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SURVEY ARTICLE

Scaling limits of interacting diffusions in domains

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Abstract We survey recent effort in establishing the hydrodynamic limits and the fluctuation limits for a class of interacting diffusions in domains. These systems are introduced to model the transport of positive and negative charges in solar cells. They are general microscopic models that can be used to describe macroscopic phenomena with coupled boundary conditions, such as the population dynamics of two segregated species under competition. Proving these two types of limits represents establishing the functional law of large numbers and the functional central limit theorem, respectively, for the empirical measures of the spatial positions of the particles. We show that the hydrodynamic limit is a pair of deterministic measures whose densities solve a coupled nonlinear heat equations, while the fluctuation limit can be described by a Gaussian Markov process that solves a stochastic partial differential equation.

Keywords Hydrodynamic limit, fluctuation, interacting diffusion, reflected diffusion, Dirichlet form, non-linear boundary condition, coupled partial differential equation, martingales, stochastic partial differential equation, Guassian process
MSC 60F17, 60K35, 60H15, 92D15

1 Introduction

Interacting particle systems is a family of mathematical models that are widely used in describing diverse phenomena, such as ecological systems [20], population dynamics [21,35,37], chemical reactions [34], super-conductivity [40], quantum dynamics [22], and fluid dynamics [25]. A principal theme in investigating these phenomena is *to connect the microscopic mechanisms of the systems with the collective behaviors that emerge in the macroscopic scale* under suitable space-time scalings. The remarkable power of these models in

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illuminating this connection has long been recognized, but proving rigorous results is usually quite challenging.

With motivation to model and analyze the transport of positive and negative charges in solar cells, two different but related stochastic interacting particle systems are introduced in [11] and [12]. Here is an informal description of the models. We model a solar cell by a domain in $\mathbb{R}^d$ that is divided into two adjacent sub-domains D_+ and D_- by an interface I, a $(d-1)$-dimensional hypersurface. Domains D_+ and D_- represent the hybrid medium that confine the positive and the negative charges, respectively. See Fig. 1 for an illustration. At microscopic level, positive and negative charges are initially modeled by N independent reflected Brownian motions (or more generally, reflected diffusions) with drift on D_+ and on D_-, respectively[1)]. These random motions model the transport of positive (resp. negative) charges under an applied electric potential field. Besides, there is a harvest region $\Lambda_\pm \subset \partial D_\pm \setminus I$ that absorbs (harvests) $\pm$ charges, respectively, whenever it is being visited. Furthermore, these two types of particles annihilate each other in pairs at a certain rate when they come close to each other near the interface I. This interaction models the annihilation, trapping, recombination, and separation phenomena of the charges. We remark that these systems can also model the population dynamics of two segregated species under competition.

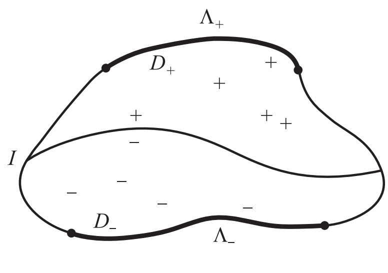

Fig. 1 I = Interface, $\Lambda_\pm$ = harvest sites

To connect the microscopic mechanism and the macroscopic evolution of the systems, we derive the *hydrodynamic limit* and the *fluctuation limit* for these models, in [11,12] and [13,14], respectively. Proving these two types of limits represents establishing the *functional law of large numbers* and the *functional central limit theorem*, respectively, for the *time trajectory* of the spatial densities of the particles in the systems. More precisely, we investigate the asymptotic behavior (when $N \to \infty$) of the empirical measure of positive and negative charges:

$$\mathfrak{X}_t^{N,+}(\mathrm{d}x) := \frac{1}{N}\sum_{\alpha:\,\zeta_\alpha>t} \mathbf{1}_{X_\alpha^+(t)}(\mathrm{d}x), \quad \mathfrak{X}_t^{N,-}(\mathrm{d}y) := \frac{1}{N}\sum_{\beta:\,\zeta_\beta>t} \mathbf{1}_{X_\beta^-(t)}(\mathrm{d}y).$$

1) In [11], they are actually modeled by N biased random walks on lattices inside D_+ and D_- that serve as discrete approximation of reflected Brownian motions with drifts.

Here, $\mathbf{1}_y(\mathrm{d}x)$ stands for the Dirac measure concentrated at the point y and ζ_α for the life time of particle indexed by α. So $\zeta_\alpha > t$ (resp. $\zeta_\beta > t$) denotes the condition that the particle X_α^+ (resp. X_β^-) is alive at time t. We show that, under a suitable scaling (to be explained in detail in each model in the sequel) and an appropriate condition on the initial configuration, the pair of random measures $(\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-})$ converge in distribution to a deterministic pair of measures which are absolutely continuous with respect to the Lebesgue measure. Furthermore, the densities satisfy a system of partial differential equations (PDEs) that has non-linear boundary interaction terms at the interface. See Theorems 2.1 and 2.4 below for precise statements of the hydrodynamic results. We further study the fluctuation of the empirical measure around the hydrodynamic limit and establish that the fluctuation limit is a continuous Gaussian Markov process that solves a stochastic partial differential equation (SPDE). The latter is a nonlinear version of the Langevin equation. See Theorem 3.2 below for a precise statement of the fluctuation result.

