THREE-DIMENSIONAL SHEAR DRIVEN TURBULENCE WITH NOISE AT THE BOUNDARY

WAI-TONG LOUIS FAN, MICHAEL JOLLY, AND ALI PAKZAD

ABSTRACT. We consider the incompressible 3D Navier-Stokes equations subject to a shear induced by noisy movement of part of the boundary. The effect of the noise is quantified by upper bounds on both the expected value and the variance of the dissipation rate. The expected value estimate recovers the bound in [14] for the deterministic case. The movement of the boundary is given by an Ornstein–Uhlenbeck process; a potential for over-dissipation is noted if the OU process were replaced by the Wiener process.

1. INTRODUCTION

Noise is added to turbulence models for a variety of reasons, both practical and theoretical. For example, the onset of turbulence is often related to the randomness of background movement [34]. In any turbulent flow there are unavoidably perturbations in boundary conditions and material properties; see [39, Chapter 3]. The addition of noise in a physical model can be interpreted as a perturbation from the model. There is considerable evidence supporting the stabilization of solutions by noise (see, e.g., [1, 9, 20, 28]). However, the effect of noise in turbulent flow is far from completely understood.

This paper concerns the Kolmogorov dissipation law associated with the incompressible Navier-Stokes equations (NSE) in a 3-dimensional box $D = [0, L]^2 \times [0, h]$ subject to a shear induced by noisy movement of one wall. Specifically, we consider the following differential equation,

$$\begin{aligned} du + (u \cdot \nabla u - \nu \Delta u + \nabla p)\, dt = 0, \\ \nabla \cdot u = 0, \end{aligned} \tag{1.1}$$

with random boundary conditions given by the following:

$$\begin{aligned} &u \text{ is } L-\text{periodic in the } x_1 \text{ and } x_2 \text{ directions}, \\ &u(x_1, x_2, 0, t) = (\mathbb{X}_t, 0, 0)^\top \quad \text{and} \quad u(x_1, x_2, h, t) = (0, 0, 0)^\top, \end{aligned} \tag{1.2}$$

where $\nu > 0$ is a fixed real parameter representing the viscosity, and $\mathbb{X}_t = \mathbb{X}_t(\omega) : \Omega \to \mathbb{R}$, $t \in \mathbb{R}_+$, is a given continuous-in-time stochastic process. The stochastic processes $u(x_1, x_2, x_3, t; \omega)$ and $p(x_1, x_2, x_3; \omega)$ represent respectively the velocity field and the pressure.

The Kolmogorov dissipation law is tied to a phenomenon in turbulence called the energy cascade, which can be explained in 3 main steps. 1− In the absence of a body force, the kinetic energy is introduced into the large scales of the fluid between the parallel plates by the effects of the moving plate. This energy is called *energy input.* 2− The large eddies break up into smaller eddies through vortex stretching over an *intermediate range*, where the energy is transferred to smaller scales and the energy dissipation due to the viscous force is negligible. 3− At small enough scales (expected to be $\sim \mathrm{Re}^{-3/4}$, where Re is the Reynolds number defined in (1.3)) *dissipation dominates* and the energy in those smallest scales decays to zero exponentially fast.

Based on the above description the dissipation is effective at the end of a sequence of processes. Therefore, the rate of dissipation, which measures the amount of energy lost by the viscous force, is determined by the first process in the sequence, which is the energy input. The persistent force driving the shear flow is the motion of the bottom wall $\{(x_1, x_2, 0) : (x_1, x_2) \in [0, L]^2\}$. The time averaged energy dissipation rate must balance the drag exerted by the walls on the fluid. In terms of the characteristic speed U, the large eddies have energy of order U^2 and time scale $\tau = h/U$, so the rate of energy input can be scaled as $U^2/\tau = U^3/h$. This suggests the Kolmogorov dissipation law for time-averaged energy dissipation rate ε (Kolmogorov 1941);

$$\varepsilon \sim \frac{U^3}{h}.$$

Here $a \sim b$ means $a \lesssim b$ and $b \lesssim a$; $a \lesssim b$ means $a \le c\, b$ for a nondimensional universal constant c.

The energy dissipation rate has been widely studied in the literature in the deterministic case [7, 12, 16, 18, 25, 27, 30, 31, 36–38]. Doering and Constantin proved in [14] a rigorous asymptotic bound directly from the Navier-Stokes equations. Their bound is of the form

$$\varepsilon \lesssim \frac{U^3}{h}\,, \quad \text{as } \mathrm{Re} \to \infty, \quad \text{where} \quad \mathrm{Re} = Uh/\nu\,. \tag{1.3}$$

In this paper we derive an upper bound on the expected value of the energy dissipation rate as well as its variance in terms of characteristics of an Ornstein-Uhlenbeck process that is moving part of the boundary. Our estimate recovers (1.3) in the limit as the variance of the noise tends to 0. The key to the analysis is the choice of a stochastic background flow and the treatment of a stochastic integral (with respect to the Wiener process) as a local martingale.

Since the work of Bensoussan and Temam [3] in 1973, there has been substantial advance in understanding the stochastic Navier-Stokes equations, see for example [2, 5, 6, 34, 35, 43] and the references therein. Recently in [11], the exact dissipation rate is obtained for the stochastically forced Navier-Stokes equations under an assumption of energy balance. In all those works the equation always contains noise as a forcing term. Other than the analysis of symmetries of a passive scalar advected by a shear flow in which a boundary moves as a stochastic process in [8], to the best of our knowledge, there is no other work concerning the equations of the motion with stochastic boundary conditions.

Organization of this paper. In section 2, we will introduce the necessary notation and preliminary results needed in the proceeding sections. In section 3, we will state the main result of this work. We will set up an almost sure bound starting form the energy equation in section 4. From there, we will derive an upper bound on the mean value and variance of the energy dissipation respectively in sections 5 and 6. The concluding Section 7 contains some open problems in this direction.

2. Definitions and Notations

We take $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$ to be a complete, filtered probability space equipped with a filtration $\mathcal{F} = \{\mathcal{F}_t; t \in \mathbb{R}_+\}$. Let $W = (W_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion (a.k.a. the Wiener process) adapted to $\mathcal{F}$. The L^2 norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$ respectively, while all other norms will be labeled with subscripts.

