TRANSLATION SURFACES AND KONTSEVICH-ZORICH MONODROMY GROUPS

FELIX FILOZOV AND JAEDON RICH

Abstract. A translation surface is a collection of polygons in $\mathbb{R}^2$ with an even number of sides where we identify pairs of parallel edges of equal length by translation. A square-tiled surface is a special case of a translation surface that arises as a collection of unit squares. Our focus is on a particular square-tiled surface called Eierlegende Wollmilchsau (EW). As a surface in $\mathbb{R}^2$, a certain subgroup of $SL(2,\mathbb{R})$ maps EW to itself. We investigate how this group acts on the curves on EW. The representation of this group is called the Kontsevich-Zorich monodromy group, which is used to study the dynamics of surfaces in the space of translation surfaces. We also consider countable families of EW's, and show that their KZ monodromy groups are non-arithmetic. We work out examples of full KZ monodromy groups in the cases where this is computationally feasible, and for the general case we use properties of ramified covers to obtain results for an entire family of translation surfaces.

Contents

1. Introduction 1
Acknowledgements 4
2. Definitions/Background 5
2.1. Decomposition of H_1 into $H_1^{st} \oplus H_1^{(0)}$ 7
3. Main results 11
3.1. Our Family of Translation Surfaces 11
3.2. A Zero-Holonomy Basis 11
3.3. Veech Group Elements 13
3.4. Computing Representations 15
3.5. Covering Maps 16
3.6. Lifting Properties 19
3.7. Arithmeticity 22
References 28

1. Introduction

A *translation surface* is a collection of polygons in $\mathbb{R}^2$ with an even number of sides where we identify pairs of parallel edges of equal length by translation. For example, the torus (Figure 1) and the double torus (Figure 2) are both translation surfaces. A *square-tiled surface* is a translation surface made of a finite collection of unit squares (Figure 3, and Figure 4).

Date: August 16, 2024.

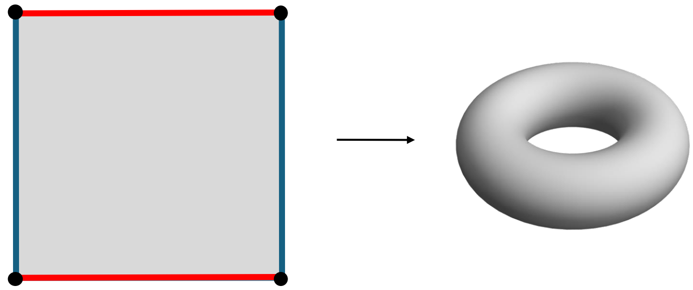


FIGURE 1. A square with parallel sides glued together results in a torus.

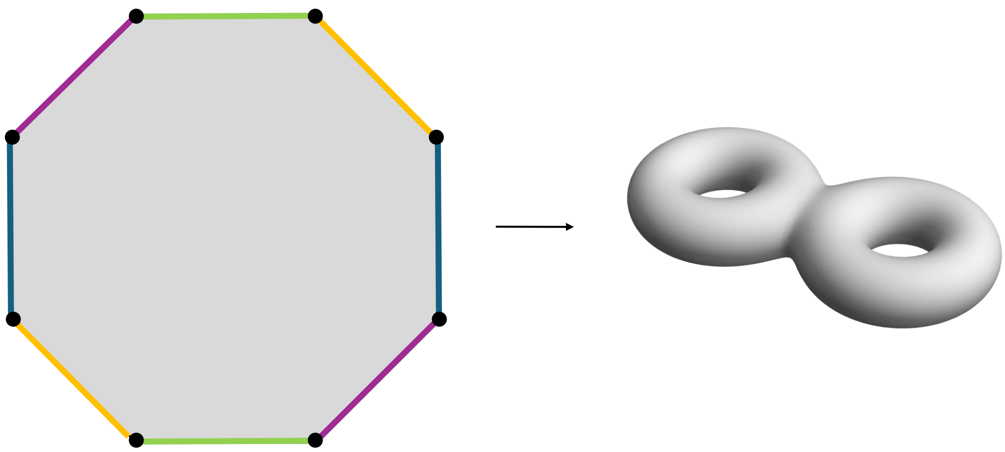


FIGURE 2. An octagon with parallel sides glued together results in a double torus.

We are interested in the moduli space of translation surfaces, therefore we define an equivalence class of translation surfaces. Two translation surfaces are *equivalent* if one translation surface can be cut and reglued (by translation) into the other, without rotation, reflection, or scaling.

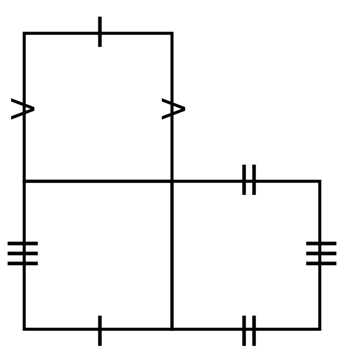


FIGURE 3. A square-tiled surface (denoted *L-Shape*) with corresponding marked sides glued together results in a double torus.

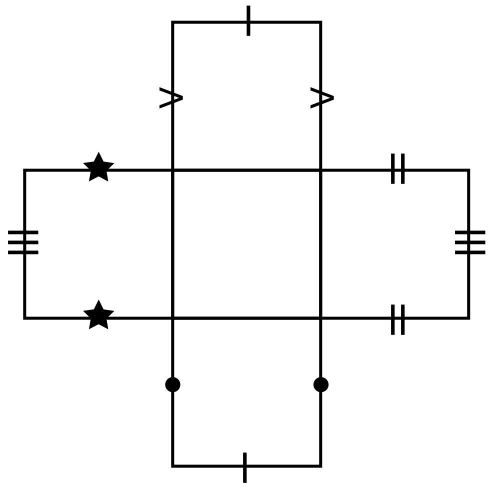


FIGURE 4. A square-tiled surface with corresponding marked sides glued together also results in a double torus.

Since a translation surface has a polygonal presentation in $\mathbb{R}^2$, one can apply a matrix from $GL(2,\mathbb{R})$ to the space of translation surfaces. Note that parallel lines remain parallel under the $GL(2,\mathbb{R})$-action, hence polygons with pairs of parallel sides are mapped to polygons with pairs of parallel sides. To preserve the area of a surface, we restrict to the action of $SL(2,\mathbb{R})$.

The stabilizer of the $SL_2(\mathbb{R})$-action on a surface X is called the *Veech group* of that surface, denoted $\Gamma(X)$. There are surfaces with large Veech groups in the sense that they are lattices in $SL_2(\mathbb{R})$. These surfaces include the square-tiled surfaces.

We observe that $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$ is an example of a stabilizing element in the Veech group of the unit torus. Indeed, after applying the element to the surface, we can get an equivalent square-tiled surface. By cutting and gluing by translation (Figure 5), we can recover the original polygonal representation of the torus as a translation surface. Moreover, the group $SL(2,\mathbb{Z})$ is the Veech group of the square torus.

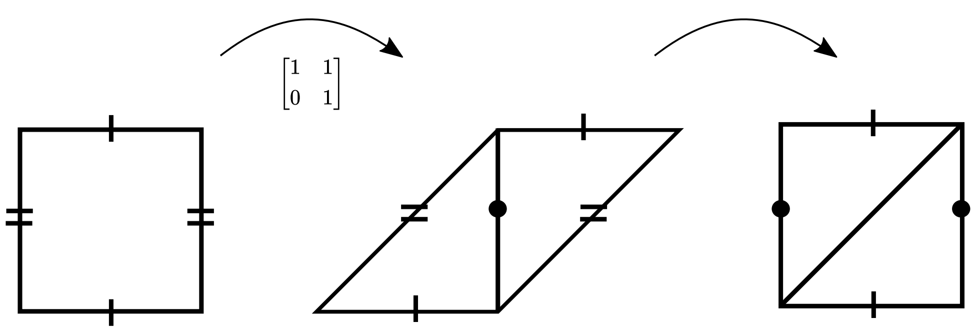


FIGURE 5. Applying a stabilizer to the torus, and regluing, results in a torus.

Equivalently, we can define the Veech group of a surface as the collection of derivatives of the group of affine diffeomorphisms on the surface, denoted Aff(X). The affine diffeomorphisms of a translation surface X are diffeomorphisms of the surface such that the function can be represented as a linear map in charts. This perspective of the Veech group is particularly important to us since most of the complexity of the group of affine maps of a square-tiled surface is captured by the collection of derivatives. Indeed, the following is an exact sequence:

$$0 \to \mathrm{Aut}(X) \to \mathrm{Aff}(X) \xrightarrow{\mathrm{D}} \Gamma(X) \to 0$$

Moreover, the kernel of the derivative map D is finite and corresponds to the collection of automorphisms of the translation structure (which act as translations on the surface).

A surface is fixed in the moduli space under the action of the Veech group of that surface. However, $\mathrm{Aff}(X)$ is a collection of maps on the surface, and consequently, the group acts nontrivially on homology classes of curves on the underlying surface. The *Kontsevich–Zorich monodromy group* encodes this information (Section **??**).

In this project, we are interested in a specific family of square-tiled surfaces that arise as covers of a *Eierlegende Wollmilchsau* (originally studied in [For06], [HS08]), which we will call EW for short, (Figure 6). We construct a sequence of surfaces in attempt to find surfaces that answer the following question from [GRLS24].

Question 1.1 ([GRLS24])**.** *Do there exist surfaces of arbitrarily large genus with a highly nonsimple spectrum?*

While a lot of research has been done in finding surfaces with a completely degenerate Lyapunov spectrum, not much is known about surfaces with highly nonsimple Lyapunov spectrum.

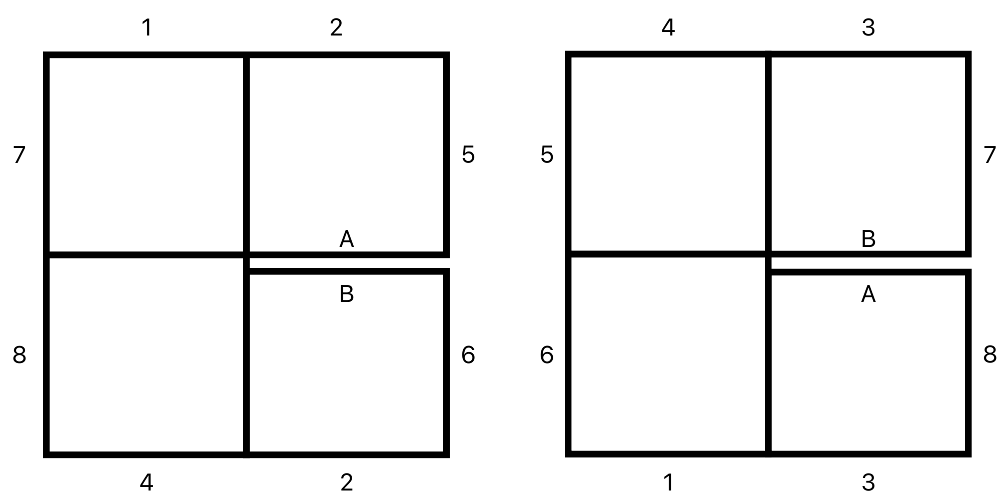


FIGURE 6. Eierlegende Wollmilchsau. All squares are unit squares. The gap is there to emphasize that sides A and B are not identified.

The main result of our project is partial progress in understanding the Kontsevich–Zorich monodromy groups of the surfaces in this family. In Section 2, we define our tools and show how they are used on EW. In Section 3, we carry out our computations on our family of surfaces using tools developed in Section 2.