The only geometric restrictions that we impose are that D_+ and D_- be adjacent bounded Lipschitz domains in $\mathbb{R}^d$, that the interface I be rectifiable (which can be possibly disconnected), and that the harvest sites $\Lambda_\pm$ be regular with respect to the underlying reflected diffusion of each particle (see [12]). We refer the readers to [11–14] for the most rigorous formulation and generalizations of the results.

Interacting particle systems is an exciting and active research area in mathematics. We refer the reader to standard references such as [9,30,36] and the references therein for its history and a wealth of results. We will provide more specific literature review in each of the two main sections below and point out how our work fit into it. The two main sections, Sections 2 and 3, are devoted to hydrodynamic limits and to fluctuation limits, respectively.

2 Hydrodynamic limits

The study of hydrodynamic limits dates back to the work of J. Clerk Maxell and L. Boltzmann, founders of the kinetic theory of gases. Later, D. Hilbert formulated the question of hydrodynamic limits as a mathematical problem, and presented it as an example in his sixth problem for the mathematical treatment of the axioms of physics. From the probabilistic or statistical point of view, proving hydrodynamic limits corresponds to establishing functional law of large numbers for the *evolution in time* of the empirical measure of some attributes (such as position, genetic type, and spin type) of the individuals in the systems. It reveals fascinating connections between the microscopic stochastic systems and deterministic partial differential equations that describe the macroscopic pictures. It also provides approximations via stochastic models to some partial differential equations that are hard or impossible to solve directly.

Since Hilbert formulated his sixth problem in year 1900, there have been many different lines of research on stochastic particle systems. Various models were studied and different techniques were developed to establish hydrodynamic limits. Here, we cite the *entropy method* [27] and the *relative entropy method* [40] as general methods, and mention the non-gradient techniques and attractiveness techniques [30], among many other important techniques. Unfortunately, these methods do not seem to work directly for our model due to the singular interactions near the interface. Among all the most extensively studied models, *reaction-diffusion systems* in [3,16,18,32] and *Fleming-Viot type systems* in [7,8,26] are relatively close to ours. Nevertheless, our models distinguish themselves due to the coupling effect near the boundary which leads to nonlinear heat equations. So new approaches and techniques are called for to analyze these systems. Our models are non-equilibrium systems for which boundary effects are visible in the macroscopic scale.

2.1 Hydrodynamic limit for interacting random walk model

In this section, we summarize and discuss the main results from [11], which are the hydrodynamic result and the propagation of chaos result for a random walk version of the interacting diffusion models described in the introduction. To focus on the interaction near the interface I, we assume that there is no harvest site in this model. We first describe the stochastic model, which we call 'Annihilating Random Walk Model'.

2.1.1 Annihilating random walk model

We approximate $D_\pm$ by a square lattice $D^\varepsilon_\pm = D_\pm \cap \varepsilon\mathbb{Z}^d$ of edge length ε (as in Fig. 2), where ε is an N-dependent parameter such that $N\varepsilon^d = 1$ and N is the initial number of particles in each of D^ε_+ and D^ε_-. Each particle in $D^\varepsilon_\pm$ performs biased random walk speeded up by a factor d/ε^2 (diffusive scaling). More precisely, the one-step transition probabilities is chosen in such a way that the motion approximating a reflected Brownian motion with gradient drift $\frac{1}{2}\nabla(\log\rho_\pm)$, where $\rho_\pm$ is a positive continuously differentiable function on $\overline{D}_\pm$. Note that $\rho_\pm = 1$ corresponds to the particular case when there is no drift. When a pair of particles of different types are close to each other (which must happen near I, such as when they are at (z_+, z_-) in Fig. 2), they annihilate each other at a rate of order λ/ε, where $\lambda > 0$ is a fixed parameter. Here, we say that an event happens at rate r if the time of occurrence is an exponential random variable of parameter r (in particular, the probability of occurrence in a short amount of time t is $rt + o(t)$, where $o(t)/t \to 0$ as $t \to 0$). Overall and intuitively speaking, according to a random time clock which runs with a speed

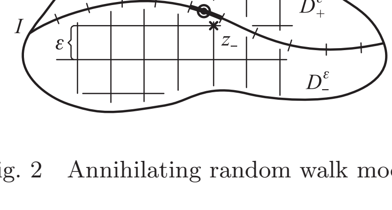

Fig. 2 Annihilating random walk model

proportional to the number of pairs (one particle of each type) of distance ε, we annihilate a pair (picked uniformly among those pairs of distance less than ε) with an exponential rate of parameter λ/ε.