As is tacitly assumed with most work on shear flow in the deterministic case (see, for instance, [22, Section 2]), we take the path-wise solution to be regular enough to satisfy the energy equality. That is, $u = (u_t)_{t\in\mathbb{R}_+}$ is a stochastic process defined on $(\Omega, \mathcal{A}, \mathbb{P})$, adapted to the filtration $\mathcal{F}$, and for $\mathbb{P}$-almost all sample points $\omega \in \Omega$, u is an element in $L^2_{loc}(\mathbb{R}_+; W^{1,2}(D)) \cap L^\infty_{loc}(\mathbb{R}_+; L^2(D))$ that solves (1.1)-(1.2) in the classical deterministic case where $\mathbb{X}_t$ is replaced by a constant speed, U. Since $u \in L^2_{loc}(\mathbb{R}_+; W^{1,2}(D))$ almost surely, we have

$$\int_0^t \|\nabla u\|^2 \, ds < \infty \quad \mathbb{P} - a.s. \quad \text{for all } t \in \mathbb{R}_+. \tag{2.1}$$

In experiments, it is natural to take a long but fixed time interval $[0, T]$ and compute the time-average

$$\langle \epsilon \rangle_T := \frac{1}{|D|} \frac{1}{T} \int_0^T \nu \|\nabla u(t, \cdot, \omega)\|^2_{L^2} \, dt \; . \tag{2.2}$$

It is shown in [22] that the effect of T in finite-time averages of physical quantities in turbulence theory, including the energy dissipation rate, can be controlled by parameters such as Re. In our setting, this finite-time average in (2.2) is a random variable whose mathematical expectation can be approximated by taking average over a number of samples in the experiments.

Definition 2.1. We take the time-averaged expected energy dissipation rate for (1.1)-(1.2) is defined by

$$\varepsilon := \limsup_{T\to\infty} \mathbb{E}[\langle \epsilon \rangle_T] \; . \tag{2.3}$$

Our main result, Theorem 3.1 below, is an upper bound for ε in terms of the characteristics of the noise added to the movement of the boundary. Moreover, to assess the deviation from the expectation, we also obtain an upper bound for

$$\limsup_{T\to\infty} Var[\langle \epsilon \rangle_T].$$

Remark 2.1. We note that by Fatou's lemma

$$\limsup_{T\to\infty} \mathbb{E}[\langle\epsilon\rangle_T] \le \mathbb{E}\left[\limsup_{T\to\infty}\langle\epsilon\rangle_T\right].$$

Hence our upper bound on ε defined in (2.3) does not imply one when the order of the lim sup and expectation are reversed.

Definition 2.2. The *Ornstein–Uhlenbeck process* is the strong solution to the Itô stochastic differential equation

$$d\mathbb{X}_t = \theta(U - \mathbb{X}_t)dt + \sigma dW_t,$$

where W_t denotes the Wiener process, and $\theta > 0$ and $\sigma > 0$ are parameters. Hence $\mathbb{X}_t$ is explicitly given by

$$\mathbb{X}_t = \mathbb{X}_0\, e^{-\theta t} + U\,(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)}\, dW_s \tag{2.4}$$

and has stationary distribution given by the normal distribution with mean U and variance $\frac{\sigma^2}{2\theta}$. If the initial distribution satisfies $\mathbb{X}_0 \sim N(U, \frac{\sigma^2}{2\theta})$, then $\mathbb{X}_t \sim N(U, \frac{\sigma^2}{2\theta})$ for all $t \ge 0$ and we say $\mathbb{X}$ is a *stationary OU* process.

Intuitively, the OU process is a Wiener process plus a tendency to move towards a location U, where the tendency is greater when the process is further away from that location. In (2.4), θ is the decay-rate which measures how strongly the system reacts to perturbations, and σ^2 is the variation or the size of the noise. If the Ornstein–Uhlenbeck process (2.4) is stationary, then it can be represented in terms of a time-dependent Wiener process $\mathbb{X}_t = U + \frac{\sigma}{\sqrt{2\theta}} e^{-\theta t} W_{e^{2\theta t}}$, where W_t is the standard Wiener process and the equality is in distribution.

3. Statement of the Results

Theorem 3.1. *Suppose the stochastic process u satisfies (1.1) and boundary conditions (1.2), where $\mathbb{X}_t$ is a stationary Ornstein–Uhlenbeck process* (2.4)*. Assume that the initial condition u_0 is such that $\mathbb{E}(\|u_0\|^2) < \infty$. Then the energy dissipation rate* (2.3) *satisfies*

$$\varepsilon := \limsup_{T\to\infty} \mathbb{E}[\langle\epsilon\rangle_T] \lesssim \left(\frac{1}{Re} + \frac{U}{\theta\, h} + \frac{1}{Re^2}\frac{h\,\theta}{U}\right)\sigma^2 + \frac{\left(U^4 + U^2\left(\frac{\sigma^2}{\theta}\right) + \left(\frac{\sigma^4}{\theta^2}\right)\right)^{3/4}}{h}. \tag{3.1}$$

Moreover, the variance of $\langle\epsilon\rangle_T$ satisfies

$$\begin{aligned}\limsup_{T\to\infty} Var[\langle\epsilon\rangle_T] &\le \limsup_{T\to\infty} \mathbb{E}[\langle\epsilon\rangle_T^2]\\ &\lesssim \frac{1}{h^2}\Bigg\{\frac{\nu^2\sigma^4}{U^2} + \left[U^6 + U^4\frac{\sigma^2}{\theta} + U^2\left(\frac{\sigma^2}{\theta}\right)^2 + \left(\frac{\sigma^2}{\theta}\right)^3\right] + \\ &\qquad U^2\left[U^4 + U^2\left(\frac{\sigma^2}{\theta}\right) + \left(\frac{\sigma^2}{\theta}\right)^2\right] + \frac{\nu^4\theta^4}{U^6}\left(\frac{\sigma^2}{\theta}\right)^2\Bigg\}.\end{aligned} \tag{3.2}$$

In the above estimate on the mean of dissipation rate (3.1), as the variance σ of the disturbance from U tends to 0, we recover the upper bound in Kolmogorov's dissipation law,

$$\lim_{\sigma\to 0} \varepsilon \le \frac{U^3}{h},$$

which is also consistent with the rate proven for the Navier-Stokes equations in [14]. The constants suppressed by the use of $\lesssim$ in (3.1) and (3.2) are explicitly given in (5.5) and (6.10).

Remark 3.2. Since U is the mean velocity of the bottom wall, $\mathbb{X}_t$ has the dimension of velocity. Therefore, θ scales as $\frac{1}{\text{time}}$, and σ has dimension $\frac{\text{velocity}}{\sqrt{\text{time}}}$. Therefore, one can check that the results in Theorem 3.1 are also dimensionally consistent.

4. An almost sure bound for the energy dissipation

The difficulty in the analysis of the shear flow (1.2) is due to the effect of the random nonhomogeneous boundary condition on the flow. We overcome this difficulty by constructing a carefully chosen stochastic background flow. This construction is based on the Hopf extension [24].