Acknowledgements. This report is based on work supported by NSF grant DMS-2051032, which we gratefully acknowledge. We would also like to express special thanks to the Department of Mathematics at Indiana University for hosting the program.

2. Definitions/Background

We are interested in understanding how the Affine group acts on first homology group on particular covers of EW. We define the *first homology group* of a translation surface X, denoted $H_1(X)$, as the abelianization of the fundamental group of X, i.e, $H_1(X) := \pi_1(X)^{ab}$. Since the Euler characteristic of a translation surface X is $\chi(X) = 2 - 2g$, we define the *genus* to be g. Before introducing the covers we will be studying, we will walk through a few examples that detail how this action manifests.

We start with a slightly nontrivial surface called the L-Shape surface (Figure 3). Since L-Shape is genus 2, there are four basis curves in the first homology group (Figure 7).

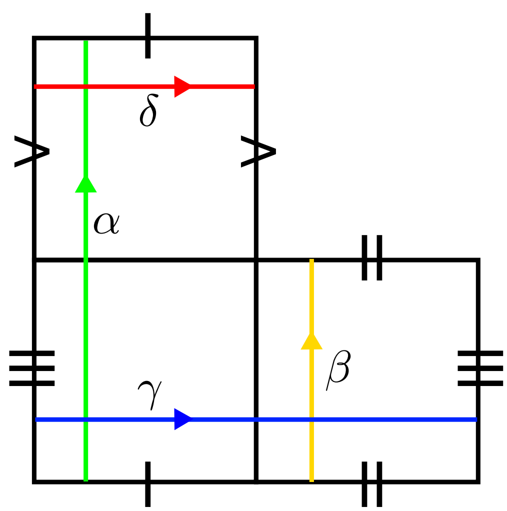


FIGURE 7. L-Shape with basis vectors.

The Veech group of the L-Shape is $\Gamma = \langle \left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right] \rangle \lneq SL(2, \mathbb{Z})$. By applying $\left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$ to L-Shape, we are in effect looking at the action of a particular affine diffeomorphism of the surface, and we can observe how the basis curves change under this map (Figure 8).

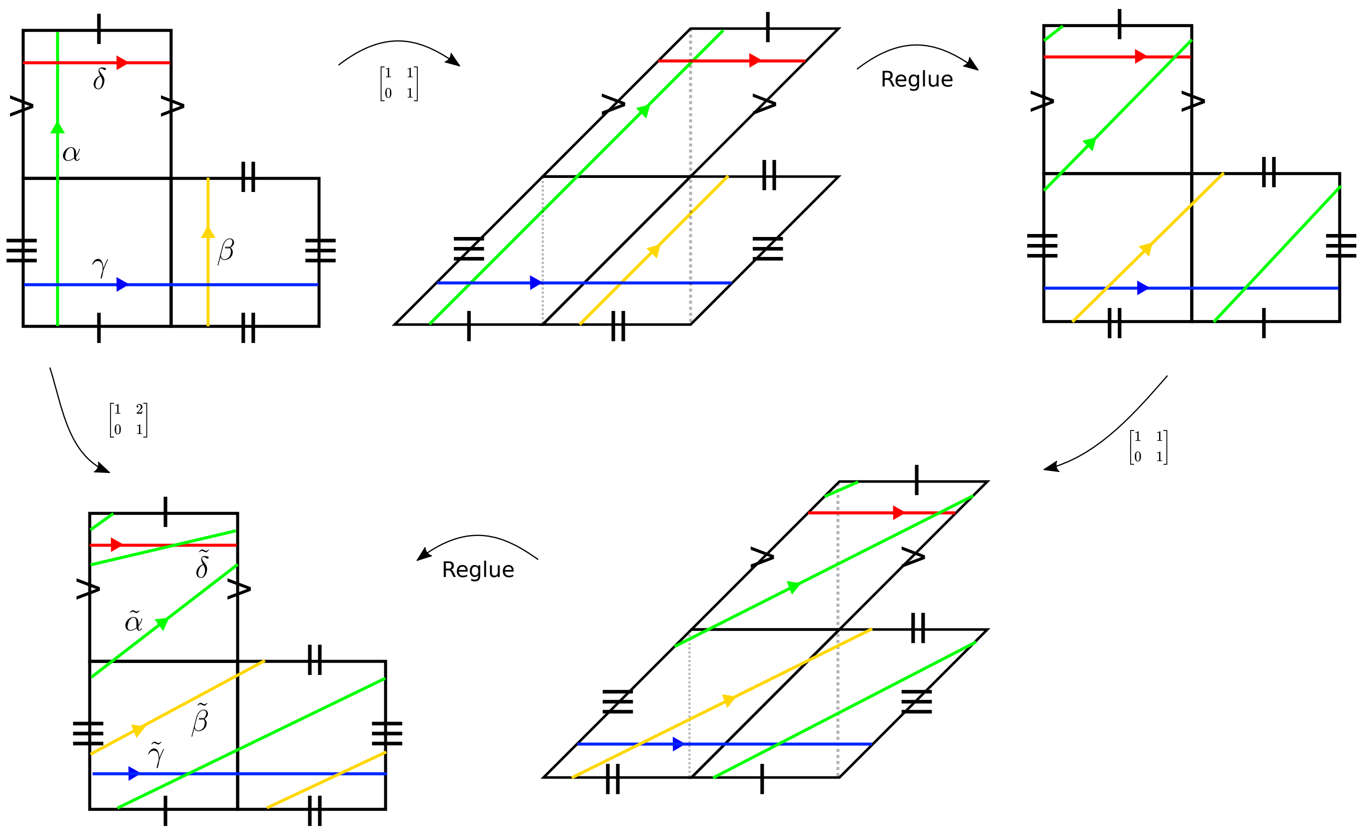


FIGURE 8. L-Shape, and its basis, acted on by stabilizer $\left[\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right]$.

We write the transformed curves in terms of the basis

$$\begin{aligned}\tilde{\alpha} &= \alpha + \gamma + 2\delta\\ \tilde{\beta} &= \beta + \gamma\\ \tilde{\gamma} &= \gamma\\ \tilde{\delta} &= \delta\end{aligned}$$

Let $\{\alpha, \beta, \gamma, \delta\}$ be an ordered basis of the first homology group, then with respect to this basis, we encode the induced action of the affine group as a representation $\rho : \mathrm{Aff}(X) \to \mathrm{End}(H_1(X;\mathbb{R}))$. The representation of the element $\left[\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right]$ is

$$\rho\left(\begin{bmatrix}1&2\\0&1\end{bmatrix}\right) = \begin{bmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\2&0&0&1\end{bmatrix}.$$

Similarly,

$$\rho\left(\begin{bmatrix}1&0\\2&1\end{bmatrix}\right) = \begin{bmatrix}1&0&1&1\\0&1&2&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

2.1. **Decomposition of H_1 into $H_1^{st} \oplus H_1^{(0)}$.** Let X be a squared-tiled surface tiled by n squares. For each square i, where $i \in \{1, \ldots, n\}$, consider the cycles σ_i and ζ_i (Figure 9). The *tautological plane* [GS23], $H_1^{st}(X)$, is the homology group spanned by the cycles $\sum \sigma_i$ and $\sum \zeta_i$. The tautological plane is so named because it duplicates the original information of the Veech group: any automorphism of the surface will act trivially on the tautological plane, and any other affine map will act as its derivative, the Veech group element, on the tautological plane.

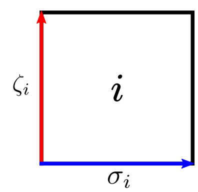


FIGURE 9. For each square i, we consider the cycles σ_i and ζ_i.

We define the *zero-holonomy group*, $H_1^{(0)}(X) := \{\gamma \in H_1(X) | \langle \gamma, \alpha \rangle = 0, \forall \alpha \in H_1^{st}(X)\}$, where $\langle \gamma, \alpha \rangle$ is the intersection form, which counts the (algebraic) number of intersections between two curves. By *algebraic*, we mean that we not only consider the number of intersections but also consider the orientation of curves at each point of intersection.

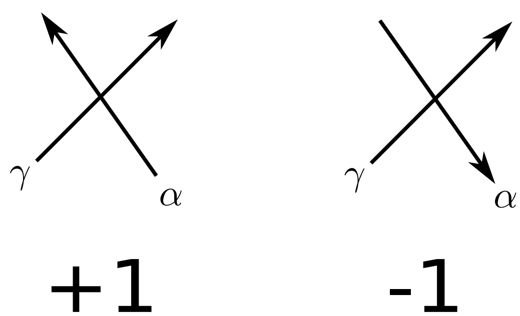


FIGURE 10. $\langle \gamma, \alpha \rangle = 1$ if γ and α intersect as on the left. $\langle \gamma, \alpha \rangle = -1$ if γ and α intersect as on the right. In short, $\langle \dot{,} \rangle$ is a skew-symmetric form.

We note the decomposition
$$H_1(X) = H_1^{st}(X) \oplus H_1^{(0)}(X).$$
Each subspace is invariant under the affine group of X. In other words, a curve in the tautological plane stays in the tautological plane after the action of a stabilizer. The same holds for the zero-holonomy group. Furthermore, since the representation of the affine group respects the intersection form, which is dual to a symplectic (volume) form on the cohomology side, we have that our representation $\rho : \mathrm{Aff}(X) \to \mathrm{Sp}(H_1(X; \mathbb{R}))$, i.e. all of our affine maps are represented as a symplectic matrix with respect to the intersection form.

The *Kontsevich–Zorich monodromy group* is the group generated by the group representations of the Veech group on the zero-holonomy subspace. From now on, we call this *KZ monodromy group*.

We will compute the KZ monodromy group of EW. Consider EW with a basis for $H_1^{(0)}(X)$ as shown in Figure 11.

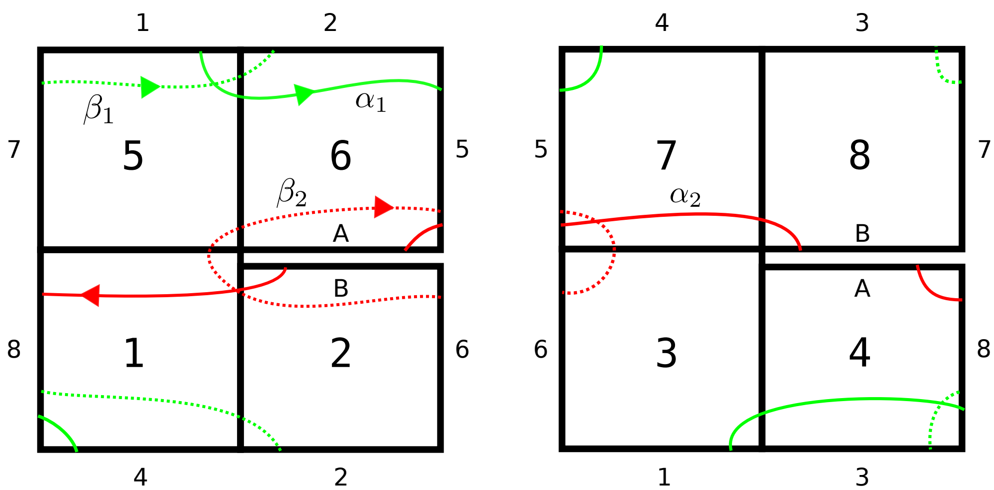


Figure 11. EW with basis vectors for the zero-holonomy vector space.

We apply the stabilizer $\left[\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right]$, and we keep track of how the basis curves move as we reglue the surface (Figure 12, Figure13, and Figure 14).

