURE 8. Rolls exhibited by the stationary solution $u(x,t) = 0.817\cos(-25x_1 - 25\sqrt{3}x_2)$

We look now at the Jacobian (15.37) evaluated at this solution. Plugging in coefficients, we get

$$J(p_4) = \begin{pmatrix} -470 & 65.68 & 65.68 \\ 65.68 & -470 & 65.68 \\ 65.68 & 65.68 & -470 \end{pmatrix}. \tag{7.55}$$

The eigenvalues of this matrix are $(-\frac{13392}{25}, -\frac{13392}{25}, -\frac{-8466}{25})$, all of which are stable. Thus, the stationary solution is stable and can be written as

(7.56)
$$u_7(x,t) = -0.134[\cos(50x_1) + \cos(-25x_1 - 25\sqrt{3}x_2) + \cos(25x_1 - 25\sqrt{3}x_2)].$$

FIGURE 8 shows a graph of this solution. Notice that the circles are hexagonally-packed, in contrast with the square-packed circles of FIGURE 4. In fact, hexagonally-packed circles (HPC) are not normally observed under the regular Cahn-Hilliard model on rectangular domains. It is the uniqueness of our lattice structure and the high multiplicity of the critical eigenvalue that allows for this pattern to emerge. We consider the long-range interaction model in the next section.

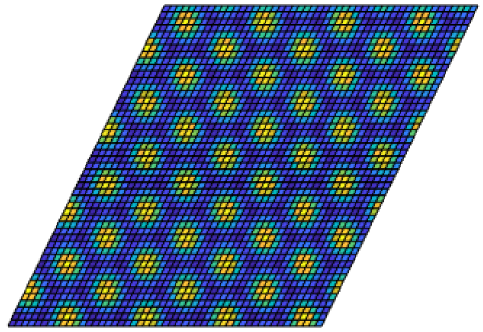

Figure 9. Hexagonally-packed circles exhibited by the stationary solution $u(x,t) = -0.213(\cos(50x_1) + \cos(-25x_1 - 25\sqrt{3}x_2) + \cos(25x_1 - 25\sqrt{3}x_2))$

8. Long-Range Interaction

Assume the same lattice structure as mentioned in section 1, but now consider any solution to the boundary value problem

$$\begin{aligned} u_t &= -\Delta^2 u - \lambda\Delta u - \sigma u + \Delta(\gamma_2 u^2 + \gamma_3 u^3), (x,t) \in \mathbb{R}^2 \times \mathbb{R}^+, \\ u(x+k,t) &= u(x,t), k \in L^*, \\ u(x,0) &= \phi(x), \\ \int_U u(x,t)dx &= 0. \end{aligned} \tag{8.1}$$

In this case, the linear operator L is given by $Lu = -\Delta^2 u - \lambda\Delta u - \sigma u$, the eigenvalues of L are

$$\begin{aligned} \beta_{n_1 n_2}(\lambda) &= -|n_1k_1 + n_2k_2|^4 + \lambda|n_1k_1 + n_2k_2|^2 - \sigma \\ &= -|n_1k_1 + n_2k_2|^2(|n_1k_1 + n_2k_2|^2 - \lambda) - \sigma. \end{aligned} \tag{8.2}$$

To find the first mode that goes unstable, observe that if $\beta_{n_1n_2}(\lambda) = 0$, then $\lambda = |n_1k_1 + n_2k_2|^2 + \frac{\sigma}{|n_1k_1+n_2k_2|^2}$ Thus, the critical value λ_0 is given by

$$\lambda_0 = \min_{k \in L^* \setminus \{(0,0)\}} (|n_1k_1 + n_2k_2|^2 + \frac{\sigma}{|n_1k_1 + n_2k_2|^2}). \tag{8.3}$$

Let $S \subset \mathbb{Z}^2 \setminus \{(0,0)\}$ such that $S = \{(n_1,n_2) | (|n_1k_1 + n_2k_2|^2 + \frac{\sigma}{|n_1k_1+n_2k_2|^2}) = \lambda_0\}$. It can be seen that depending on parameter values, the possible values of the cardinality of S are any even natural number. The critical vector $k^c = n_1^c k_1 + n_2^c k_2$ is related to the long-range interaction term in the manner $|k^c|^2 \sim \sqrt{\sigma}$. As σ gets larger, $|k^c|^2$ increases leading to richer and more complex patterns.

Assume going forward that $\#S = 2$. Then,

$$S = \{(n_1^c, n_2^c), (-n_1^c, -n_2^c)\}. \tag{8.4}$$

Thus, we have $\beta_{n_1n_2}(\lambda_0) < 0$ for all $(n_1, n_2) \in \mathbb{Z}^2 \setminus (S \cup \{(0,0)\})$. Thus, with critical value λ_0, the eigenvalue

$$\begin{aligned} \beta_{n_1^c n_2^c}(\lambda) &= -|n_1^c k_1 + n_2^c k_2|^2(|n_1^c k_1 + n_2^c k_2|^2 - \lambda) - \sigma \\ &= -|k_c|^2(|k_c|^2 - \lambda) - \sigma, \end{aligned} \tag{8.5}$$

has multiplicity two with

$$\begin{aligned} e_1 &= e^{i(k_1^c \cdot x)}, \quad e_2 = e^{-i(k_1^c \cdot x)} \\ E_1^\lambda &= span\{e_1, e_2\} \\ E_2^\lambda &= \overline{span\{e_3, e_4, ...\}}. \end{aligned} \tag{8.6}$$