Stochastic Background Flow. Our key idea here is to choose the boundary layer thickness $\delta = \delta_t(\omega)$ in the background flow to be random and time-dependent, namely,

$$\delta = \delta_t(\omega) = \frac{\nu}{2(|\mathbb{X}_t(\omega)| + U)}. \tag{4.1}$$

We then let

$$\phi(\omega) = \phi_t(x_3;\omega) = \begin{cases} \left(1 - \dfrac{x_3}{\delta_t(\omega)}\right)\mathbb{X}_t(\omega) & \text{if } 0 \le x_3 \le \delta_t(\omega) \\ 0 & \text{otherwise} \end{cases}, \tag{4.2}$$

and define the stochastic background flow $\Phi = \Phi_t(x_1, x_2, x_3;\omega)$ as:

$$\Phi(\omega) := (\phi(\omega), 0, 0)^\top. \tag{4.3}$$

The boundary layer is denoted by

$$D_\delta = (0, L)^2 \times (0, \delta).$$

Note that $\delta \in (0, h)$ if $\frac{\nu}{2U} < h$, and that for all $U \ge 0$,

$$\frac{1}{4} \le \frac{1}{2} - \frac{\delta\,|\mathbb{X}_t|}{2\nu} \le \frac{1}{2}.$$

Remark 4.1. It is worth mentioning that Φ is a divergence free stochastic vector field which also satisfies the non-homogeneous boundary conditions (1.2), by our construction in (4.2). In addition, δ in (4.1) is determined so as to absorb a term in (4.13).

The key idea to estimate the mean value of the dissipation rate is to decompose the velocity,

$$u = v + \Phi,$$

where Φ is a stochastic, incompressible background field (4.3), carrying the inhomogeneities of the problem and v is a fluctuating incompressible field which is unforced and hence of arbitrary amplitude. Therefore using (1.1) and (4.2) we have

$$dv = du - d\Phi = -(u\cdot\nabla u - \nu\Delta u + \nabla p)\,dt + \begin{cases} \big((\frac{x_3}{\delta}-1)d\mathbb{X}_t,\, 0,\, 0\big)^\top & \text{if } 0\le x_3\le\delta \\ 0 & \text{otherwise} \end{cases}.$$

Now use the Itô's product rule to obtain

$$v\cdot dv = \frac{1}{2}d(v\cdot v) - \frac{1}{2}(\frac{x_3}{\delta}-1)^2\,d\langle\mathbb{X}\rangle_t\,, \quad \text{for} \quad 0\le x_3\le\delta. \tag{4.4}$$

Inserting $u = v+\Phi$ in (1.1), we find the stochastic process v satisfies,

$$dv + d\Phi = -(v\cdot\nabla v + v\cdot\nabla\Phi + \Phi\cdot\nabla v + \Phi\cdot\nabla\Phi - \nu\Delta v - \nu\Delta\Phi + \nabla p)\,dt,$$
$$\nabla\cdot v = 0.$$

The boundary conditions for v are periodic in the x_1 and x_2 directions while in the x_3 direction,

$$v(x_1,x_2,0,t) = v(x_1,x_2,h,t) = 0\,.$$

The energy equation for v, obtained by taking the dot product of v with the above stochastic equation, integrating over D, and integrating by parts, is

$$\begin{aligned}\underbrace{\int_D v\cdot dv\,dx}_{\text{I}} + \underbrace{\int_D v\cdot d\Phi\,dx}_{\text{II}} = \Big(&-\underbrace{(v\cdot\nabla v, v)}_{\text{III}} - \underbrace{(v\cdot\nabla\Phi, v)}_{\text{IV}} - \underbrace{(\Phi\cdot\nabla v, v)}_{\text{V}} \\ &- \underbrace{(\Phi\cdot\nabla\Phi, v)}_{\text{VI}} - \nu\|\nabla v\|^2 - \underbrace{\nu(\nabla v, \nabla\Phi)}_{\text{VII}}\Big)\,dt.\end{aligned} \tag{4.5}$$

We shall estimate each numbered term in (4.5).

Term I. Using Proposition 5.1 (ii) and (4.4) together with a direct calculation, we have

$$\begin{aligned}\int_D v\cdot dv\,dx &= \frac{1}{2}d\|v\|^2 - \frac{1}{2}\int_D\Big[(\frac{x_3}{\delta}-1)^2 d\langle\mathbb{X}\rangle_t\Big]\,dx \\ &= \frac{1}{2}d\|v\|^2 - \frac{1}{2}\int_{D_\delta}\Big[(\frac{x_3}{\delta}-1)^2 dx\Big]\,d\langle\mathbb{X}\rangle_t \\ &= \frac{1}{2}d\|v\|^2 - \frac{1}{6}\delta L^2\,\sigma^2 dt.\end{aligned} \tag{4.6}$$

Term II. From (2.4) it follows that

$$\begin{aligned}\int_D v\cdot d\Phi\,dx &= \int_{D_\delta} v_1\cdot d\phi\,dx \\ &= \Big[\int_{D_\delta} v_1(1-\frac{x_3}{\delta})\,dx\Big]\,d\mathbb{X}_t \\ &= \Big[\int_{D_\delta} v_1(1-\frac{x_3}{\delta})\,dx\Big]\,\theta(U-\mathbb{X}_t)dt \,+\, \sigma\,\Big[\int_{D_\delta} v_1(1-\frac{x_3}{\delta})\,dx\Big]\,dW_t.\end{aligned} \tag{4.7}$$

Term III. Using the incompressibility of v, along with integration by parts, we get

$$(v\cdot\nabla v,\, v)=0.$$

Term IV. Applying the Cauchy-Schwarz inequality (twice), we first estimate as

$$\begin{aligned}\left|\int_0^L\int_0^L v_1\,v_3 dx_1 dx_2\right| &= \left|\int_0^L\int_0^L\int_0^{x_3}\frac{\partial v_1}{\partial x_3}(x_1,x_2,\xi)\;d\xi\int_0^{x_3}\frac{\partial v_3}{\partial x_3}(x_1,x_2,\eta)\;d\eta\,dx_1dx_2\right|\\ &\le x_3\|\frac{\partial v_1}{\partial x_3}\|\,\|\frac{\partial v_3}{\partial x_3}\|\,.\end{aligned}$$

Using this together with Young's inequality, we have

$$\begin{aligned}|(v\cdot\nabla\Phi,v)| = \left|\int_{D_\delta} v_1v_3\frac{\partial\phi}{\partial x_3}dx\right| &\le \left|\frac{\mathbb{X}_t}{\delta}\right|\,\left|\int_0^L\int_0^L\int_0^\delta v_1v_3\,dx_1dx_2dx_3\right|\\ &= \left|\frac{\mathbb{X}_t}{\delta}\right|\,\left|\int_0^\delta\left[\int_0^L\int_0^L v_1v_3\,dx_1dx_2\right]dx_3\right|\\ &\le \left|\frac{\mathbb{X}_t}{\delta}\right|\,\left|\int_0^\delta\left[x_3\|\frac{\partial v_1}{\partial x_3}\|\,\|\frac{\partial v_3}{\partial x_3}\|\right]dx_3\right|\\ &= \left|\frac{\mathbb{X}_t}{\delta}\right|\,\frac{\delta^2}{2}\,\|\frac{\partial v_1}{\partial x_3}\|\,\|\frac{\partial v_3}{\partial x_3}\|\\ &\le \frac{\delta}{2}|\mathbb{X}_t|\left[\frac{1}{2}\|\frac{\partial v_1}{\partial x_3}\|^2+\frac{1}{2}\|\frac{\partial v_3}{\partial x_3}\|^2\right]\\ &\le \frac{\delta}{2}\,|\mathbb{X}_t|\,\|\nabla v\|^2.\end{aligned}\tag{4.8}$$