The rigorous formulation of the particle system is captured by its generator $\mathfrak{L}^\varepsilon$ stated in [11].

2.1.2 Hydrodynamic limit and propagation of chaos

It is clear that $(\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-})_{t\geqslant 0}$ is a continuous time Markov process with state space

$$\mathfrak{M} := M_{\geqslant 0}(\overline{D}_+) \times M_{\geqslant 0}(\overline{D}_-),$$

where $M_{\geqslant 0}(E)$ denotes the space of non-negative Borel sub-probability measures on a topological space E. In what follows, '$\xrightarrow{d}$' stands for convergence in law as $N \to \infty$, and $D([0,T],E)$ is the Skorokhod space of *càdlàg* paths in E (see [2]). The following result on hydrodynamic limit was obtained in [11].

Theorem 2.1 *Under appropriate and mild conditions on the initial configuration* $(\mathfrak{X}_0^{N,+}, \mathfrak{X}_0^{N,-})$, *we have, for any* $T > 0$,

$$(\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-}) \xrightarrow{d} (u_+(t,x)\rho_+(x)\mathrm{d}x, u_-(t,y)\rho_-(y)\mathrm{d}y) \quad \textit{in } D([0,T],\mathfrak{M}),$$

where (u_+, u_-) *is the solution of the following coupled nonlinear heat equations*:

$$\begin{cases} \dfrac{\partial u_+}{\partial t} = \dfrac{1}{2}\Delta u_+ + \dfrac{1}{2}\nabla(\log\rho_+)\cdot\nabla u_+ & \textit{in } (0,\infty)\times D_+, \\ \dfrac{\partial u_+}{\partial \boldsymbol{n}_+} = \dfrac{\lambda}{\rho_+}u_+u_-\mathbf{1}_I & \textit{on } (0,\infty)\times \partial D_+, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \dfrac{\partial u_-}{\partial t} = \dfrac{1}{2}\Delta u_- + \dfrac{1}{2}\nabla(\log\rho_-)\cdot\nabla u_- & \textit{in } (0,\infty)\times D_-, \\ \dfrac{\partial u_-}{\partial \boldsymbol{n}_-} = \dfrac{\lambda}{\rho_-}u_+u_-\mathbf{1}_I & \textit{on } (0,\infty)\times \partial D_-, \end{cases} \tag{2.2}$$

where $\boldsymbol{n}_\pm$ *is the inward unit normal vector field of* $D_\pm$ *and* $\mathbf{1}_I$ *is the indicator function on* I.

The above result tells us that for any fixed time $t > 0$, the probability distribution of a randomly picked particle in $D_\pm^\varepsilon$ at time t is close to $c_\pm(t)u_\pm(t,x)$ when N is large, where

$$c_\pm(t) = \bigg(\int_{D_\pm} u_\pm(t)\bigg)^{-1}$$

is a normalizing constant. In fact, the above convergence holds at the level of the path space. That is, the full trajectory (and hence, the joint law at different times) of the particle profile converges to the deterministic scaling limit described by (2.1) and (2.2).

What about the joint distribution of more than one particle? The answer is provided by our second main result in [11], which says that *propagation of chaos* holds true for our system: when the number of particles tends to infinity, their positions appear to be independent of each other, and hence, the joint distribution can be factorized.

Theorem 2.2 *Suppose that n and m unlabeled particles in D_+^ε and D_-^ε, respectively, are chosen uniformly among the living particles at time t. Then, as $N \to \infty$, the probability joint distribution for their positions converges to*

$$c_{(n,m)}(t) \prod_{i=1}^{n} u_+(t, r_i) \prod_{j=1}^{m} u_-(t, s_j)$$

uniformly on compact sets, where $c_{(n,m)}(t)$ is a normalizing constant.

We point out here that *scaling* is an important and ubiquitous concept in stochastic interacting particle systems. The heuristic reason for the choice of the scaling λ/ε for the per-pair annihilation rate is to guarantee that, in the limit $N \to \infty$, a non-trivial proportion of particles is killed during the time interval $[0, t]$. Since diffusive particles typically spread out in space, the number of pairs near the interface is of order $N^2 \varepsilon^{d+1}$ (because there are $N\varepsilon$ particles in D_+ near I, and each of them is near to $N\varepsilon^d$ particles in D_-). Hence, the expected number of pairs killed within t units of time is about

$$N^2 \varepsilon^{d+1} \frac{\lambda t}{\varepsilon} = \lambda N t$$

when $t > 0$ is small (here, we used the scaling $N\varepsilon^d = 1$). This implies that a non-trivial proportion of particles are annihilated in any open time interval, and hence, accounts for the boundary term in the hydrodynamic limit.