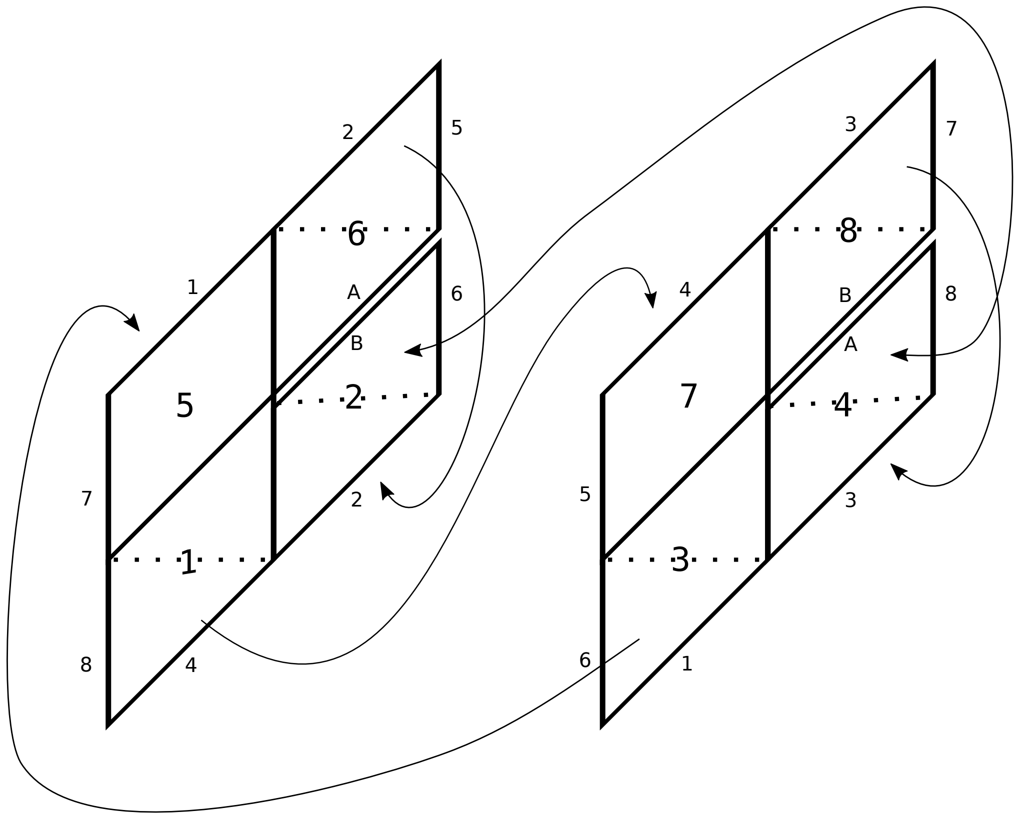


FIGURE 12. The action of $\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]$ together with regluing instruction to get back EW.

We observe the following: $\langle\alpha_1,\tilde{\alpha}_1\rangle = 1$, $\langle\beta_1,\tilde{\alpha}_1\rangle = -1$, $\langle\alpha_2,\tilde{\alpha}_1\rangle = -1$, and $\langle\beta_2,\tilde{\alpha}_1\rangle = 1$. Since the intersection form is linear, and $\tilde{\alpha}_1 = c_1\alpha_1 + c_2\beta_1 + c_3\alpha_2 + c_4\beta_2$, we get a system of equations

$$
\begin{aligned}
c_1\langle\alpha_1,\alpha_1\rangle + c_2\langle\alpha_1,\beta_1\rangle + c_3\langle\alpha_1,\alpha_2\rangle + c_4\langle\alpha_1,\beta_2\rangle &= 1 = \langle\alpha_1,\tilde{\alpha}_1\rangle\\
c_1\langle\alpha_1,\alpha_1\rangle + c_2\langle\alpha_1,\beta_1\rangle + c_3\langle\alpha_1,\alpha_2\rangle + c_4\langle\alpha_1,\beta_2\rangle &= -1 = \langle\beta_1,\tilde{\alpha}_1\rangle\\
c_1\langle\alpha_1,\alpha_1\rangle + c_2\langle\alpha_1,\beta_1\rangle + c_3\langle\alpha_1,\alpha_2\rangle + c_4\langle\alpha_1,\beta_2\rangle &= -1 = \langle\alpha_2,\tilde{\alpha}_1\rangle\\
c_1\langle\alpha_1,\alpha_1\rangle + c_2\langle\alpha_1,\beta_1\rangle + c_3\langle\alpha_1,\alpha_2\rangle + c_4\langle\alpha_1,\beta_2\rangle &= 1 = \langle\beta_2,\tilde{\alpha}_1\rangle
\end{aligned}
$$

We conclude that $\tilde{\alpha}_1 = \frac{1}{2}(\alpha_1 + \beta_1 + \alpha_2 + \beta_2)$.

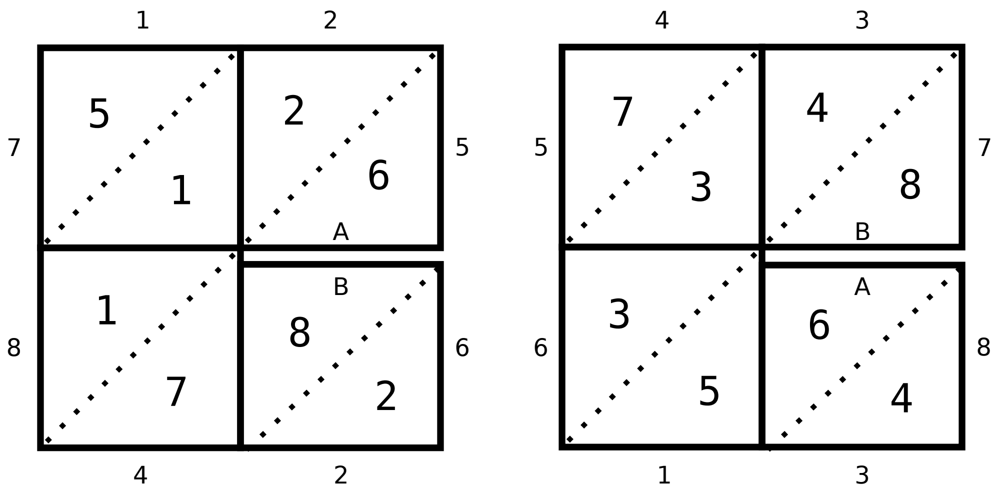


FIGURE 13. Applying the gluing instructions from Figure 12, and an automorphism, which fixes the bottom left cone point of box 1.

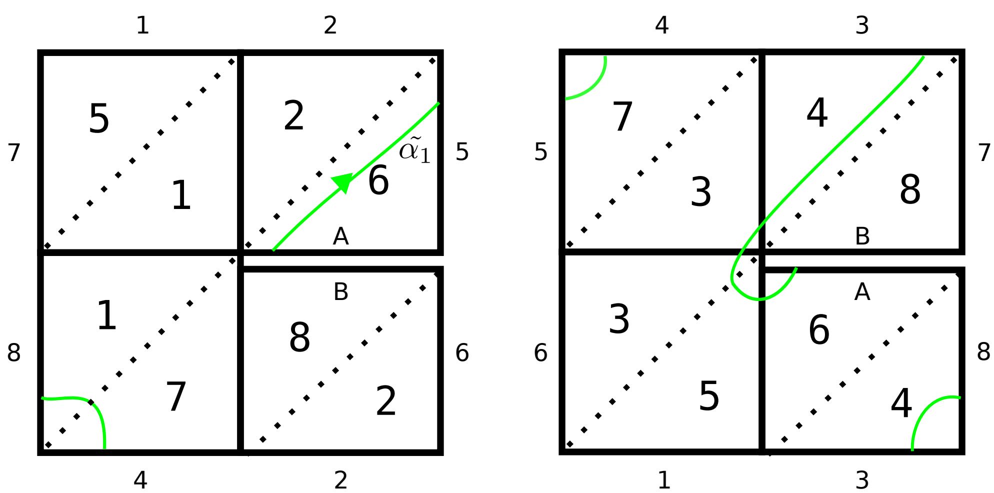


FIGURE 14. We observe how α_1 changes by the transformations from Figure 12, and Figure 13.

When performing a similar computation for other basis curves, we calculate that the group representation of $\left[\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right]$ is

$$\rho\left(\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & 1 & 1 & 1\\ 1 & -1 & -1 & 1\end{bmatrix}.$$

Similarly,

$$\rho\left(\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{bmatrix}.$$

3. Main results

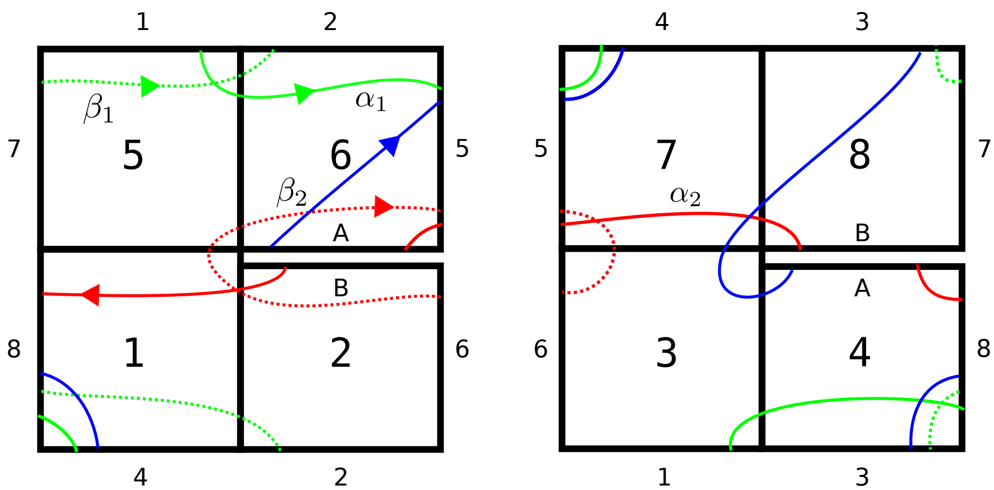


FIGURE 15. The curve $\tilde{\alpha}_1$ is superimposed onto EW before transformations in Figure 12, and Figure 13.