Term V. Using integration by parts, one can show that,

$$(\Phi\cdot\nabla v,v)=0.$$

Term VI. A pointwise calculation leads to $\Phi\cdot\nabla\Phi=0$, hence,

$$(\Phi\cdot\nabla\Phi,v)=0.$$

Term VII. Direct calculation shows that

$$\|\frac{\partial\phi}{\partial x_3}\| = \left(\frac{L^2}{\delta}\right)^{\frac{1}{2}}\;|\mathbb{X}_t|.$$

Therefore using the Cauchy-Schwarz inequality and Young's inequality, we find

$$
\begin{aligned}
|\nu(\nabla v, \nabla \Phi)| &\leq \nu \int_\Omega \left|\frac{\partial \phi}{\partial x_3}\right| \left|\frac{\partial v_1}{\partial x_3}\right| dx \\
&\leq \nu \|\frac{\partial \phi}{\partial x_3}\| \, \|\frac{\partial v_1}{\partial x_3}\| \\
&\leq \nu \left(\frac{L^2}{\delta}\right)^{\frac{1}{2}} |\mathbb{X}_t| \, \|\nabla v\| \\
&\leq \frac{\nu}{\delta} L^2 |\mathbb{X}_t|^2 + \frac{\nu}{4}\|\nabla v\|^2.
\end{aligned} \tag{4.9}
$$

Using the estimates for all the seven terms above in (4.5) yields,

$$
\begin{aligned}
&\frac{1}{2} d\|v\|^2 + \frac{3\nu}{4}\|\nabla v\|^2 \, dt + \sigma \left[\int_{D_\delta} v_1(1 - \frac{x_3}{\delta}) \, dx\right] dW_t \\
&\leq \frac{1}{6}\delta L^2 \, \sigma^2 dt + \left|\int_{D_\delta} v_1(1 - \frac{x_3}{\delta}) \, dx\right| \theta \, |U - \mathbb{X}_t| dt + \left[\frac{\delta}{2}|\mathbb{X}_t| \, \|\nabla v\|^2 + \nu L^2 \frac{|\mathbb{X}_t|^2}{\delta}\right] dt.
\end{aligned} \tag{4.10}
$$

The second term on the right hand side of inequality (4.10) can be bounded above as follows. Since v_1 vanishes on the bottom wall, we express $v_1(x_1, x_2, x_3)$ as $\int_0^{x_3} \frac{\partial v_1}{\partial \zeta}(x_1, x_2, \zeta) \, d\zeta$, and apply the Cauchy-Schwarz inequality twice,

$$
\begin{aligned}
\left|\int_{D_\delta} v_1 \left(1 - \frac{x_3}{\delta}\right) \, dx\right| &= \left|\int_0^L \int_0^L \int_0^\delta \left(1 - \frac{x_3}{\delta}\right) \int_0^{x_3} \frac{\partial v_1}{\partial x_3}(x_1, x_2, \xi) \, d\xi \, dx\right| \\
&\leq \int_0^L \int_0^L \int_0^\delta \left(1 - \frac{x_3}{\delta}\right) x_3^{1/2} \left(\int_0^{x_3} \left|\frac{\partial v_1}{\partial x_3}\right|^2 \, d\xi\right)^{1/2} \, dx \\
&\leq \frac{4}{15}\delta^{3/2} \int_0^L \int_0^L \left(\int_0^\delta \left|\frac{\partial v_1}{\partial x_3}\right|^2 \, d\xi\right)^{1/2} \, dx_1 \, dx_2 \\
&\leq \frac{1}{3}\delta^{3/2} L \|\nabla v\|.
\end{aligned} \tag{4.11}
$$

Now applying this and then Young's inequality,

$$
\left|\int_{D_\delta} v_1(1 - \frac{x_3}{\delta}) \, dx\right| \theta \, |U - \mathbb{X}_t| \leq \frac{\theta}{3}\delta^{\frac{3}{2}} L \|\nabla v\| \, |U - \mathbb{X}_t| \leq \frac{\nu}{4}\|\nabla v\|^2 + \frac{1}{9\nu}\delta^3 L^2 (U - \mathbb{X}_t)^2 \theta^2. \tag{4.12}
$$

Hence inserting estimate (4.12) in (4.10), and collecting terms, we have the following stochastic equation.

$$
\begin{aligned}
&\frac{1}{2} d\|v\|^2 + \left(\frac{1}{2} - \frac{\delta \, |\mathbb{X}_t|}{2\nu}\right) \nu \|\nabla v\|^2 dt + \sigma \left[\int_{D_\delta} v_1 \left(1 - \frac{x_3}{\delta}\right) \, dx\right] dW_t \\
&\leq \left[\frac{1}{6} L^2 \delta \, \sigma^2 + \nu L^2 \frac{|\mathbb{X}_t|^2}{\delta} + \frac{1}{9\nu}\delta^3 L^2 (U - \mathbb{X}_t)^2 \theta^2\right] dt.
\end{aligned} \tag{4.13}
$$

The stochastic differential inequality (4.13) is interpreted in its integral form.

With our choice of δ in (4.1), the stochastic integral inequality (4.13) gives, for all $T \geq 0$,

$$\begin{aligned}&\frac{1}{2}\|v(T)\|^2 - \frac{1}{2}\|v_0\|^2 + \frac{1}{4}\int_0^T \nu\|\nabla v\|^2 dt + \int_0^T \sigma\left[\int_{D_\delta} v_1\left(1-\frac{x_3}{\delta}\right)dx\right]dW_t \\ \leq &\frac{\nu}{12}L^2\int_0^T \frac{\sigma^2}{|\mathbb{X}_t|+U}\,dt \;+\; 2L^2\int_0^T |\mathbb{X}_t|^2(|\mathbb{X}_t|+U)\,dt \;+\; \frac{1}{72}L^2\nu^2\theta^2\int_0^T \frac{(U-\mathbb{X}_t)^2}{(|\mathbb{X}_t|+U)^3}\,dt.\end{aligned} \tag{4.14}$$

Note that we can ignore the term $\frac{1}{2}\|v(T)\|^2$ on the left of (4.14) to obtain the following almost sure upper bound for the energy dissipation. Lemma 4.2 is the main result of this section.