2.1.3 Key ideas of proofs

We need first to make sense of the notion 'solution' of the coupled PDEs. This is taken care of by [11, Proposition 2.19], which was proved by using a probabilistic representation and a Banach fixed point argument. Let $X^\pm$ be a reflected Brownian motion with gradient drift $\frac{1}{2} \nabla(\log \rho_\pm)$ in $\overline{D}^\pm$. Denote by $\{P_t^\pm;\, t \geqslant 0\}$ and $p^\pm(t, x, y)$ the transition semigroup and the transition density of $X^\pm$ with respect to the symmetrizing measure $\rho_\pm(x)\mathrm{d}x$, respectively. That is,

$$P_t^\pm f(x) = \mathbb{E}^x[f(X_t^\pm)] = \int_{D_\pm} f(y) p^\pm(t, x, y) \rho_\pm(y) \mathrm{d}y.$$

Proposition 2.3 [11, Proposition 2.19] *Let $T > 0$. Suppose*

$$u_+(0) = f \in C(\overline{D}_+), \quad u_-(0) = g \in C(\overline{D}_-).$$

There is a unique element $(u_+, u_-) \in C([0, T] \times \overline{D}_+) \times C([0, T] \times \overline{D}_-)$ that *satisfies the coupled integral equation*

$$\begin{cases} u_+(t,x) = P_t^+ f(x) - \dfrac{\lambda}{2}\displaystyle\int_0^t \int_I p^+(t-r,x,z)[u_+(r,z)u_-(r,z)]\rho_+(y)\mathrm{d}\sigma(z)\mathrm{d}r, \\ u_-(t,y) = P_t^- g(y) - \dfrac{\lambda}{2}\displaystyle\int_0^t \int_I p^-(t-r,y,z)[u_+(r,z)u_-(r,z)]\rho_-(y)\mathrm{d}\sigma(z)\mathrm{d}r. \end{cases} \tag{2.3}$$

Moreover, (u_+, u_-) *admits the following probabilistic representation*:

$$\begin{cases} u_+(t,x) = \mathbb{E}^x\big[f(X_t^+)\mathrm{e}^{-\lambda\int_0^t u_-(t-s,X_s^+)\mathrm{d}L_s^+}\big] \\ u_-(t,y) = \mathbb{E}^y\big[g(X_t^-)\mathrm{e}^{-\lambda\int_0^t u_+(t-s,X_s^-)\mathrm{d}L_s^-}\big], \end{cases} \tag{2.4}$$

where $L^\pm$ *is the boundary local time of* $X^\pm$ *on the interface* I.

The functions (u_+, u_-) satisfying equation (2.3) are called a weak solution of (2.1) and (2.2), as it can be shown that they are weakly differentiable and solve the equations in the distributional sense.

To establish hydrodynamic limit result (Theorem 2.1), we employ the classical tightness plus finite dimensional distribution approach. In fact, propagation of chaos (Theorem 2.2) is a crucial step in identifying the limit in Theorem 2.1. A key step in our proof of Theorem 2.2 is [11, Section 3.5], which establishes uniqueness of solution for the infinite system of equations satisfied by the correlation functions of the particles in the limit $N \to \infty$. Such infinite system of equations is sometimes called *BBGKY-hierarchy*[1)] in statistical physics. Our limiting BBGKY hierarchy involves boundary terms on the interface, which is new to the literature. The limiting BBGKY hierarchy can be constructed as follows: since u_+, u_-, and u_+u_- are the candidates of the $(1,0)$, $(0,1)$, and $(1,1)$ correlation functions for the limiting empirical distribution, the first equation in (2.4) can be viewed as an integral equation relating the $(1,0)$ and the $(1,1)$ correlation functions. Applying (2.4) again, we obtain an integral equation which relates u_+u_- to $u_+^2u_-$ and $u_+u_-^2$. Keep iterating using (2.4), we obtain the limiting BBGKY hierarchy. We show that the correlation functions of every subsequential limit of $(\mathfrak{X}^+, \mathfrak{X}^-)$ satisfies the same limiting BBGKY hierarchy. The crux of the difficulty in [11] is to establish the existence and uniqueness of solutions to this limiting BBGKY hierarchy. With such uniqueness, we conclude that the propagation of chaos holds for our stochastic system, from which we can deduce the hydrodynamic limit result. Our proof of uniqueness involves a representation and manipulations of the hierarchy in terms of trees. This technique is related to but different from that in [22] which used Feynman diagrams. The techniques developed in limiting [11] is potentially useful in the study of other stochastic models involving coupled differential equations.

1) BBGKY stands for N. N. Bogoliubov, Max Born, H. S. Green, J. G. Kirkwood, and J. Yvon, who derived this type of hierarchy of equations in the 1930s and 1940s in a series of papers.