3.1. **Our Family of Translation Surfaces.**

The presentation of EW (Figure 6) that we are working with lends itself naturally to generalization. EW consists of two sets of four squares, but we can construct a new surface using n sets of four squares, with gluing as follows:

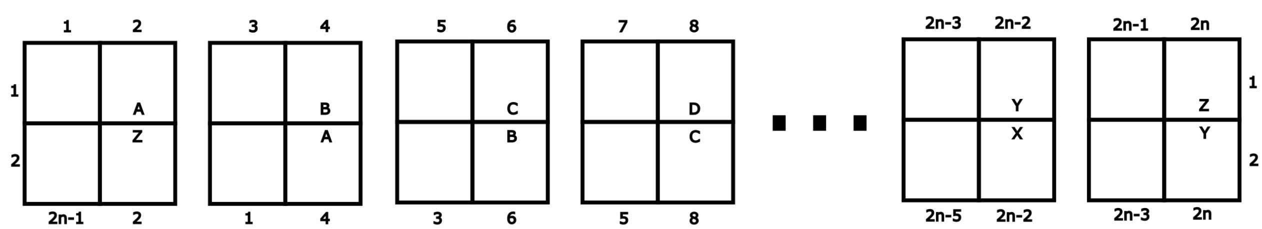


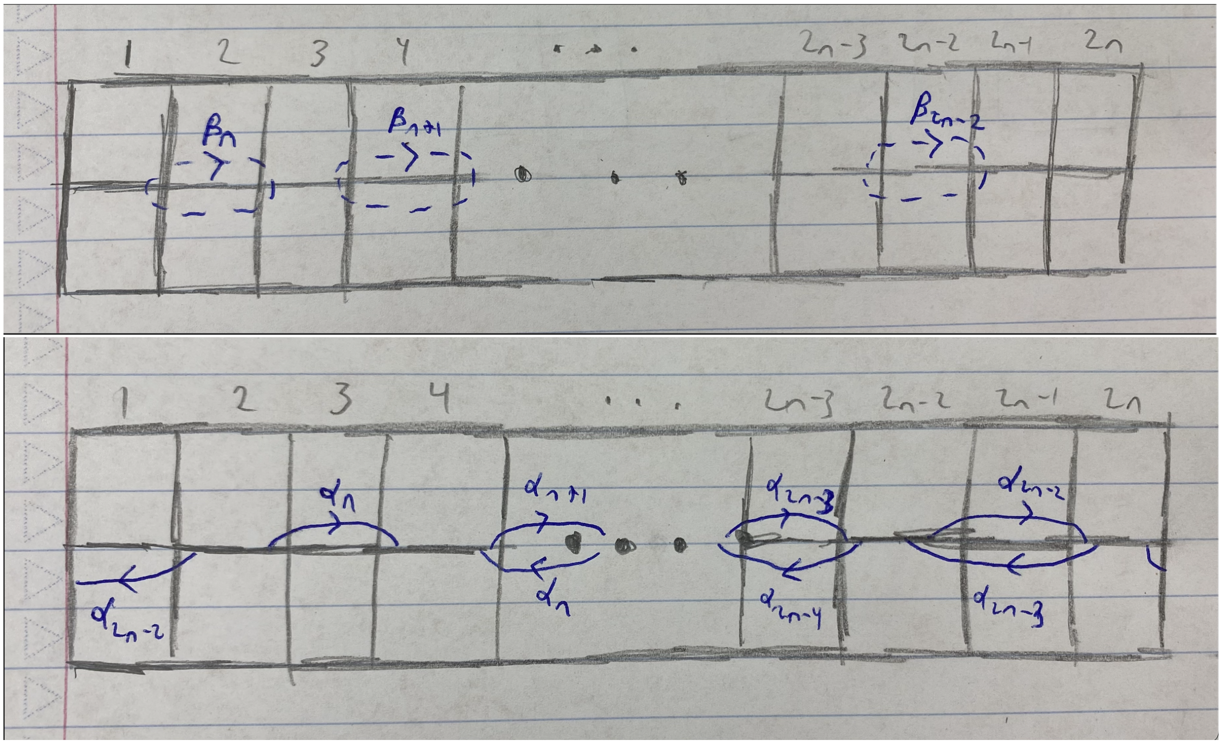
FIGURE 16. The general gluing pattern for the nth member of our family.

where adjacent vertical edges are glued together (so that one may draw the entire surface as a single 2 by $2n$ rectangle, without the vertical gaps). Call the surface with n sets of four squares X_n, so that EW $= X_2$. Note X_1 gives the torus.

The surface X_n has genus $g_n = 2n - 1$. There are four cone points each with total angle $2\pi n$.

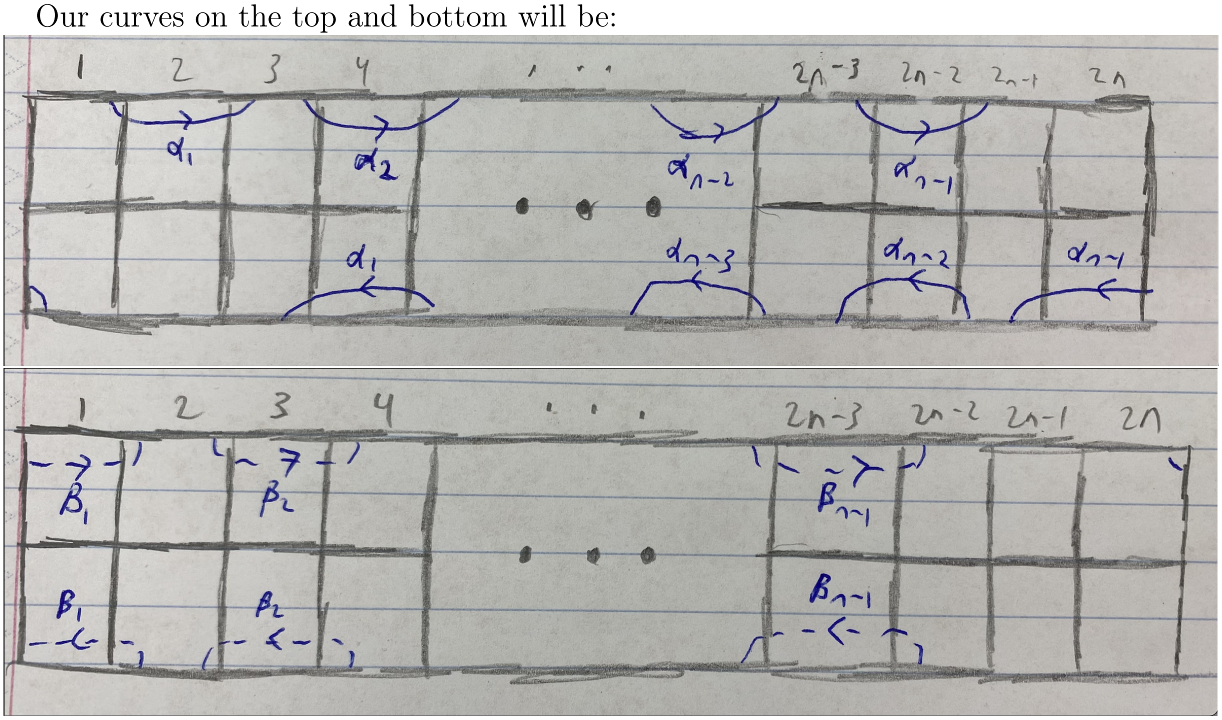
3.2. **A Zero-Holonomy Basis.**

In order to carry out our computations, we need a basis to work with. In particular, if we find a basis specifically for the zero-holonomy subspace (as opposed to one for the entire homology), this will avoid additional computations. The zero-holonomy basis we work with can be partitioned into two groups, curves along the center and curves along the top and bottom. For our curves along the center, we have the following:



There are two types of curve here. The first are the dashed curves, which are simply loops circling every slit but the last. The second set of curves consist of two pieces, one running between adjacent slits from left to right on top and one running between the corresponding adjacent slits on the bottom from right to left so as to close the loop. Again this is done beginning on every slit except the last.

Our curves on the top and bottom will be:



Again, beginning at the top we have two sets of curves. First the solid curves, running from the first square to the third, the third to the fifth, the fifth to the seventh, etc, until we reach the end. Each of these loops is then closed in the bottom by a corresponding segment running from right to left. For instance, the first curve in the top left runs left from the first square to the third square, and the second piece of that curve, in the bottom squares, runs from right to left from the fifth square in the bottom row to the third square, closing the loop.

The dashed curves are simple loops circling every other top and bottom edge. That is, the first dashed curve is a loop around the top left edge, the next is a loop around the third top edge, and so on, stopping at the fourth-to-last edge.

To verify that this is a basis, we can check that the intersection form is invertible. X_n has genus $2n-1$, thus the zero-holonomy space has $2g-2=4n-4$ curves. From left to right along the top, label the solid curves by $\alpha_1,\ldots,\alpha_{n-1}$ and the dashed curves on the top by $\beta_1,\ldots,\beta_{n-1}$. From left to right label the solid curves in the center by $\alpha_n,\ldots,\alpha_{2n-2}$ and the dashed curves by $\beta_n,\ldots,\beta_{2n-2}$. Order the basis by $\alpha_1,\beta_1,\alpha_2,\beta_2,\ldots,\alpha_{2n-2},\beta_{2n-2}$. For $1\le i\le n-1$ and $n\le j\le 2n-2$, α_i and β_i have no intersections with α_j and β_j. Thus our intersection matrix will be block-diagonal with two blocks. In fact, these two blocks will be negatives of each other. Let C be the block corresponding to the intersections of the center curves (that is, the intersections of the curves $\alpha_n,\ldots,\alpha_{2n-2}$ and $\beta_n,\ldots,\beta_{2n-2}$, which given the ordering of our basis will appear as the block in the lower right of the intersection form Ω). That is,

$$\Omega=\begin{bmatrix}-C & 0\\ 0 & C\end{bmatrix}$$

with C having dimensions $n-1\times n-1$. We claim C has the following form. Consider the $2n\times 2n$ matrix where every element is zero except the following:

If $i\equiv 1 \pmod 2$, then the ith row has a -1 in columns $i+1$ and $i+5 \pmod{2n}$, and a 2 in column $i+3$.

If $i\equiv 0 \pmod 2$, then the ith row has a 1 in columns $i-1$ and $i-5$, and a -2 in column $i-3$.

Note the columns are specified mod $2n$, but the rows and columns are indexed beginning at 1, so the position 0 mod $2n$ corresponds to the last column rather than the first.

In terms of this $2n\times 2n$ matrix, we claim C is the upper left $2n-2\times 2n-2$ submatrix.

With some effort one can show that C is indeed invertible, from which it follows that the intersection form is invertible.

To exhibit an example of the above construction of the matrix C, and because we will use it later to compute representations on X_4, we present the matrix C in the $n=4$ case ($g_4=7$):

$$C=\begin{bmatrix}0 & -1 & 0 & 2 & 0 & -1\\ 1 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1 & 0 & 2\\ -2 & 0 & 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & -1\\ 1 & 0 & -2 & 0 & 1 & 0\end{bmatrix}$$

3.3. **Veech Group Elements.** In this section we will determine some Veech group elements across our family of surfaces. By computing the Veech groups through Sage for the first several dozen surfaces, we find that the number of generators of each Veech group, as well as the index, grows rapidly (roughly quadratically) but seemingly without a nice explicit form.

One other thing we observe but have not been able to prove is that for n odd, the Veech groups of X_n and X_{2n} are precisely the same.