Lemma 4.2. *With probability one, the following inequality holds for all $T > 0$.*

$$\int_0^T \nu\|\nabla v\|^2 dt + 4M_T \leq 2\|v_0\|^2 + Y_T, \tag{4.15}$$

where

$$Y_T := \int_0^T \frac{\nu}{3}L^2\sigma^2\frac{1}{|\mathbb{X}_t|+U} \;+\; 8L^2\left(|\mathbb{X}_t|^3 + U|\mathbb{X}_t|^2\right) \;+\; \frac{1}{18}L^2\nu^2\theta^2\,\frac{(U-\mathbb{X}_t)^2}{(|\mathbb{X}_t|+U)^3}\;\;dt \tag{4.16}$$

and

$$M_T := \int_0^T \sigma\left[\int_{D_\delta} v_1(1-\frac{x_3}{\delta})\,dx\right]dW_t. \tag{4.17}$$

Remark 4.3. The term $\left[\int_{D_\delta} v_1(1-\frac{x_3}{\delta})\,dx\right]dW_t$ in (4.7) is the first time that we need to make sense of an Itô integral. Assumption (2.1) and (4.11) ensure that M_T is a local martingale with quadratic variation,

$$\langle M\rangle_T = \sigma^2\int_0^T\left[\int_{D_\delta} v_1(1-\frac{x_3}{\delta})\,dx\right]^2 dt \tag{4.18}$$

$$\leq \sigma^2\int_0^T \delta^3L^2\|\nabla v\|^2 dt \tag{4.19}$$

$$= \frac{\nu^3L^2\sigma^2}{8}\int_0^T \frac{\|\nabla v\|^2}{(|\mathbb{X}_t|+U)^3}\,dt, \tag{4.20}$$

where the last equality follows from the definition of δ in (4.1).

Remark 4.4. Consider the stochastic integral. For each $n \geq 1$ we let

$$\tau_n := \inf\left\{t \geq 0 : \int_0^t \frac{\|\nabla v\|^2}{(|\mathbb{X}_t|+U)^3}\,ds > n\right\}.$$

The stochastic integral $\int_0^{T\wedge\tau_n} \frac{\|\nabla v\|}{(|\mathbb{X}_t|+\frac{\nu}{h})^{\frac{3}{2}}}\,dW_t$ (as a process indexed by T) is a martingale and hence

$$\mathbb{E}[M_{T\wedge\tau_n}] = \mathbb{E}\left[\int_0^{T\wedge\tau_n} \frac{\|\nabla v\|}{(|\mathbb{X}_t|+U)^{\frac{3}{2}}}\,dW_t\right] = 0 \quad \text{for all } T \in [0,\infty),\; n \in \mathbb{N}. \tag{4.21}$$

From (4.15), for all $n \geq 1$, we have $\mathbb{P}$-a.s.,

$$\int_0^{T\wedge\tau_n} \nu\|\nabla v\|^2 dt + M_{T\wedge\tau_n} \leq 2\|v_0\|^2 + Y_{T\wedge\tau_n}. \tag{4.22}$$

Taking expectation $\mathbb{E}$ on both sides, applying (4.21), and observe that Y_t is nondecreasing in t, we obtain

$$\mathbb{E}\int_0^{T\wedge\tau_n} \nu\|\nabla v\|^2 dt \leq \mathbb{E}[2\|v_0\|^2 + Y_T]. \tag{4.23}$$

By Assumption (2.1), the process $\langle M\rangle_t$ does not blow up in the sense that $\lim_{n\to\infty}\tau_n = +\infty$ almost surely. Hence $\mathbb{E}\int_0^{T\wedge\tau_n}\|\nabla v\|^2 dt \to \mathbb{E}\int_0^T \|\nabla v\|^2 dt$ as $n\to\infty$ and therefore

$$\mathbb{E}\int_0^{T} \nu\|\nabla v\|^2 dt \leq \mathbb{E}[2\|v_0\|^2 + Y_T]. \tag{4.24}$$

5. Estimation of the Mean Value

With Remark 4.4 in mind, the rest of proof is as follows. To construct the estimate on $\mathbb{E}[\langle\epsilon\rangle_T]$, take the expected value of (4.15) over $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$, then average it over $[0,T]$, and finally take the limit superior as $T\to\infty$.

We shall estimate the four terms in Y_t (4.16) separately in (5.1:5.4). To this end we need the following standard properties for the stationary OU process and Gaussian random variables (for a proof and additional properties see [17]).

Proposition 5.1. *Let $\mathbb{X}_t$ be a stationary Ornstein–Uhlenbeck process* (2.4)*. The following hold for all $t\geq 0$.*

(i) $\mathbb{X}_t \sim N(U, \frac{\sigma^2}{2\theta})$,
(ii) $\langle\mathbb{X}\rangle_t = \sigma^2 t$, *where* $\langle\mathbb{X}\rangle_t$ *is the quadratic variation of* $\mathbb{X}$,
(iii) $\mathbb{E}\left[|\mathbb{X}_t|^2\right] = \frac{\sigma^2}{2\theta}$,
(iv) $\mathbb{E}\left[|\mathbb{X}_t|^4\right] = U^4 + 6U^2\left(\frac{\sigma^2}{2\theta}\right) + 3\left(\frac{\sigma^2}{2\theta}\right)^2$ *and therefore* $\mathbb{E}\left[|U-\mathbb{X}_t|^4\right] = 3\left(\frac{\sigma^2}{2\theta}\right)^2$,
(v) $\mathbb{E}\left[|\mathbb{X}_t|^6\right] = U^6 + 15U^4\frac{\sigma^2}{2\theta} + 45U^2\left(\frac{\sigma^2}{2\theta}\right)^2 + 15\left(\frac{\sigma^2}{2\theta}\right)^3$.

For the first term on the right of (4.16), simply from $|\mathbb{X}_t|\geq 0$, we have

$$\frac{1}{T}\,\mathbb{E}\left[\int_0^T \frac{\sigma^2}{|\mathbb{X}_t|+U}\,dt\right] \leq \frac{1}{T}\,\mathbb{E}\left[\int_0^T \frac{\sigma^2}{U}\,dt\right] = \frac{1}{U}\,\sigma^2\,. \tag{5.1}$$

Recall that, $\mathbb{X}_t$ has normal distribution with mean U and variance $\frac{\sigma^2}{2\theta}$ for all $t\in\mathbb{R}_+$ under $\mathbb{P}$. By Fubini's theorem and Jensen's inequality, the second term is estimated as

$$\begin{aligned}\frac{1}{T}\,\mathbb{E}\left[\int_0^T |\mathbb{X}_t|^3\,dt\right] &= \frac{1}{T}\int_0^T\int_\Omega |\mathbb{X}|^3\,dP\,dt \leq \frac{1}{T}\int_0^T\left(\int_\Omega |\mathbb{X}|^4\,dP\right)^{3/4} dt\\ &\leq \left(\mathbb{E}|\mathbb{X}_t|^4\right)^{3/4} = \left(U^4 + 6U^2\left(\frac{\sigma^2}{2\theta}\right) + 3\left(\frac{\sigma^2}{2\theta}\right)^2\right)^{3/4}.\end{aligned} \tag{5.2}$$

Also by Fubin's theorm, the third term can be written as

$$\frac{1}{T}\,\mathbb{E}\left[\int_0^T |\mathbb{X}_t|^2\,dt\right] = \mathbb{E}\left[|\mathbb{X}_t|^2\right] = \frac{\sigma^2}{2\theta}. \tag{5.3}$$