In addition, two new tools for discrete approximation of random walks in domains are developed in this article. Namely, the local central limit theorem (local CLT) for reflected random walk on bounded Lipschitz domains (see [11, Section 3.5]) and the 'discrete surface measure' ([11, Lemma 2.4]). We believe that these tools are potentially useful in many discrete schemes which involve reflected Brownian motions. Weak convergence of simple random walk on $D^{\varepsilon}_{\pm}$ to reflected Brownian motion (RBM) has been established for general bounded domains in [5,6]. However, we need a local convergence result which guarantees that the convergence rate is uniform up to the boundary. For this, we establish the local CLT. We further generalize the weak convergence result and the local limit theorem to deal with RBMs with gradient drift. The proof of the local CLT is based on a 'discrete relative isoperimetric inequality' which leads to the Poincaré inequality and the Nash inequality. The crucial point is that these two inequalities are uniform in ε (scaling of lattice size) and is invariant under the dilation of the domain $D \mapsto aD$.

2.2 Hydrodynamic limit for interacting diffusion model

While the interacting random walk model in Section 2.1 is more amenable to computer simulation, it is subject to technical restrictions associated with the discrete approximations of both the diffusions performed by the particles and the underlying domains $D_{\pm}$. Furthermore, that model does not consider harvest of charges. To address these issues, a new continuous state stochastic model was introduced and investigated in [12]. This new model, which we refer to as the 'Annihilating Diffusion Model' in this article, is different from that of [11] in three ways:

(i) the particles perform *reflected diffusions* on continuous state spaces rather than random walks over discrete state spaces,

(ii) particles are absorbed (harvested) at some regions (harvest sites) away from the interface I, and

(iii) the annihilation mechanism near I is different.

The annihilating diffusion model allows more flexibility in modeling the underlying spatial motions performed by the particles and in the study of their various properties. In particular, it is more convenient to work with when we study the fluctuation limit, which is the subject of [14].

2.2.1 Annihilating diffusion model

Reflected diffusions are natural mathematical objects to study. After all, the random motions of the pollen grains observed by Robert Brown in year 1827 were reflected on the boundary of the water tank. Suppose that $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain, $\rho \in W^{1,2}(D) \cap C(\overline{D})$ is a strictly positive function, and $\mathbf{a} = (a^{ij})$ is a symmetric, bounded, uniformly elliptic $d \times d$ matrix-valued function with $a^{ij} \in W^{1,2}(D)$ for each i, j. It is well known (see [1,10,15]) that the bilinear form $(\mathscr{E}, W^{1,2}(D))$ defined by

$$\mathscr{E}(f,g) := \frac{1}{2}\int_D \mathbf{a}\nabla f(x) \cdot \nabla g(x)\rho(x)\mathrm{d}x$$

$$:= \frac{1}{2} \int_D \sum_{i,j=1}^{d} a^{ij}(x) \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_i}(x) \rho(x) \mathrm{d}x \tag{2.5}$$

is a regular Dirichlet form in $L^2(D, \rho(x)\mathrm{d}x)$, and hence, has an associated Hunt process X (unique in distribution). Furthermore, X is a continuous strong Markov process with symmetrizing measure ρ and has infinitesimal generator

$$\mathscr{A} := \frac{1}{2\rho} \nabla \cdot (\rho \mathbf{a} \nabla).$$

Intuitively, X behaves like a diffusion process associated to the second order elliptic differential operator $\mathscr{A}$ in the interior of D, and is instantaneously reflected at the boundary in the inward conormal direction $\boldsymbol{\nu} := \mathbf{a}\boldsymbol{n}$, where $\boldsymbol{n}$ is the inward unit normal. See Chen [10] for the Skorokhod representation for X, which tells us precise pathwise properties of X. We call X an $(\mathbf{a}, \rho)$-reflected diffusion or an $(\mathscr{A}, \rho)$-reflected diffusion. A special but very important case is when $\mathbf{a}$ is the identity matrix and $\rho = 1$, in which X is called a *reflected Brownian motion* (RBM).

Now, we describe our annihilating diffusion model (Fig. 3). Suppose that we are also given a harvest region $\Lambda_\pm \subset \partial D_\pm \setminus I$ that absorbs (harvests) $\pm$ charges. Let N be the initial number of particles in each of D_+ and D_-. Each particle in $D_\pm$ moves as an $(\mathbf{a}_\pm, \rho_\pm)$-reflected diffusion in $D_\pm$ and is absorbed upon hitting $\Lambda_\pm$. Moreover, when two particles of different types are of a small distance δ_N, they disappear with intensity $\lambda/(N\delta_N^{d+1})$. The reason for the choice of the above scaling is the same as that for the annihilating random walk model: to guarantee that a nontrivial proportion of particles is annihilated in any open time interval.

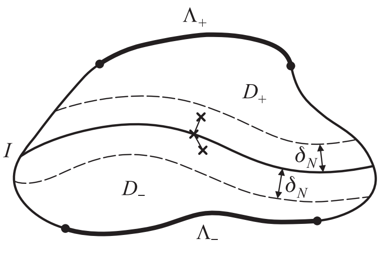

Fig. 3 Annihilating diffusion model

2.2.2 Hydrodynamic limit

Based on a strengthened result in geometric measure theory obtained in [12, Lemma 7.2], we developed a new and direct approach to prove the following hydrodynamic limit in [12].