Proposition 3.1. *$-I$ is in the Veech group of every X_n.*

Proof. It is essentially immediate to check that $-I$ is in the Veech group for every member of the family; after applying the matrix, simply cut and reglue the two leftmost squares of Figure 16 onto the very right. □

Proposition 3.2. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ *is in the Veech group of every X_n.*

Proof. This follows by essentially the same gluing pattern as was used for X_2 (see Figure 12). After applying the transformation and dividing the parallelograms into triangles as was done there, then for each parallelogram do the following: translate the top right triangle down to the bottom right, translate the bottom left triangle onto the top left of the parallelogram to the left, and translate the triangle containing the lower part of the slit into the corresponding position in the parallelogram to the left. Note that for the leftmost parallelogram things "wrap around", so moving a triangle to the parallelogram to its left means moving that triangle to the rightmost parallelogram. □

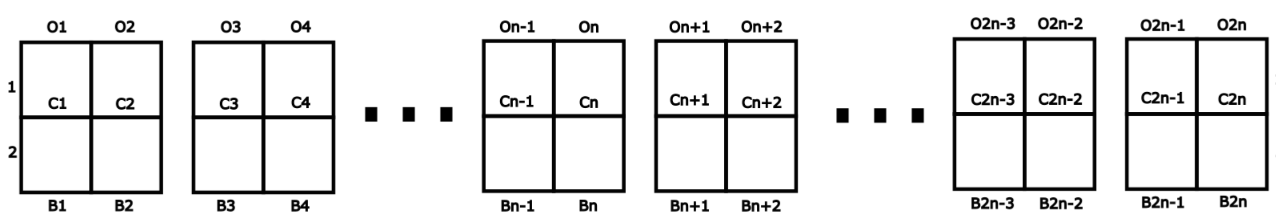
For n even, Sage also tells us that $\begin{bmatrix} n-1 & -n \\ 1 & -1 \end{bmatrix}$ is in the Veech group of X_n. One notices, however, that

$$\begin{bmatrix} n-1 & -n \\ 1 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1},$$

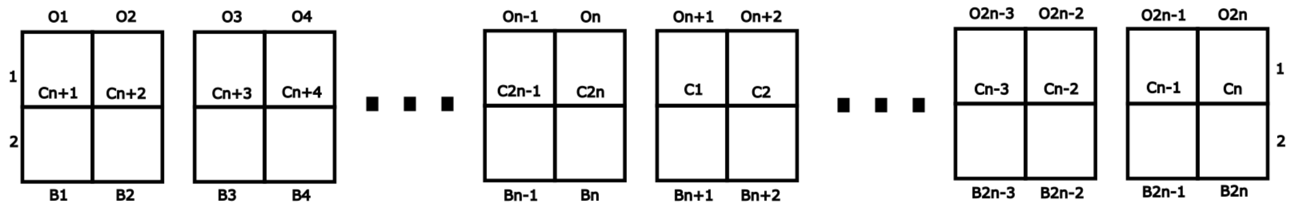
so to show that this is an element of the Veech group, it suffices to show

Proposition 3.3. $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ *is in the Veech group of X_n for n even.*

Proof. To apply this matrix, it will be convenient to view our surfaces as one long 2 by $2n$ rectangle in the manner described earlier. After applying $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, note that we obtain a parallelogram where the top left corner is directly above the bottom right corner. This means we can cut our surface down the vertical line connecting these corners, then reglue the two resulting triangles by joining them along their diagonal to once again obtain a rectangular configuration. In particular, after regluing we get the following transformation (keeping track simply of where the top, center, and bottom edges are sent):



$\downarrow$



One can check that this preserves the structure of the translation surface. □

Proposition 3.4. $\begin{bmatrix}1 & 2n\\0 & 1\end{bmatrix}$ *is in the Veech group of every* X_n.

This is obtained by a very similar cutting and regluing process.

Additionally, for completeness we will state here the following fact, which we will use and restate later: for $d|n$, the Veech group of X_n is contained in the Veech group of X_d.

3.4. **Computing Representations.** Suppose we want to compute the representation of a matrix M. To do so, we define a matrix T where the ijth entry is the intersection of the ith basis curve with the image of the jth basis curve under M. Then $\rho(M) = \Omega^{-1}T$.
Sometimes this is unnecessary, if we can see visually where a curve is sent, e.g., when M is acts as a horizontal shear map. For instance, after applying $\begin{bmatrix}1 & 2n\\0 & 1\end{bmatrix}$, it will be clear that each curve is simply mapped to itself, thus $\rho(\begin{bmatrix}1 & 2n\\0 & 1\end{bmatrix}) = I$. Similarly, when applying $\begin{bmatrix}1 & n\\0 & 1\end{bmatrix}$ it is clear that our curves along the top and bottom are fixed, while the center curves (from our depiction of the transformation above) are each shifted to their counterpart $\frac{n}{2}$ to the right. For instance, $\alpha_n \mapsto \alpha_{n+\frac{n}{2}}$, and $\alpha_{2n-2} \mapsto \alpha_{n-2+\frac{n}{2}}$ (taking into account that the indexing only goes up to $2n-2$ and we must 'wrap around'). From this one can write down a matrix for the transformation.
By contrast, when computing the representation of $\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$, it is not easy to simply visualize this and we must count intersections and multiply by the inverse of the intersection form. For instance, we computed the representation of $\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$ on X_4 by transforming each curve one by one and finding their intersections with the original basis curves. The resulting matrix of intersections (what we called T above) was

$$T = \begin{pmatrix}
1 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & -1 & 0\\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0\\
-1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 1\\
1 & -1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0\\
0 & 1 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & -1\\
-1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1\\
-1 & 0 & 1 & -1 & 0 & 1 & 1 & -1 & 0 & 1 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & 0 & 1\\
-1 & -1 & 1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0\\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & -1\\
0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & -1 & -1 & 1 & 1
\end{pmatrix}$$

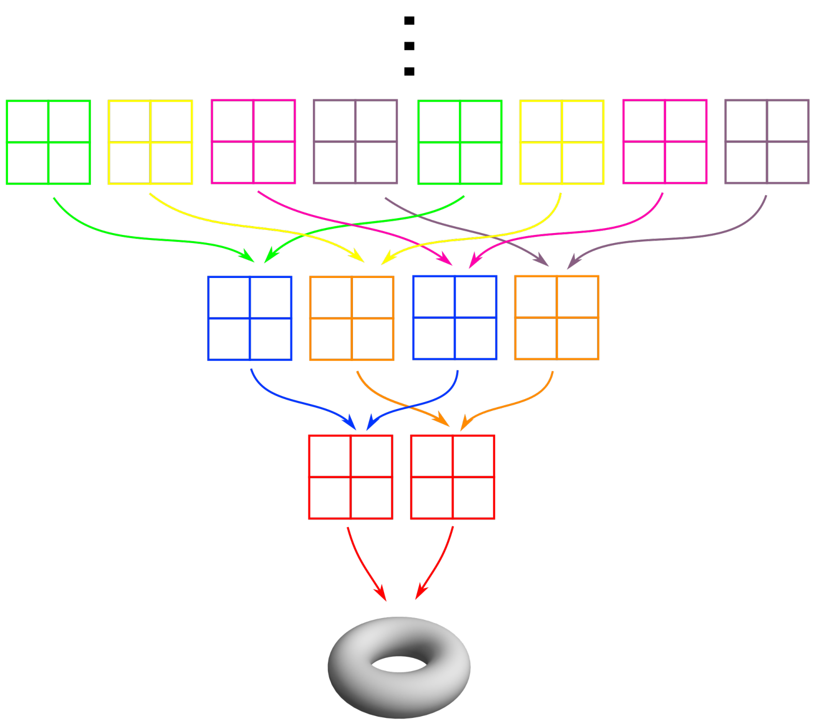
while

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 \end{pmatrix},$$

allowing us to compute

$$\rho\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\right) = \Omega^{-1}T = \frac{1}{4}\begin{pmatrix} 1 & -3 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 \\ -1 & 3 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 \\ -2 & -2 & 2 & -2 & 2 & 2 & -2 & -2 & 2 & -2 & 2 & 2 \\ -2 & 2 & -2 & 2 & 2 & -2 & -2 & 2 & 2 & -2 & 2 & -2 \\ -1 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & -1 & -1 & 3 & -1 \\ 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & -3 \\ 3 & 3 & -1 & -1 & -1 & -1 & 3 & 3 & -1 & -1 & -1 & -1 \\ 1 & -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 & 1 \\ 2 & 2 & 2 & 2 & -2 & -2 & 2 & 2 & 2 & 2 & -2 & -2 \\ -2 & -2 & 2 & -2 & 2 & 2 & 2 & -2 & 2 & 2 & -2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & 3 \end{pmatrix}$$

3.5. **Covering Maps.** For all $d|n$, there is a natural choice of cover $p : X_n \to X_d$ given in the following way: beginning from the left, we can partition the n blocks of X_n into $\frac{n}{d}$ groups of d consecutive squares. Map each of these $\frac{n}{d}$ into X_d simply by inclusion. For instance, when $n = 2^k$ we have



This covering map induces a homomorphism $p_* : H_1(X_n) \to H_1(X_d)$, which we can evaluate by simply looking at the images of curves under p. Clearly any zero-holonomy curve will remain zero-holonomy after being projected from X_n to X_d, so p_* restricts to a map $p_* : H_1^{(0)}(X_n) \to H_1^{(0)}(X_d)$. Let's compute this map explicitly using our basis curves.
While we will be using the same curves as before, in order to get a nice matrix form for p_* we will be reordering (and relabeling) our basis. We have curves $\alpha_1, \ldots, \alpha_{2n-2}$ and $\beta_1, \ldots, \beta_{2n-2}$. The curves $\alpha_1, \ldots, \alpha_{n-1}$ and $\beta_1, \ldots, \beta_{n-1}$ (which are the curves along the top and bottom of X_n) will continue to be called $\alpha_1, \ldots, \alpha_{n-1}$ and $\beta_1, \ldots, \beta_{n-1}$. However $\alpha_n, \ldots, \alpha_{2n-2}$ will now be $\gamma_1, \ldots, \gamma_{n-1}$, respectively, and $\beta_n, \ldots, \beta_{2n-2}$ will now be $\delta_1, \ldots, \delta_{n-1}$, respectively. In terms of these new labels, the ordering of our basis will be $\alpha_1, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{n-1}, \gamma_1, \ldots, \gamma_{n-1}, \delta_1, \ldots, \delta_{n-1}$, with each of these four groups corresponding to the natural grouping of our curves. Note that all the α_i are translates of each other, all the β_i are translates of each other, et cetera.

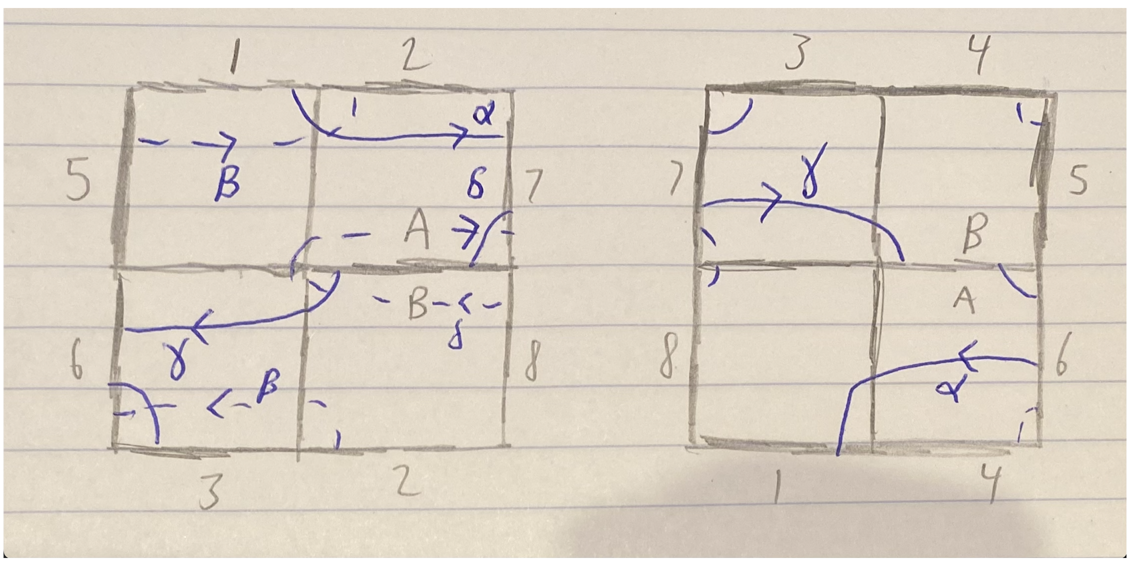


Figure 17. New basis labeling on EW

We will find that each of these four groups behave essentially the same. That is, what holds for the α_i will also hold for the $\beta_i, \gamma_i, \delta_i$. Thus in order to avoid repeating ourselves for every result, we will use e_i as a placeholder for an arbitrary basis element. That is, e_1 can stand for $\alpha_1, \beta_1, \gamma_1$ or δ_1. Note that when there are several e_i in a single place, all the e_i should be interpreted as the same letter. For instance, $e_2 + e_3$ can only mean $\alpha_2 + \alpha_3, \beta_2 + \beta_3, \gamma_2 + \gamma_3$, or $\delta_2 + \delta_3$.
In order to compute p_*, it suffices to look in X_n at each of the $\frac{n}{d}$ groups of d individually. The first $\frac{n}{d} - 1$ of these groups look the same, and our projection is as follows. The first $d-1$ α, β, γ, and δ curves in this group of d are mapped to the $d-1$ α, β, γ, and δ curves, respectively, in X_d. The single remaining α, β, γ, and δ curves are not mapped to a basis curve in X_d. Recall that in constructing our basis, for each e_i we were simply repeating a pattern in every square except the last, thus obtaining $d-1$ of each type of curve among the d squares of X_d. The image of our remaining curves is where the dth copy of the $\alpha_i, \beta_i, \gamma_i, \delta_i$ would have been. What remains is to express these final curves in terms of our basis.