Since $U-\mathbb{X}_t$ is a centered normal variable with variance $\frac{\sigma^2}{2\theta}$, we again interchange the order of integration to obtain the following bound on the fourth term of Y_t in (4.16)

$$\frac{1}{T}\,\mathbb{E}\left[\int_0^T \frac{(U-\mathbb{X}_t)^2}{(|\mathbb{X}_t|+U)^3}\,dt\right] \le \frac{1}{U^3}\,\frac{1}{T}\,\mathbb{E}\left[\int_0^T |U-\mathbb{X}_t|^2\,dt\right] = \frac{1}{U^3}\,\mathbb{E}\left[|U-\mathbb{X}_t|^2\right] = \frac{1}{U^3}\,\frac{\sigma^2}{2\theta}. \tag{5.4}$$

Now take the expectation of (4.15), divide by T and $|D| = L^2h$, and use the above estimates (5.1), (5.2), (5.3), (5.4), and (4.21) to obtain

$$\varepsilon \le \left(\frac{1}{3}\frac{\nu}{U\,h} + 4\,\frac{U}{\theta\,h} + \frac{1}{36}\,\frac{\nu^2\theta}{h\,U^3}\right)\sigma^2 + \frac{8\left(U^4 + 6U^2\left(\frac{\sigma^2}{2\theta}\right) + \frac{3}{4}\left(\frac{\sigma^4}{\theta^2}\right)\right)^{3/4}}{h}\ , \tag{5.5}$$

which also can be represented as

$$\varepsilon \le \left(\frac{1}{3}\frac{1}{\mathrm{Re}} + 4\,\frac{U}{\theta\,h} + \frac{1}{36}\,\frac{1}{\mathrm{Re}^2}\,\frac{h\,\theta}{U}\right)\sigma^2 + \frac{8\left(U^4 + 6U^2\left(\frac{\sigma^2}{2\theta}\right) + \frac{3}{4}\left(\frac{\sigma^4}{\theta^2}\right)\right)^{3/4}}{h}. \tag{5.6}$$

Remark 5.2. One could replace U in (4.1) with the dimensionally consistent term $\frac{\nu}{h}$. Proceeding as before, one arrives at

$$\varepsilon \lesssim \left(1 + \frac{\nu}{h^2\theta} + \frac{h^2\theta}{\nu}\right)\sigma^2 + \frac{8\left(U^4 + U^2\left(\frac{\sigma^2}{2\theta}\right) + Big(\frac{\sigma^4}{\theta^2})\right)^{3/4}}{h}. \tag{5.7}$$

This avoids the singularity in (5.5) as $U \to 0$, but replaces it with one as $\nu \to 0$.

Remark 5.3. Based on our current analysis, if we were to instead take $\mathbb{X}_t$ to be Brownian motion, i.e., $\mathbb{X}_t = W_t$, this would result in a potential over-dissipation of the model, since,

$$\frac{1}{T}\,\mathbb{E}\left[\int_0^T |\mathbb{X}_t|^2\,dt\right] = \frac{1}{T}\int_0^T \mathbb{E}\left[W_t^2\right]\,dt = \frac{1}{2}T \to \infty, \qquad \text{as} \qquad T \to \infty.$$

Remark 5.4. If $\theta \to 0$, the estimate (5.5) also allows for over-dissipation of the model $\mathbb{E}[\langle\varepsilon\rangle] \to \infty$. This observation is also consistent with Remark 5.3 because while $\theta \to 0$, the Ornstein–Uhlenbeck process $\to$ the Wiener process in (2.4).

6. Estimation of the Variance

Define, for $T > 0$,

$$\mathcal{E}_T := \int_0^T \nu\|\nabla v\|^2 dt. \tag{6.1}$$

Lemma 4.2 tells us that

$$\mathcal{E}_T \leq 2\|v_0\|^2 + Y_T + |M_T|. \tag{6.2}$$

The variance of $\mathcal{E}_T$ is bounded above by the second moment, and

$$\mathbb{E}[|\mathcal{E}_T|^2] \leq 3\,\mathbb{E}\left[4\|v_0\|^2 + |Y_T|^2 + |M_T|^2\right]. \tag{6.3}$$

6.1. **Bound** $\mathbb{E}[|M_T|^2]$**.** From (4.18) we have

$$\begin{aligned}\mathbb{E}[|M_T|^2] = \mathbb{E}[\langle M\rangle_T] \leq& \frac{\nu^3 L^2\,\sigma^2}{8}\int_0^T \frac{\|\nabla v\|^2}{(|\mathbb{X}_t|+U)^3}\,dt\\ \leq& \frac{\nu^2 L^2\,\sigma^2}{8U^3}\mathbb{E}[\mathcal{E}_T]. \end{aligned}\tag{6.4}$$

6.2. **Bound** $\mathbb{E}[|Y_T|^2]$**.** We apply the Cauchy-Schwarz inequality to (4.16) to obtain

$$|Y_T|^2 \leq T\int_0^T \left[\frac{\nu}{3}L^2\sigma^2\,\frac{1}{|\mathbb{X}_t|+U} + 8L^2\left(|\mathbb{X}_t|^3 + U|\mathbb{X}_t|^2\right) + \frac{1}{18}L^2\nu^2\theta^2\,\frac{(U-\mathbb{X}_t)^2}{(|\mathbb{X}_t|+U)^3}\right]^2 dt. \tag{6.5}$$

Hence

$$\mathbb{E}[|Y_T|^2] \leq T\int_0^T \mathbb{E}\left[\frac{\nu}{3}L^2\sigma^2\,\frac{1}{|\mathbb{X}_t|+U} + 8L^2\left(|\mathbb{X}_t|^3 + U|\mathbb{X}_t|^2\right) + \frac{1}{18}L^2\nu^2\theta^2\,\frac{(U-\mathbb{X}_t)^2}{(|\mathbb{X}_t|+U)^3}\right]^2 dt \tag{6.6}$$

$$= T^2\,\mathbb{E}\left[\frac{\nu}{3}L^2\sigma^2\,\frac{1}{|\mathbb{X}_t|+U} + 8L^2\left(|\mathbb{X}_t|^3 + U|\mathbb{X}_t|^2\right) + \frac{1}{18}L^2\nu^2\theta^2\,\frac{(U-\mathbb{X}_t)^2}{(|\mathbb{X}_t|+U)^3}\right]^2 \tag{6.7}$$

because $\mathbb{X}_t$ has normal distribution with mean U and variance $\frac{\sigma^2}{2\theta}$ for all t. By the Cauchy-Schwarz inequality $(\sum_{i=1}^n a_i)^2 \leq n\sum_{i=1}^n a_i^2$ with $n=4$, which applied to the above expectation gives