Theorem 2.4 *Suppose that* $\liminf_{N\to\infty} N\delta_N^d \in (0, \infty]$, $\delta_N \to 0$, *and* $(\mathfrak{X}_0^{N,+}, \mathfrak{X}_0^{N,-})$ *converges in distribution. Then for any* $T > 0$, *we have*

$$(\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-}) \xrightarrow{d} (u_+(t,x)\rho_+(x)\mathrm{d}x, u_-(t,y)\rho_-(y)\mathrm{d}y) \quad \textit{in } D([0,T], \mathfrak{M}),$$

where (u_+, u_-) *is the solution of the coupled heat equations*

$$\frac{\partial u_\pm}{\partial t} = \mathscr{A}^\pm u_\pm$$

on $D_\pm$, *with Dirichlet boundary condition* $u_\pm = 0$ *on* $\Lambda_\pm$ *and with the following nonlinear coupled boundary condition*:

$$\frac{\partial u_\pm}{\partial \boldsymbol{\nu}_\pm} = \frac{\lambda}{\rho_\pm} u_+ u_- \mathbf{1}_I \quad \text{on } \partial D_\pm \setminus \Lambda_\pm,$$

where $\boldsymbol{\nu}_\pm := \boldsymbol{a}_\pm \boldsymbol{n}_\pm$ *is the inward conormal vector field of* $\partial D_\pm$.

As an immediate application of Theorem 2.4, we get an analytic formula for the asymptotic mass of positive charges harvested during the time interval $[0, T]$, which is

$$1 - \int_{D_+} u_+(T, x)\rho_+(x)\mathrm{d}x - \lambda \int_0^T \int_I u_+(s, z)u_-(s, z)\mathrm{d}\sigma(z)\mathrm{d}s.$$

The condition $\liminf_{N\to\infty} N\delta_N^d \in (0, \infty]$ is an upper bound for the rate at which the annihilations distance δ_N tends to 0. Such kind of condition is necessary by the following reason: the dimension of I is $d+1$, lower than that of $D_+ \times D_-$. So we can choose δ_N small enough so that particles of different types cannot 'see' each other in the limit $N \to \infty$, resulting a decoupled linear system of PDEs with Dirichlet boundary condition on $\Lambda_\pm$ and Neumann boundary condition on $\partial D_\pm \setminus \Lambda_\pm$. See [12, Example 5.3] for a rigorous statement and proof. Note that Theorem 2.4 is valid for the case

$$\liminf_{N\to\infty} N\delta_N^d = \infty$$

(a high density assumption). This is important because in the fluctuation result in Theorem 3.2 below, we require

$$\liminf_{N\to\infty} N\delta_N^{2d} > 0.$$

2.2.3 Key ideas of proofs

The existence and uniqueness of solution for the coupled heat equation appeared in Theorem 2.4 follows from the same probabilistic representation and fixed point argument as in the case for the annihilating random walk model. In fact, we have an analogous proposition, with each reflected diffusion $X^\pm$ being replaced by reflected diffusion *killed upon hitting the harvest site* $\Lambda_\pm$.

Based on a strengthened version of [24, Theorem 3.2.39] from geometric measure theory, we develop a direct approach to establish the hydrodynamic limit. This approach avoids going through the delicate BBGKY hierarchy as described in Section 2.1.3. More precisely, [24, Theorem 3.2.39] asserts that

$$\lim_{\delta\to 0} \frac{\mathscr{H}^{2d}(I^\delta)}{c_{d+1}\delta^{d+1}} = \mathscr{H}^{d-1}(I), \tag{2.6}$$

where

$$I^\delta := \{(x,y) \in D_+ \times D_- \colon |x-z|^2 + |y-z|^2 < \delta^2 \text{ for some } z \in I\},$$

c_{d+1} is the volume of the unit ball in $\mathbb{R}^{d+1}$, and $\mathscr{H}^m$ is the m-dimensional Hausdorff measure. In [12, Lemma 7.2], this has been strengthened to

$$\lim_{\delta\to 0} \int \ell_\delta(x,y) f(x,y) \mathrm{d}x\mathrm{d}y = \int_I f(z,z) \mathrm{d}\mathscr{H}^{d-1}(z) \tag{2.7}$$

uniformly in f from any equi-continuous family in $C(\overline{D}_+ \times \overline{D}_-)$, where

$$\ell_\delta(x,y) := \frac{1}{c_{d+1}\delta^{d+1}} \mathbf{1}_{I^\delta}(x,y).$$

This uniform convergence, together with the condition $\liminf_{N\to\infty} N\delta_N^d \in (0,\infty]$, leads us to the key observation that

$$\begin{aligned} &\lim_{\delta\to 0}\lim_{N\to\infty} \mathbb{E}\int_0^T \langle \ell_\delta(x,y), \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-}\rangle \mathrm{d}s \\ &\quad= \lim_{N\to\infty}\lim_{\delta\to 0} \mathbb{E}\int_0^T \langle \ell_\delta(x,y), \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-}\rangle \mathrm{d}s, \end{aligned} \tag{2.8}$$

where

$$\langle \ell_\delta(x,y), \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-}\rangle := \int_{\overline{D}_+}\int_{\overline{D}_-} \ell_\delta(x,y) \mathfrak{X}_s^{N,+}(\mathrm{d}x) \mathfrak{X}_s^{N,-}(\mathrm{d}y).$$

This interchange of limit in turn allows us to characterize the mean of any subsequential limit of $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ by comparing the integral equations satisfied by the hydrodynamic limit with its stochastic counterpart.