Lemma 3.5. *In X_d, call these additional copies of each curve $\alpha_d, \beta_d, \gamma_d, \delta_d$. Then $e_d = -\sum_{i=1}^{d-1} e_i$.*

Proof. This is proven simply by counting intersections and verifying that the expressions on both sides of the equality have the same intersection with every basis element. $\square$

We therefore know precisely what p_* does to each block of d in X_n in terms of terms of the basis for X_d. To distinguish between our basis curves in X_n and X_d, denote the curves in X_n by e_i^n and the curves in X_d by e_i^d. Then the first $d-1$ of the e_i^n are mapped to the corresponding e_i^d, and the last e_i^n is mapped to $-\sum_{i=1}^{d-1} e_i^d$. In our basis, this means p_* restricted to these blocks and restricted to just one set of e_i is of the form

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix}_{(d-1)\times d}$$

Recall that this is for the first $\frac{n}{d}-1$ blocks of d squares in X_n. The final block of d squares does not contain this last column, as it only has $d-1$ of each type of curve. Putting all this together, we have that restricted to just the e_i^n, p_* is of the form

$$B = \begin{pmatrix} P & \cdots & P & I_{(d-1)\times(d-1)} \end{pmatrix}_{(d-1)\times(n-1)}$$

where there are $\frac{n}{d}-1$ copies of P.
Then in our full basis, on all of the $\alpha_i^n, \beta_i^n, \gamma_i^n, \delta_i^n$, we have that

$$p_* = \begin{pmatrix} B & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & B \end{pmatrix}_{(4d-4)\times(4n-4)}$$

where each copy of B corresponds to the $\alpha, \beta, \gamma, \delta$ curves, respectively.
Let's compute the kernel of p_*. We see that with the ordering of our basis that we have chosen, p_* is in reduced row echelon form, and in particular to find the kernel it suffices to find the kernel of B, which is also in reduced row echelon form. Let $e_1, \ldots, e_{n-1}$ be the basis in the domain for B. Then $e_d, \ldots, e_{n-1}$ are the free variables. Let $x = \sum x^i e_i$ be an arbitrary element. Then setting $Bx = 0$ gives us a set of $d-1$ equations to specify all x in our basis. Reading off the kth row of this matrix equation, we get for $1 \le k \le d-1$ that

$$x^k + \sum_{j=1}^{\frac{n}{d}-1} x^{jd+k} - \sum_{l=1}^{\frac{n}{d}-1} x^{ld} = 0.$$

That is,

$$x^k = \sum_{j=1}^{\frac{n}{d}-1} x^{ld} - x^{jd+k}.$$

For each $d \le i \le n-1$ we therefore get a basis vector for the kernel: if $d|i$, then $e_i + \sum_{j=1}^{d-1} e_j$ is a basis vector for the kernel, and if $d \nmid i$, then $e_i - e_j$ is a basis vector for the kernel where $1 \le j \le d-1$ is the remainder after dividing i by d. This gives the entire kernel.

3.6. **Lifting Properties.** In this section, we use lifting properties to compute the KZ monodromy group and certain Lyapunov exponents of surfaces that arise as covers of other surfaces.

The following from [GJ00] holds in much greater generality, but we will state it only as we need it here.

Theorem 3.6 ([GJ00], Theorem 2)**.** *Let S_1, S_2 be square-tiled surfaces with Veech groups $\Gamma(S_1), \Gamma(S_2)$, respectively. If there exists an affine covering (see [GJ00], Definition 4) $\phi : S_1 \to S_2$, then for some $g \in \mathrm{SL}_2(\mathbb{R})$ we have that $\Gamma(S_1) \cap g\Gamma(S_2)g^{-1}$ is a finite index subgroup of both $\Gamma(S_1)$ and $\Gamma(S_2)$.*

For us, the theorem takes the following form:

Theorem 3.7. *If $d|n$, then there is a subgroup of $\Gamma(X_n)$ that has finite index in $\Gamma(X_d)$, and for any $M \in \Gamma(X_n) \cap \Gamma(X_d)$ we have the following commutative diagram:*

$$\begin{tikzcd} X_n \arrow[r, "M"] \arrow[d, "p"'] & X_n \arrow[d, "p"] \\ X_d \arrow[r, "M"'] & X_d \end{tikzcd}$$

which induces

$$\begin{tikzcd} H_1^{(0)}(X_n) \arrow[r, "\widetilde{\rho(M)}"] \arrow[d, "p_*"'] & H_1^{(0)}(X_n) \arrow[d, "p_*"] \\ H_1^{(0)}(X_d) \arrow[r, "\rho(M)"'] & H_1^{(0)}(X_d) \end{tikzcd}$$

For the properties we will be studying, examining a finite index subgroup as opposed to the full group will still yield precisely the same information. Note that for conciseness we will speak as if we actually have the entire group as opposed to just a finite index subgroup.

Lemma 3.8. $\ker p_*$ *is a $\widetilde{\rho(M)}$-stable subspace.*

Proof. Diagram chasing. Let $x \in \ker p_*$ be arbitrary. Then $0 = \rho(M) \circ p_*(x) = p_* \circ \widetilde{\rho(M)}(x)$, thus $\widetilde{\rho(M)}(x)$ is in the kernel of p_*. $\square$

Lemma 3.9. *If V is a $\widetilde{\rho(M)}$-stable subspace, then $V^\perp$, the orthogonal complement of V in $H_1^{(0)}(X_n)$ with respect to the symplectic form, is also a $\widetilde{\rho(M)}$-stable subspace.*

Proof. Let Ω be the symplectic form. We know $\widetilde{\rho(M)}$ is an element of $\mathrm{Sp}(H_1^{(0)}(X_n))$. Thus if $x \in V^\perp := \{x \in H_1^{(0)}(X_n) \mid \Omega(x, y) = 0 \ \forall y \in V\}$, then

$$\forall y \in V, \ \ \Omega(\widetilde{\rho(M)}(x), y) = \Omega(x, \widetilde{\rho(M)}^{-1}(y)) = 0,$$

where the last equality holds because $y \in V \implies \widetilde{\rho(M)}^{-1}(y) \in V$, and we have $x \in V^\perp$. Thus $\widetilde{\rho(M)}(x) \in V^\perp$, as claimed. $\square$

Corollary 3.10. $(\ker p_*)^\perp$ *is a stable subspace of the KZ-monodromy group.*

We have a basis for the kernel of p_*, and now we will produce a basis for its complement. The dimension of this complement is $\dim H_1^{(0)}(X_d) = 4d-4$. For each $1 \le i \le d-1$, I claim $\sum_{j=0}^{\frac{n}{d}-1} e_{i+jd}$ is a basis vector in the complement. As e runs across $\alpha, \beta, \gamma, \delta$, this gives us our $4d-4$ curves. These curves are trivially linearly independent, as they are each sums of entirely disjoint collections of basis elements. Thus to show that they are a basis for $(\ker p_*)^\perp$, one only needs show that they are orthogonal to each basis element of the kernel, which is easily done by inspection, as most of the curves never cross at all.

Proposition 3.11. $\mathrm{KZ}(X_n)\big|_{(\ker p_*)^\perp} = \mathrm{KZ}(X_d)$ *(up to appropriate choice of basis).*

Proof. Recall each basis vector of $(\ker p_*)^\perp$ is of the form $\sum_{j=0}^{\frac{n}{d}-1} e_{i+jd}$ for some $1 \le i \le d-1$. It is easy to see that $p_*\left(\sum_{j=0}^{\frac{n}{d}-1} e^n_{i+jd}\right) = \frac{n}{d} e^d_i$, thus if we multiply our basis by $\frac{d}{n}$ to get a new basis of the form $\{\frac{d}{n}\sum_{j=0}^{\frac{n}{d}-1} e_{i+jd}\}$, then p_* sends each of these basis elements to a distinct basis element of X_d. Thus, with the appropriate ordering on our new basis of $(\ker p_*)^\perp$, we have that $p_*\big|_{(\ker p_*)^\perp}$ acts by the identity. Our commutative diagram on the level of homology, restricted to $(\ker p_*)^\perp$, then gives the desired result.
Without specifying a basis, we know abstractly that p_* surjective means there exists a section $\sigma : H_1^{(0)}(X_d) \to H_1^{(0)}(X_n)$ satisfying $p_* \circ \sigma = I$ and $\sigma \circ p_*\big|_{(\ker p_*)^\perp} = I$, thus by including σ in our commutative diagram

$$\begin{tikzcd}
H_1^{(0)}(X_n) \arrow[r, "\widetilde{\rho(M)}"] \arrow[d, "p_*"', shift right] & H_1^{(0)}(X_n) \arrow[d, "p_*"] \\
H_1^{(0)}(X_d) \arrow[r, "\rho(M)"'] \arrow[u, "\sigma"', shift right, bend right] & H_1^{(0)}(X_d)
\end{tikzcd}$$

we get that for any $M \in \Gamma(X_n)$, we have $\rho_d(M) = p_* \circ \rho_n(M) \circ \sigma$, and then by restricting to $(\ker p_*)^\perp$ and using $\sigma = (p_*\big|_{(\ker p_*)^\perp})^{-1}$ we get the general result. $\square$

Example 3.12. *For $n = 4, d = 2,$ and $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.*

Previously, we have computed

$$\rho_2(M) = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$\rho_4(M) = \frac{1}{4}\begin{pmatrix} 1 & -3 & 1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 \\ -1 & 3 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 \\ -2 & -2 & 2 & -2 & 2 & 2 & -2 & -2 & 2 & -2 & 2 & 2 \\ -2 & 2 & -2 & 2 & 2 & -2 & -2 & 2 & 2 & -2 & 2 & -2 \\ -1 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & -1 & -1 & 3 & -1 \\ 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & -3 \\ 3 & 3 & -1 & -1 & -1 & -1 & 3 & 3 & -1 & -1 & -1 & -1 \\ 1 & -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 & 1 \\ 2 & 2 & 2 & 2 & -2 & -2 & 2 & 2 & 2 & 2 & -2 & -2 \\ -2 & -2 & 2 & -2 & 2 & 2 & 2 & -2 & 2 & 2 & -2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & 3 \end{pmatrix}$$

Now we simply need to perform a change of basis, noting that these representations were computed with our formerly ordered basis. Our four basis vectors for $(\ker p_*)^\perp$ are $\alpha_1^4 + \alpha_3^4, \beta_1^4 + \beta_3^4, \gamma_1^4 + \gamma_3^4, \delta_1^4 + \delta_3^4$, while the basis for $\ker p_*$ itself is $\alpha_1^4 + \alpha_2^4, -\alpha_1^4 + \alpha_3^4, \beta_1^4 + \beta_2^4, -\beta_1^4 + \beta_3^4, \gamma_1^4 + \gamma_2^4, -\gamma_1^4 + \gamma_3^4, \delta_1^4 + \delta_2^4, -\delta_1^4 + \delta_3^4$. The change of basis matrix is simply writing these as columns (using the old ordering for our basis), so we get

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

for the change of basis matrix. Computing $Q^{-1}\rho_4(M)Q$ gives:

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 1 & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

and indeed we see that the top left corner, which corresponds to $(\ker p_*)^\perp$, is precisely $\rho_2(M)$.