$$\begin{aligned}&\mathbb{E}\left[\frac{\nu}{3}L^2\sigma^2\,\frac{1}{|\mathbb{X}_t|+U} + 8L^2\left(|\mathbb{X}_t|^3 + U|\mathbb{X}_t|^2\right) + \frac{1}{18}L^2\nu^2\theta^2\,\frac{(U-\mathbb{X}_t)^2}{(|\mathbb{X}_t|+U)^3}\right]^2\\ &\leq 4\left\{\frac{\nu^2L^4\sigma^4}{9U^2} + 64L^4\mathbb{E}[\mathbb{X}_t^6] + 64L^4U^2\mathbb{E}[\mathbb{X}_t^4] + \frac{L^4\nu^4\theta^4}{324U^6}\mathbb{E}[(U-\mathbb{X}_t)^4]\right\}\\ &= 4\left\{\frac{\nu^2L^4\sigma^4}{9U^2} + 64L^4\left[U^6 + 15U^4\frac{\sigma^2}{2\theta} + 45U^2\left(\frac{\sigma^2}{2\theta}\right)^2 + 15\left(\frac{\sigma^2}{2\theta}\right)^3\right] + \right.\\ &\left. 64L^4U^2\left[U^4 + 6U^2\left(\frac{\sigma^2}{2\theta}\right) + 3\left(\frac{\sigma^2}{2\theta}\right)^2\right] + \frac{L^4\nu^4\theta^4}{108U^6}\left(\frac{\sigma^2}{2\theta}\right)^2\right\} := \Theta,\end{aligned}\tag{6.8}$$

where in the last equality we used Proposition 5.1 to estimate the moments of normal random variable.

6.3. **Summarizing.** Putting the above and (6.4) into (6.3), we obtain

$$\mathbb{E}[|\mathcal{E}_T|^2] \leq 12\,\mathbb{E}\left[\|v_0\|^2\right] + 3\frac{\nu^2L^2\,\sigma^2}{8U^3}\mathbb{E}[\mathcal{E}_T] + 3T^2\Theta. \tag{6.9}$$

Applying Jensen's inequality and then Young's inequality, we find

$$\alpha \mathbb{E}[\mathcal{E}_T] \leq \alpha \sqrt{\mathbb{E}[|\mathcal{E}_T|^2]} \leq \frac{\alpha^2}{2} + \frac{\mathbb{E}[|\mathcal{E}_T|^2]}{2} \ ,$$

where $\alpha = 3\frac{\nu^2 L^2 \sigma^2}{8U^3}$ has the same dimension as that of $\mathcal{E}_T$.

Using this in (6.9), we have

$$\mathbb{E}[|\mathcal{E}_T|^2] \leq 12\, \mathbb{E}\left[\|v_0\|^2\right] + \frac{\alpha^2}{2} + \frac{\mathbb{E}[|\mathcal{E}_T|^2]}{2} + 3T^2\Theta$$

so that

$$\mathbb{E}[|\mathcal{E}_T|^2] \leq 24\, \mathbb{E}\left[\|v_0\|^2\right] + \alpha^2 + 6T^2\Theta \ ,$$

and hence

$$\limsup_{T\to\infty} \frac{\mathbb{E}[|\mathcal{E}_T|^2]}{T^2} \leq 6\Theta.$$

Recalling $|D| = L^2 h$, we have

$$\begin{aligned} \limsup_{T\to\infty} \mathbb{E}[\langle\epsilon\rangle_T^2] = \limsup_{T\to\infty} \frac{\mathbb{E}[|\mathcal{E}_T|^2]}{|D|^2 T^2} &\leq \frac{6\,\Theta}{L^4 h^2} \\ =&\frac{24}{h^2}\left\{ \frac{\nu^2\sigma^4}{9K^2} + 64\left[U^6 + 15U^4 \frac{\sigma^2}{2\theta} + 45U^2\left(\frac{\sigma^2}{2\theta}\right)^2 + 15\left(\frac{\sigma^2}{2\theta}\right)^3 \right] + \right. \\ &\left. 64U^2\left[U^4 + 6U^2\Big(\frac{\sigma^2}{2\theta}\Big) + 3\Big(\frac{\sigma^2}{2\theta}\Big)^2\right] + \frac{\nu^4\theta^4}{12U^6}\left(\frac{\sigma^2}{2\theta}\right)^2 \right\}. \end{aligned} \tag{6.10}$$

7. Conclusion and Commentary

In this paper we have derived uniform (in T) bounds for both the mean and the variance of the energy dissipation rate for solutions of the incompressible Navier–Stokes equations with a boundary wall moving as a stationary Ornstein–Uhlenbeck process. We recover the bound for the deterministic case in [14] as the variance of the OU process tends to 0. A similar argument can be used to find higher moment bounds. A novelty of our method is the construction of a carefully chosen stochastic background flow Φ that depends on the stochastic forcing, as indicated in (4.1). Our technique can be readily generalized to the case where the OU process is replaced by a general gradient system of the form

$$dX_t = -\nabla h(X_t)\, dt + \sigma\, dW_t, \tag{7.1}$$

where $\sigma > 0$. The OU process (2.4) is the case when $h(x) = -\theta(x-U)^2/2$. It is well-known that if

$$Z^{(\sigma)} := \int_{\mathbb{R}} \exp\left(-\frac{2}{\sigma^2} h(x)\right) dx < \infty \ ,$$

then 1-dimensional gradient system (7.1) has a unique invariant distribution given by the Gibbs measure

$$\frac{1}{Z^{(\sigma)}}\exp\left(-\frac{2}{\sigma^2}h(x)\right). \tag{7.2}$$

This technique can also be generalized to jump processes. The analysis herein would allow for over-dissipation of the model if the noise at the boundary were taken to be the Wiener process, as noted in Remarks 5.3 and 5.4.

Finally, it was crucial to take the limit superior in time *after* the expectation. Our estimate does not provide a bound when the operations are taken in the reverse order. It remains to find a bound in the latter case, or quantify the difference in the two expressions describing the rate of dissipation.

8. Acknowledgments

The work of M. Jolly was supported in part by NSF grant DMS-1818754. W.T. Fan is partially supported by NSF grant DMS-1804492 and ONR grant TCRI N00014-19-S-B001.

References

[1] V. Barbu, *Stabilization of Navier–Stokes Flows*, Springer, London, 2001.

[2] J. Bedrossian, M. Coti Zelati, S. Punshon-Smith, and F. Weber, *A sufficient condition for the Kolmogorov 4/5 law for stationary martingale solutions to the 3D Navier-Stokes equations*, Comm. Math. Phys. **367** (2019), no. 3, 1045–1075.

[3] A. Bensoussan and R. Temam, *Equatios stochastique du type Navier-Stokes*, J. Funct. Anal. **13** (1973), 195–222.

[4] A. Biswas, M. S. Jolly, V. R. Martinez, and E. S. Titi, *Dissipation Length Scale Estimates for Turbulent Flows: A Wiener Algebra Approach*, Journal of Nonlinear Science **24** (2014), 441-471.

[5] H. Breckner, *Galerkin approximation and the strong solution of the Navier-Stokes equation*, Journal of Applied Mathematics and Stochastic Analysis **13** (2000), no. 3, 239–259.

[6] Z Brzeźniak and S Peszat, *Infinite dimensional stochastic analysis*, Verh. Afd. Natuurkd. 1. Reeks. K. Ned. Akad. Wet., vol. 52, R. Neth. Acad. Arts Sci., Amsterdam, 2000.