The solution can thus be written as

$$u(x,t) = y_1 e_1 + y_2 e_2 + z, \tag{8.7}$$

where $z \in E_2^\lambda$ is the stable component. By similar computation, the center manifold function up to higher order terms is given by

$$\phi(x) = \frac{4\gamma_2 |k_1^c|^2 y_1^2}{-4|k_1^c|^2(4|k_1^c|^2 - \lambda) - \sigma} e^{2ik_c \cdot x} + \frac{4\gamma_2 |k_1^c|^2 y_2^2}{-4|k_1^c|^2(4|k_1^c|^2 - \lambda) - \sigma} e^{-2ik_c \cdot x}. \tag{8.8}$$

Using this function, it can be calculated that the reduced equations for this system are

$$\begin{aligned} y_{1t} &= \beta y_1 + \frac{8|k_1^c|^2 \gamma_2^2 y_1^2 y_2}{-4|k_1^c|^2(4|k_1^c|^2 - \lambda) - \sigma} - 3|k_1^c|^2 y_1^2 y_2 \gamma_3, \\ y_{2t} &= \beta y_2 + \frac{8|k_1^c|^2 \gamma_2^2 y_1 y_2^2}{-4|k_1^c|^2(4|k_1^c|^2 - \lambda) - \sigma} - 3|k_1^c|^2 y_1 y_2^2 \gamma_3. \end{aligned} \tag{8.9}$$

By letting

$$\begin{aligned} &\lambda = \lambda_0 = |k_1^c|^2 + \frac{\sigma}{|k_1^c|^2}, \quad y_1 = a_1 + a_2 i, \\ &y_2 = a_1 - a_2 i, \qquad \eta = \frac{-8|k_1^c|^4}{3(-4|k_1^c|^4 + \sigma)}\gamma_2^2 - 3|k_1^c|^2\gamma_3, \end{aligned} \tag{8.10}$$

the reduced system can be rewritten as

$$a_{1t} = \beta a_1 + \eta a_1(a_1^2 + a_2^2) + o(3), \tag{8.11}$$

$$a_{2t} = \beta a_2 + \eta a_2(a_1^2 + a_2^2) + o(3). \tag{8.12}$$

Theorem 8.1 (Transition Types with Long-Range Interaction)**.** *Assume the multiplicity of β_1 is two at $\lambda = \lambda_0 = |k_c|^2$. The following are true:*

1. If $\gamma_3 > \frac{8|k_1^c|^2}{9(4|k_1^c|^4 - \sigma)}\gamma_2^2$ the system undergoes a continuous dynamical transition (Type I) to $\Sigma_\lambda \approx S^1$ consisting of a circle of steady-states as λ crosses λ_0.

2. If $\gamma_3 < \frac{8|k_1^c|^2}{9(4|k_1^c|^4 - \sigma)}\gamma_2^2$ the system undergoes a jump dynamical transition (Type II) as λ crosses λ_0.

Proof. By analyzing the system

$$\begin{aligned} a_{1t} &= \eta a_1(a_1^2 + a_2^2), \\ a_{2t} &= \eta a_2(a_1^2 + a_2^2), \end{aligned} \tag{8.13}$$

it can be seen that all solutions tend towards the origin when $\eta < 0$ and tend away from the origin when $\eta > 0$. Thus, the transition is Type I when $\eta < 0$ and Type II when $\eta > 0$. □

By using the approximative system

$$\begin{aligned} a_{1t} &= \beta a_1 + \eta a_1(a_1^2 + a_2^2) + o(3), \\ a_{2t} &= \beta a_2 + \eta a_2(a_1^2 + a_2^2) + o(3), \end{aligned} \tag{8.14}$$

and letting $a_1^2 + a_2^2 = r^2$, this system can be rewritten as

$$r_t = \beta r + \eta r^3. \tag{8.15}$$

The nontrivial equilibrium of this system is $r = \sqrt{\frac{-\beta}{\eta}}$. The Jacobian of this system at a fixed point r is

$$J = \left(\beta + 3\eta r^2\right). \tag{8.16}$$

When $r = \sqrt{\frac{-\beta}{\eta}}$, the eigenvalue of the Jacobian is $\beta + 3\eta|\frac{\beta}{\eta}|$. If $\beta > 0$, then $\eta < 0$ must be true, which implies that this solution will be stable. Stationary solutions in this case are given by

$$\begin{aligned} u(x,t) &= y_1 e_1 + \bar{y_1}\bar{e_1} \\ &= (a_1 + ia_2)(\cos(k_c \cdot x) + i\sin(k_c \cdot x)) \\ &+ (a_1 - ia_2)(\cos(k_c \cdot x) - i\sin(k_c \cdot x)) \\ &= 2(a_1\cos(k_c \cdot x) - a_2\sin(k_c \cdot x)). \end{aligned} \tag{8.17}$$

where (a_1, a_2) run along the circle $a_1^2 + a_2^2 = r^2$. Note that solutions depend solely on the two critical vectors of the lattice in which the magnitude is least, and that the spanning vectors play no direct role besides specifying the domain.

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