Corollary 3.13. *The Lyapunov exponents of X_d lift to be Lyapunov exponents of X_n.*

This leads to a natural question: if we know the Lyapunov exponents of X_d for all $d|n$, how many Lyapunov exponents of X_n do we get from this simple lift?
Note that, for instance, when $n = 12$, we get Lyapunov exponents from X_2, X_3, and X_4, but X_2 and X_3 both share the Lyapunov exponent 1, and all the Lyapunov exponents of X_2 are also Lyapunov exponents of X_4, and we want to avoid this double counting since these exponents do not get represented multiple times.
Since the spectrum of Lyapunov exponents is symmetric (with respect to 0), it suffices to consider only the non-negative part of the spectrum. Define L_n to be the number of non-negative Lyapunov exponents of X_n which *are not* obtained by lifting Lyapunov exponents from some X_d for $d|n$. X_n has g_n non-negative Lyaponuv exponents, thus $L_n = g_n - \sum_{\substack{d|n \\ d\neq n}} L_d$ (take the total number of Lyapunov exponents and subtract off the inherited Lyapunov exponents; notice that there will be no double counting). Note we are taking $X_1 = \mathbb{T}^2$, so that all X_n are said to inherit the trivial Lyapunov exponent from the torus.
We can rewrite our recurrence by $g_n = \sum_{d|n} L_d$. Standard Möbius inversion then tells us that $L_n = \sum_{d|n} \mu(d) g_{\frac{n}{d}}$, where μ is the standard Möbius function. This becomes:

$$\begin{aligned}
L_n &= \sum_{d|n} \mu(d) g_{\frac{n}{d}} \\
&= \sum_{d|n} \mu(d) \left(2\frac{n}{d} - 1\right) \\
&= 2\sum_{d|n} \mu(d)\frac{n}{d} - \sum_{d|n} \mu(d) \\
&= 2\phi(n) - 0 \\
&= 2\phi(n)
\end{aligned}$$

where we have used standard identities of μ, and ϕ is the Euler totient function. (Note $\sum_{d|n} \mu(d) = 0$ holds only for $n > 1$; when $n = 1$ we simply have $L_1 = 1$).
This means the number of non-negative Lyapunov exponents that X_n *does* inherit from all X_d is

$$g_n - L_n = 2(n - \phi(n)) - 1.$$

Note $\liminf n - \phi(n) = 1$ but $\limsup n - \phi(n) = \infty$, so the amount of information we are getting varies wildly.

3.7. **Arithmeticity.** The KZ monodromy group $\mathrm{KZ}(X)$ of a translation surface is said to be arithmetic if it is dense (under the Zariski topology) in $\mathrm{Sp}(H_1^{(0)}(X))$ and has finite index. A translation surface is said to be arithmetic if its KZ monodromy group is arithmetic.
It is known that EW is non-arithmetic, thus $\mathrm{KZ}(X_2)$ is not dense in $\mathrm{Sp}(H_1^{(0)}(X_2))$. In other words, the elements of $\mathrm{KZ}(X_2)$ satisfy a non-trivial algebraic equation other than those obtained from the defining equation of $\mathrm{Sp}(H_1^{(0)}(X_2))$, namely $A\Omega A^T = \Omega$, where Ω is the symplectic intersection form.
For $2|n$, our proposition earlier tells us $H_1^{(0)}(X_n)$ decomposes into two components that are orthogonal with respect to Ω, which means Ω itself decomposes as a block-diagonal matrix, and the condition $A\Omega A^T = \Omega$ thus also decomposes into these blocks. Note $\Omega\big|_{(\ker p_*)^\perp}$ is precisely $\frac{n}{2}$ times the intersection form of $H_1^{(0)}(X_2)$, which one can check by inspection.

3.7.1. *Representation of the Automorphism Group.* For $n > 2$, the group of automorphisms of X_n ($\mathrm{Aut}(X_n)$) is cyclic and generated by the automorphism sending the top left square down to the bottom left. This uniquely determines an automorphism by positioning the remaining squares in the correct relative positions. If the top row of squares is numbered 1 through $2n$ and the bottom row $2n+1$ to $4n$, then we know square 2 must remain to the right of square 1, square 3 to the right of square 2, and so on, thus the entire top row moves to the bottom. Square $2n+3$ must be above square 1, and then again we simply proceed to the right in ascending order with $2n+4, 2n+5$...

We can compute a representation of $\mathrm{Aut}(X_n)$ given as the map each automorphism induces on the zero-holonomy subspace. That is, it is precisely the same procedure done before for elements of the Veech group. Call this representation ρ.

Since the automorphism group is finite, we can decompose ρ into irreducible real representations. In particular, if $\mathrm{Irr}_{\mathbb{R}}(\mathrm{Aut}(X_n))$ is the collection of irreducible real representations of the automorphism group, let $\{V_a\}_{a\in\mathrm{Irr}_{\mathbb{R}}(\mathrm{Aut}(X_n))}$ be the corresponding set of vector spaces for each representation. Then we have

$$\rho = \bigoplus_{a\in\mathrm{Irr}_{\mathbb{R}}(\mathrm{Aut}(X_n))} \bigoplus_{i=1}^{l_a} a$$

and

$$H_1^{(0)}(X_n) \simeq \bigoplus_{a\in\mathrm{Irr}_{\mathbb{R}}(\mathrm{Aut}(X_n))} V_a^{l_a},$$

where l_a is defined to be the multiplicity with which the irreducible representation a appears in ρ. We then define $W_a := V_a^{l_a}$, called isotypical components.

What are the irreducible real representations of $\mathrm{Aut}(X_n)$? The irreducible complex representations of $\mathrm{Aut}(X_n) = \mathbb{Z}_{2n}$ are precisely the $2n$ one-dimensional representations corresponding to each $2n$th root of unity. From this we find that we have two one-dimensional real representations corresponding to the two real $2n$th roots of unity (± 1). To get the remaining real representations, we group each $2n$th root of unity with its conjugate to obtain $n-1$ two-dimensional real representations. Thus we say that either $a \in \mathrm{Irr}(\mathrm{Aut}(X_n))$ is real if it is a representation corresponding to ± 1 or we say a is complex if it is a representation corresponding to a conjugate pair of non-real roots of unity.

We will use the following proposition from Matheus–Yoccoz–Zmiaikou [MYZ13], from which the notation we just used was taken.

Proposition 3.14. ([MYZ13], *Propositions 3.16, 3.17*)
If a is real, $\mathrm{Sp}(W_a) \simeq \mathrm{Sp}(l_a, \mathbb{R})$.
If a is complex, $\mathrm{Sp}(W_a) \simeq \mathrm{U}_{\mathbb{C}}(p, q)$ *for some* $p+q = l_a$*, where* $\mathrm{U}_{\mathbb{C}}(p, q)$ *is the pseudo unitary group over* $\mathbb{C}$.

We would therefore like to compute the representation ρ of $\mathrm{Aut}(X_n)$. Fortunately, it suffices to determine how ρ acts on just the generator of $\mathrm{Aut}(X_n)$. Denote by σ_n the generator of $\mathrm{Aut}(X_n)$, and its representation $\rho(\sigma_n)$. One can verify that the representation that is induced by σ_n on $H_1^{(0)}(X_n)$ is as follows:

$$\begin{aligned}
&\alpha_i \mapsto -\delta_i \\
&\beta_i \mapsto -\gamma_{i-2}; \quad \beta_1 \mapsto -\gamma_{n-1}; \quad \beta_2 \mapsto \gamma_1 + \cdots + \gamma_{n-1} \\
&\gamma_i \mapsto -\beta_{i+1}; \quad \gamma_{n-1} \mapsto \beta_1 + \cdots + \beta_{n-1} \\
&\delta_i \mapsto -\alpha_{i-1}; \quad \delta_1 \mapsto \alpha_1 + \cdots + \alpha_{n-1}
\end{aligned}$$

Thus if we let

$$\rho(\sigma_n) = \begin{bmatrix} 0 & 0 & 0 & D_n \\ 0 & 0 & C_n & 0 \\ 0 & B_n & 0 & 0 \\ A_n & 0 & 0 & 0 \end{bmatrix},$$

(so that A_n corresponds to where α_i are sent, B_n where β_i are sent, etc.) we have that

$$\begin{aligned}
A_n &= -I_{n-1} \\
B_n &= \begin{bmatrix} \vec{0}_{n-2} & D_{n-1} \\ -1 & (e_1)_{n-2}^T \end{bmatrix} \\
C_n &= \begin{bmatrix} \vec{0}_{n-2}^T & 1 \\ -I_{n-2} & \vec{1} \end{bmatrix} \\
D_n &= \begin{bmatrix} \vec{1}_{n-2} & -I_{n-2} \\ 1 & \vec{0}_{n-2}^T \end{bmatrix}
\end{aligned}$$

where $(e_i)_j$ is the j-tuple consisting of a 1 in the ith position and 0's everywhere else. $\vec{0}_j$ and $\vec{1}_j$ are the j-tuples consisting of all 0's and all 1's, respectively.
Unfortunately, while we are using subscripts to denote the dimensions of each of these components, A_n, B_n, C_n, D_n are actually $n-1$ by $n-1$ matrices.
In order to determine which irreducible representations make up ρ, it suffices to determine the eigenvalues of $\rho(\sigma_n)$. The eigenvalues are the roots of the characteristic polynomial, which by definition is $p_n(t) \coloneqq \det(tI_{4n-4} - \rho(\sigma_n))$. That is, we want to find the determinant of

$$\begin{bmatrix} tI_{n-1} & 0 & 0 & D_n \\ 0 & tI_{n-1} & C_n & 0 \\ 0 & B_n & tI_{n-1} & 0 \\ A_n & 0 & 0 & tI_{n-1} \end{bmatrix}.$$

In the following, we compute $p_n(t)$.

Theorem 3.15. $p_n(t) = \det(t^2 I_{n-1} + D_n)^2 = \left(\frac{t^{2n}-1}{t^2-1}\right)^2$.