[7] F. H. Busse, *Bounds for turbulent shear flow*, Journal of Fluid Mechanics **41** (1970), 4219–240.

[8] R. Camassa, Z. Kilic, and R. M. McLaughlin, *On the symmetry properties of a random passive scalar with and without boundaries, and their connection between hot and cold states*, Phys. D **400** (2019), 132124, 32, DOI 10.1016/j.physd.2019.05.004. MR4008029

[9] T. Caraballo, K. Liu, and X.R. Mao, *On stabilization of partial differential equations by noise*, Nagoya Math. J. **161(2)** (2001), 155–170.

[10] A. J. Chorin, *Numerical study of slightly visous flow*, J. Fluid Mech. **57** (1973), 785–796.

[11] Y. T. Chow and A. Pakzad, *On the zeroth law of turbulence for the stochastically forced Navier-Stokes equations*, arXiv:2004.08655 (2020).

[12] V. DeCaria, W. Layton, A. Pakzad, Y. Rong, N. Sahin, and H. Zhao, *On the determination of the grad-div criterion*, Journal of Mathematical Analysis and Applications **467** (2018), 1032–1037.

[13] C .R. Doering, *The 3D Navier-Stokes Problem*, Annual Review of Fluid Mechanics **41** (2009), 109-128.

[14] C .R. Doering and P. Constantin, *Energy dissipation in shear driven turbulence*, Physical review letters **69** (1992), 1648.

[15] C. R. Doering and P. Constantin, *Variational bounds on energy dissipation in incompressible flows. III. Convection*, Phys. Rev. E **53** (1996), 5957–5981.

[16] C. R. Doering and C. Foias, *Energy dissipation in body-forced turbulence*, J. Fluid Mech. **467** (2002), 289–306.

[17] J. L. Doob, *The Brownian movement and stochastic equations*, Annals of Mathematics (1942), 351–369.

[18] J. Duchon and R. Robert, *Inertial energy dissipation for weak solutions of incompressible Euler and Navier–Stokes equations*, Nonlinearity **13** (2000), 249–255.

[19] L. C. Evans, *An Introduction to Stochastic Differential Equations*, American Mathematical Society, 2013.

[20] K. Fellner, S. Sonner, B. Q. Tang, and D. D. Thuan, *Stabilisation by noise on the boundary for a Chafee-Infante equation with dynamical boundary conditions*, AIMS **24** (2019), 4055-4078.

[21] F. Flandoli and D. Gątarek, *Martingale and stationary solutions for stochastic Navier-Stokes equations*, Probab. Theory Related Fields **102** (1995), no. 3, 367–391.

[22] C. Foias, M. S. Jolly, O. P. Manley, R. Rosa, and R. Temam, *Kolmogorov theory via finite-time averages*, Phys. D **212** (2005), no. 3-4, 245–270, DOI 10.1016/j.physd.2005.10.002.

[23] U. Frisch, *Turbulence*, Cambridge University Press, Cambridge, 1995. The legacy of A. N. Kolmogorov.

[24] E. Hopf, *Lecture series of the symposium on partial differential equations*, Berkeley (1955).

[25] L. N. Howard, *Bounds on flow quantities*, Annual Review of Fluid Mechanics **4** (1972), 473–494.

[26] N. Jiang and W. J. Layton, *Algorithms and models for turbulence not at statistical equilibrium*, Computers and Mathematics with Applications **71** (2016), 2352–2372.

[27] R. R. Kerswell, *Variational bounds on shear-driven turbulence and turbulent Boussinesq convection*, Physica D **100** (1997), 355–376.

[28] A.A. Kwiecinska, *Stabilization of partial differential equations by noise*, Stoch. Proc. & Appl. **79** (1999), 179–184.

[29] A. N. Kolmogorov, *The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers*, Proc. Roy. Soc. London Ser. A **434** (1991), no. 1890, 9–13. Translated from the Russian by V. Levin; Turbulence and stochastic processes: Kolmogorov's ideas 50 years on.

[30] W. J. Layton, *Energy dissipation in the Smagorinsky model of turbulence*, Appl. Math. Lett. **59** (2016), 56–59.

[31] W. J. Layton, *Energy dissipation bounds for shear flows for a model in large eddy simulation*, Math. Comput. Modelling **35** (2002), no. 13, 1445–1451.

[32] T. M. Leslie and R. Shvydkoy, *Conditions implying energy equality for weak solutions of the Navier-Stokes equations*, SIAM J. Math. Anal. **50** (2018), no. 1, 870–890.

[33] C. Marchioro, *Remark on the energy dissipation in shear driven turbulence*, Phys. D **74** (1994), no. 3-4, 395–398.

[34] R. Mikulevicius and B.L. Rozovskii, *Stochastic Navier–Stokes equations for turbulent flows*, SIAM J. Math. Anal. **35** (2004), 1250–1310.

[35] ———, *Global L_2-solutions of stochastic Navier-Stokes equations*, Ann. Probab. **33** (2005), no. 1, 137–176.

[36] A. Pakzad, *Damping functions correct over-dissipation of the Smagorinsky Model*, Mathematical Methods in the Applied Sciences **40** (2017), no. 16, 5933–5945.

[37] A. Pakzad, *Analysis of mesh effects on turbulence statistics*, Journal of Mathematical Analysis and Applications **475** (2019), 839-860.

[38] A. Pakzad, *On the long time behavior of time relaxation model of fluids*, Phys. D **408** (2020), 132509, 6.

[39] S. B. Pope, *Turbulent flows*, Cambridge University Press, Cambridge, 2000.

[40] V. Scheffer, *Hausdorff measure and the Navier–Stokes equations*, Comm. Math. Phys. **55** (1977), 97–112.

[41] K. R. Sreenivasan, *An update on the energy dissipation rate in isotropic turbulence*, Phys. Fluids **10** (1998), no. 2, 528–529.

[42] J. C. Vassilicos, *Dissipation in turbulent flows*, Annual Review of Fluid Mechanics **47** (2014), 95-114.

[43] D. Wang and H. Wang, *Global existence of martingale solutions to the three-dimensional stochastic compressible Navier-Stokes equations*, Differential Integral Equations **28** (2015), no. 11-12, 1105–1154.

[44] X. Wang, *Approximation of stationary statistical properties of dissipative dynamical systems: time discretization*, Mathematics of Computation **79** (2010), no. 269, 259-280.

[45] X. Wang, *Effect of tangential derivative in the boundary layer on time averaged energy dissipation rate*, Physica D: Nonlinear Phenomena **144** (2000), 142–153.

[46] X. Wang, *Time-averaged energy dissipation rate for shear driven flows in* $\mathbf{R}^n$, Phys. D **99** (1997), no. 4, 555–563.

Department of Mathematics, Indiana University Bloomington, IN 47405, USA

E-mail address: waifan@iu.edu

Department of Mathematics, Indiana University Bloomington, IN 47405, USA

E-mail address: msjolly@indiana.edu

Department of Mathematics, Indiana University Bloomington, IN 47405, USA

E-mail address: apakzad@iu.edu