To see why this is true, we look at the case when $\Lambda_\pm$ are empty and the reflected diffusions are all RBMs for simplicity. Let $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-})$ be an arbitrary subsequential limit of $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$. We argue that for any $\phi \in C(\overline{D}_+)$, we have

$$\mathbb{E}[\langle \phi, \mathfrak{X}_t^{\infty,+}\rangle] = \langle \phi, u_+(t)\rangle. \tag{2.9}$$

Taking $L^2(D_+)$ inner product with ϕ on both sides of the integral equation satisfied by u_+ yields

$$\langle \phi, u_+(t)\rangle - \langle \phi, P_t^+ f(x)\rangle = -\frac{\lambda}{2}\int_0^t \int_I P_{t-r}^+\phi(z) u_+(r,z) u_-(r,z) \mathrm{d}\sigma(z)\mathrm{d}r. \tag{2.10}$$

On the other hand, by Dynkin's formula,

$$\mathbb{E}[\langle \phi, \mathfrak{X}_t^{N,+}\rangle] - \mathbb{E}[\langle P_t^+\phi, \mathfrak{X}_0^{N,+}\rangle] = -\frac{\lambda}{2}\int_0^t \mathbb{E}[\langle \ell_{\delta_N} P_{t-r}^+\phi, \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-}\rangle]\mathrm{d}r. \tag{2.11}$$

The challenge is to compare the right-hand sides of the above two equations. However, (2.8) guarantees that we can pass the limit through the integral (along some subsequence of $N \to \infty$) to deduce from (2.11) that

$$\mathbb{E}^\infty[\langle \phi, \mathfrak{X}_t^{\infty,+} \rangle] - \langle P_t^+ \phi, \mathfrak{X}_0^{\infty,+} \rangle = -\frac{\lambda}{2} \int_0^t \lim_{\varepsilon \to 0} \mathbb{E}^\infty [\langle \ell_\varepsilon P_{t-r}^+ \phi_+, \mathfrak{X}_r^{\infty,+} \otimes \mathfrak{X}_r^{\infty,-} \rangle] \mathrm{d}r. \tag{2.12}$$

Now, (2.9) follows from a standard Gronwall argument by comparing (2.10) with (2.12). Using a similar argument, we can identify the second moment of any subsequential limit, and hence, characterize any subsequential limit of $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$. Together with the tightness result, we obtain Theorem 2.4.

We remark that the condition $\liminf_{N\to\infty} N\delta_N^d \in (0,\infty]$ is not required to establish tightness of $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$, and it is not clear what are other possibilities for subsequential limits in the absence of this condition. Finally, we point out that

$$\frac{1}{c_{d+1}\delta^{d+1}} \int_0^t \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-}(I^\delta) \mathrm{d}s$$

quantifies the amount of interaction between the two types of particles, and is related (but different from) the *collision local time* introduced in [23].

3 Fluctuation limits

We next study fluctuation limit for the annihilating diffusion model[1)]. This provides a measure of extent by which the empirical measure deviates from the hydrodynamic limit and a rate of convergence for the hydrodynamic result in Theorem 2.4. Moreover, It enables us to do effective simulations in engineering situations where explicit solutions are not feasible (see (3.4) below). While the hydrodynamic limits of our models are described by deterministic PDEs, their fluctuation limits are stochastic partial differential equations (SPDEs).

One of the earliest rigorous results about fluctuation limit was proven by Itô [28,29], who considered a system of independent and identically distributed (i.i.d.) Brownian motions in $\mathbb{R}^d$ and showed that the limit is a $\mathscr{S}'$-valued Gaussian process solving a generalized Langevin equation, where $\mathscr{S}'$ is the Schwartz space of tempered distributions. Gaussian fluctuations for interacting diffusions in $\mathbb{R}^d$ are also proved. See, for example, [38]. For particles living in domains, Sznitman [39] studied the fluctuations of a conservative system of reflected diffusions. Fluctuations of the reaction-diffusion systems on the cube $[0,1]^d$ with polynomial reaction terms were studied in [3,19,32,33]. These

1) We did not try to prove a fluctuation limit result for the annihilating random walk model because we anticipate that the limit is the same as that of the annihilating diffusion model. A proof similar to that of the annihilating diffusion model can most likely be applied for the annihilating random walk mode, but the class of suitable correlation functions should be a modified version of the *v-functions* used in [3] (in which fluctuation of a lattice model was studied).

fluctuation results are valid only for dimension $d \leqslant 3$. In [13], we extend the functional analytic framework of [32] to deal with more general domains and reflected diffusions killed by a time-dependent potential. Our fluctuation results hold for all dimensions $d \geqslant 1$ and the covariance structures of our fluctuation limits have boundary integral terms that capture the boundary interactions in the fluctuation level.