Proof. We invoke the fact that

$$\det \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \det(P)\det(S - RP^{-1}Q).$$

Thus let

$$P = S = \begin{bmatrix} tI_{n-1} & 0 \\ 0 & tI_{n-1} \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & D_n \\ C_n & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & B_n \\ A_n & 0 \end{bmatrix}$$

so that

$$tI_{4n-4} - \rho(\sigma_n) = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}.$$

Now, $P^{-1} = t^{-1}I_{2n-2}$, so

$$S - RP^{-1}Q = tI_{2n-2} - t^{-1}RQ = t^{-1}I_{2n-2}(t^2 I_{2n-2} - RQ) = P^{-1}(t^2 I_{2n-2} - RQ),$$

which gives

$$\det(P)\det(S - RP^{-1}Q) = \det(P)\det(P^{-1})\det(t^2 I_{2n-2} - RQ) = \det(t^2 I_{2n-2} - RQ).$$

$$RQ = \begin{bmatrix} 0 & B_n \\ A_n & 0 \end{bmatrix} \begin{bmatrix} 0 & D_n \\ C_n & 0 \end{bmatrix} = \begin{bmatrix} B_n C_n & 0 \\ 0 & A_n D_n \end{bmatrix}.$$

Trivially $A_n D_n = -D_n$. Next,

$$\begin{aligned} B_n C_n &= \begin{bmatrix} \vec{0}_{n-2} & D_{n-1} \\ -1 & (e_1)^T_{n-2} \end{bmatrix} \begin{bmatrix} \vec{0}^T_{n-2} & 1 \\ -I_{n-2} & \vec{1} \end{bmatrix} \\ &= \begin{bmatrix} \vec{0}_{n-2}\vec{0}^T_{n-2} + D_{n-1}(-I_{n-2}) & \vec{0}_{n-2}(1) + D_{n-1}\vec{1}_{n-2} \\ (-1)\vec{0}^T_{n-2} + (e_1)^T_{n-2}(-I_{n-2}) & (-1)(1) + (e_1)^T_{n-2}\vec{1}_{n-2} \end{bmatrix} \\ &= \begin{bmatrix} -D_{n-1} & (e_{n-2})_{n-2} \\ -(e_1)^T_{n-2} & 0 \end{bmatrix} \\ &= -D_n \end{aligned}$$

where the only nontrivial term there is $D_{n-1}\vec{1}_{n-2}$, but notice that this is just the column vector where the ith entry is the sum of the entries in the ith row of D_{n-1}, and notice that every row of D_{n-1} has precisely a 1 and a -1 except for the last which only has a 1, thus every entry is zero except the last, which is one. One then also checks that our resulting block matrix is precisely $-D_n$.
Thus

$$p_n(t) = \det(t^2 I_{2n-2} - RQ) = \det \begin{bmatrix} t^2 I_{n-1} + D_n & 0 \\ 0 & t^2 I_{n-1} + D_n \end{bmatrix} = \det(t^2 I_{n-1} + D_n)^2.$$

We have

$$t^2 I_{n-1} + D_n = \begin{bmatrix} t^2+1 & -1 & 0 & \cdots & \cdots & 0 \\ 1 & t^2 & -1 & \ddots & \ddots & 0 \\ 1 & 0 & t^2 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & t^2 & -1 \\ 1 & 0 & \cdots & \cdots & 0 & t^2 \end{bmatrix}$$

To take the determinant of this matrix, expand by minors along the top row. We get

$$\det(t^2 I_{n-1} + D_n) = (t^2+1)t^{2(n-2)} + \det \begin{bmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 \\ 1 & t^2 & -1 & \ddots & \ddots & 0 \\ 1 & 0 & t^2 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & t^2 & -1 \\ 1 & 0 & \cdots & \cdots & 0 & t^2 \end{bmatrix}_{n-2}$$

Define

$$M_{n-2} := \det \begin{bmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 \\ 1 & t^2 & -1 & \ddots & \ddots & 0 \\ 1 & 0 & t^2 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & t^2 & -1 \\ 1 & 0 & \cdots & \cdots & 0 & t^2 \end{bmatrix}_{n-2} .$$

Then by expanding along the top row we get $M_{n-2} = t^{2(n-3)} + M_{n-3}$. It is trivial to compute $M_2 = t^2 + 1$, from which we get

$$M_{n-2} = \sum_{j=0}^{n-3} t^{2j}.$$

Thus

$$\det(t^2 I_{n-1} + D_n) = (t^2+1)t^{2(n-2)} + M_{n-2} = \sum_{j=0}^{n-1} t^{2j} = \frac{t^{2n}-1}{t^2-1}.$$

Finally we can conclude

$$p_n(t) = \det(t^2 I_{n-1} + D_n)^2 = \left(\frac{t^{2n}-1}{t^2-1}\right)^2.$$

□

Corollary 3.16. *The eigenvalues of $\rho(\sigma_n)$ are precisely two copies of every non-real $2n$th root of unity.*

Thus we see that ρ consists of two copies of every two-dimensional real representation of $\mathrm{Aut}(X_n)$.
Recalling Proposition 2.16, we have

$$H_1^{(0)}(X_n) = \bigoplus_{a \text{ complex}} W_a,$$

with $W_a = V_a^2$ and $\mathrm{Sp}(W_a) \simeq \mathrm{U}_{\mathbb{C}}(p,q)$ for some $p+q=2$.
With further investigation, one can pin-down what p and q are. (To appear in a future project.) Regardless, at this point we know $\dim(U_{\mathbb{C}}(p,q)) = (p+q)^2$, thus $\dim(\mathrm{Sp}(W_a)) = 4$.
As a consequence, we show that no member in our family of surfaces is arithmetic.

Theorem 3.17. *Let X_n be a covering of Eierlegende Wollmilchsau constructed as in Figure 16. Then no X_n is arithmetic.*

Proof. We have $\mathrm{KZ}(X_n)\big|_{W_a} \subseteq \mathrm{Sp}(W_a)$, thus $\overline{\mathrm{KZ}(X_n)\big|_{W_a}} \subseteq \mathrm{Sp}(W_a)$, where this closure is taken in the Zariski topology. This means when restricted to each W_a, the Zariski closure of the KZ monodromy group has dimension at most 4. There are $n-1$ copies of W_a, thus the Zariski closure of the KZ monodromy group has dimension at most $4n-4$. But $H_1^{(0)}(X_n)$ has dimension $4n-4$, thus $\mathrm{Sp}(H_1^{(0)}(X_n))$ has dimension $2(2n-2)^2 + (2n-2) > 4n-4$, thus $\mathrm{KZ}(X_n)$ cannot be dense in $\mathrm{Sp}(H_1^{(0)}(X_n))$ for any n, and so no member of this family is arithmetic. $\square$

We can specify things further through our liftings. Recall that for any n and any $d|n$, with $p : X_n \to X_d$ our covering map, we have $\mathrm{KZ}(X_n)\big|_{(\ker(p_d)_*)^\perp} \simeq \mathrm{KZ}(X_d)$ and $\rho(\sigma_n)\big|_{(\ker(p_d)_*)^\perp} \simeq \rho(\sigma_d)$ (the latter is not explicitly given by what we have done, but can be seen by inspection). This means that the portion of $\mathrm{KZ}(X_n)$ which we know least about is $\bigcap_{\substack{d|n \\ d<n}} \ker(p_d)_*$, and that our two decompositions of $H_1^{(0)}(X_n)$ (one based on the kernels of the maps induced from our coverings and the other based on the irreducible real representations making up the representation of our automorphism group) will agree in the sense that one is a refinement of the other.

That is, each W_a for $a \in \mathrm{Irr}(\mathrm{Aut}(X_n))$ is precisely either a subspace of some $(\ker(p_d)_*)^\perp$ or of $\bigcap_{\substack{d|n \\ d<n}} \ker(p_d)_*$, and we know that W_a is a subspace of $(\ker(p_d)_*)^\perp$ if and only if a is a non-real $2d$th root of unity. This means $(\ker(p_d)_*)^\perp$ consists precisely of the W_a corresponding to a a non-real $2n$th root of unity but not a $2d$th root of unity for any d. How many such a are there?

Every $2n$th root of unity is some power of ζ_{2n}, a primitive $2n$th root of unity. That is, $\zeta_{2n}, \zeta_{2n}^2, \ldots, \zeta_{2n}^{2n}$ is the complete list of $2n$th roots of unity. For $m \in \{1, \ldots, 2n\}$, we have that ζ_{2n}^m is a primitive $\frac{2n}{\gcd(m,2n)}$th root of unity, and by extension it is a $2d$th root of unity if $\frac{2n}{\gcd(m,2n)}$ divides $2d$. We want to know for how many m we do not get a $2d$th root of unity, so we want to find when $\frac{2n}{\gcd(m,2n)}$ does not divide $2d$? We know $\frac{2n}{\gcd(m,2n)}$ always divides $2n$. Thus it is either n, $2n$, a proper divisor of n, or twice a proper divisor of n.

There are two cases. First, suppose n even. This means that n is of the form $2d$ for d a divisor of n. Looking at the four possibilities we listed, this means the case of $\frac{2n}{\gcd(m,2n)} = n$ falls under $\frac{2n}{\gcd(m,2n)}$ being a twice a proper divisor of n. Thus we have three possibilities: $\frac{2n}{\gcd(m,2n)}$ is $2n$, a proper divisor of n, or twice a proper divisor of n. In the latter two cases, we established that this means ζ_{2n}^m is a $2d$th root of unity. In the former case, it means ζ_{2n}^m is a primitive $2n$th root of unity. Thus the $2n$th roots of unity that are not $2d$th roots of unity are precisely the primitive $2n$th roots of unity, of which there are $\phi(2n) = 2\phi(n)$.

If n is odd, n is not of the form $2d$ for d a proper divisor of n. Thus by going through the same cases, we see that any ζ_{2n}^m which is not a $2d$th root of unity is precisely either a primitive $2n$th root or a primitive nth root. There are $\phi(2n)$ of the former and $\phi(n)$ of the latter. Thus we have $\phi(2n) + \phi(n) = 2\phi(n)$ such roots.

We see, then, that in either case we have $2\phi(n)$ distinct roots of $p_n(t)$ that are not roots of any $p_d(t)$.

Another way we could have arrived at this result is as follows: We computed earlier that the number of non-negative Lyapunov exponents found in $\bigcap_{\substack{d|n \\ d<n}} \ker(p_d)_*$ is $2\phi(n)$. Thus

the dimension of this space is $4\phi(n)$. We know this subspace can be partitioned into 1-dimensional irreducible complex representations that come in conjugate pairs, and that each one appears exactly twice. Thus there must be $2\phi(n)$ distinct complex representations, that is, $2\phi(n)$ distinct complex roots of $p_n(t)$.
(A third way we could have arrived at this result would be to simply emulate our proof concerning the number of Lyapunov exponents that are lifted into this subspace; we would have obtained precisely the same recurrence relation)
So, we know we have $2\phi(n)$ distinct complex roots of $p_n(t)$ corresponding to this subspace, which gives us $\phi(n)$ 2-dimensional irreducible real representations each with multiplicity 2. Thus the closure of the KZ monodromy group restricted to this subspace has dimension bounded above by $4\phi(n)$, while $\mathrm{Sp}(\bigcap_{\substack{d|n \\ d<n}} \ker(p_d)_*)$ has dimension $8\phi(n)^2 + 2\phi(n)$.

All together, we have shown that each component in our decomposition of $H_1^{(0)}(X_n)$ (the decomposition using the kernels of p_*) by itself fails to be arithmetic.

References

[For06] Giovanni Forni, *On the Lyapunov exponents of the Kontsevich-Zorich cocycle*, Handbook of dynamical systems. Vol. 1B, Elsevier B. V., Amsterdam, 2006, pp. 549–580. MR 2186248

[GJ00] Eugene Gutkin and Chris Judge, *Affine mappings of translation surfaces: geometry and arithmetic*, Duke Mathematical Journal **103** (2000), no. 2, 191 – 213.

[GRLS24] Rodolfo Gutiérrez-Romo, Dami Lee, and Anthony Sanchez, *Kontsevich–zorich monodromy groups of translation covers of some platonic solids*, Groups, Geometry, and Dynamics (2024).

[GS23] Xun Gong and Anthony Sanchez, *An arithmetic kontsevich–zorich monodromy of a symmetric origami in genus 4*, New York Journal of Mathematics **29** (2023).

[HS08] Frank Herrlich and Gabriela Schmithüsen, *An extraordinary origami curve*, Math. Nachr. **281** (2008), no. 2, 219–237. MR 2387362

[MYZ13] Carlos Matheus, Jean-Christophe Yoccoz, and David Zmiaikou, *Homology of origamis with symmetries*, 2013.

Department of Mathematics, City University of New York (Brooklyn College), Brooklyn, NY 11210 United States
Email address: `felix.filozov26@bcmail.cuny.edu`

Department of Mathematics, University of Washington, Seattle, Washington 98195 United States
Email address: `rich034@uw.edu`