To avoid unnecessary complications, from now on, we assume that in the annihilating diffusion model, the harvest sites are empty sets and the underlying particles perform RBMs.

3.1 Fluctuations for Robin boundary model

In this subsection, we summarize and discuss the main results from [13]. As a preliminary step to understand the fluctuation for the annihilating diffusion model, we consider a simpler model which consists of a single type of particles moving as independent RBMs in a bounded domain $D \subset \mathbb{R}^d$ which has a chance of being killed by a time dependent potential function $q(t,x)$ when it is near the boundary ∂D. More precisely, when there are N particles initially, the killing intensity for each particle is

$$q_N(t,x) := \frac{1}{\delta_N} \mathbf{1}_{D^{\delta_N}}(x) q(t,x),$$

where $q(t,x)$ is a given time dependent non-negative continuous function and

$$D^\delta := \{x \in D \colon \mathrm{dist}(x, \partial D) < \delta\}.$$

We coin this model the name 'Robin boundary model' since it can be easily shown that its hydrodynamic limit is the solution to the heat equation with Robin boundary condition

$$\frac{\partial u}{\partial \boldsymbol{n}} = qu \quad \text{on } \partial D,$$

where $\boldsymbol{n}$ is the inward unit normal vector field of D.

Despite its simplicity, the Robin boundary model poses new difficulties and leads to new results when we study its fluctuation. The fluctuation of the empirical measure $\mathfrak{X}^N$ at time t is defined by

$$\mathscr{Y}_t^N(\phi) := \sqrt{N}\,(\langle \mathfrak{X}_t^N, \phi\rangle - \mathbb{E}\langle \mathfrak{X}_t^N, \phi\rangle),$$

where

$$\langle \mathfrak{X}_t^N, \phi\rangle := \frac{1}{N} \sum_{\alpha\colon \zeta_\alpha > t} \phi(X_\alpha(t))$$

is the integral of an observable (or test function) ϕ with respect to the measure $\mathfrak{X}_t^N$. Intuitively, when $\phi = \mathbf{1}_K$ is an indicator function of a subset $K \subset \overline{D}$, $\langle \mathfrak{X}_t^N, \phi\rangle$ is the mass of particles in K (which is the number of particles in K divided by N). In this case, $\mathscr{Y}_t^N(\phi)$ is the fluctuation of the mass of particles in K at time t.

Even in this simple setting, it is nontrivial to obtain satisfactory answers to the following natural questions.

(1) What is the state space for $\mathscr{Y}_t^N$? This space should posses a topology which allows us to make sense of convergence of $\mathscr{Y}^N$, if it does converge. Observe that although $\mathscr{Y}^N$ acts on $L^2(D)$ linearly, it is not a bounded operator in general.

(2) Does $\mathscr{Y}^N$ converges? If so, what can we say about the limit?

We answer these two questions fully in [13] for symmetric diffusions with Robin boundary condition. It turns out that the state space of the process $(\mathscr{Y}_t^N)_{t\geqslant 0}$ is a Hilbert distribution space $\mathscr{H}$ which strictly contains $L^2(D)$.

Our fluctuation result in [13] for the Robin boundary model contains the convergence result and the properties of the limit, which is shown to be decomposable into an independent sum of a 'transportation part' and a 'white noise part' (see (3.1) below). The 'transportation part' is governed by the evolution operator $\{Q_{s,t}\}_{s\leqslant t}$ generated on $C(\overline{D})$ by the backward PDE

$$\frac{\partial v}{\partial s} = -\frac{1}{2}\Delta v \quad \text{in } (0,t)\times D$$

with Robin boundary condition

$$\frac{\partial v}{\partial \boldsymbol{n}} = qv \quad \text{on } (0,t)\times \partial D.$$

We denote by $\mathbf{U}_{(t,s)}$ the operator on $\mathscr{H}$ defined by

$$\langle \mathbf{U}_{(t,s)}\mu, \phi\rangle := \langle \mu, Q_{s,t}\phi\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the dual paring extending the inner product on $L^2(D)$.

Theorem 3.1 [13, Theorem 1.5] *Suppose that $T > 0$ and the initial positions of particles are i.i.d. with distribution $u_0(x)\mathrm{d}x$, where $u_0 \in C(\overline{D})$. Then $\mathscr{Y}^N \xrightarrow{d} \mathscr{Y}$ in $D([0,T],\mathscr{H})$, where $\mathscr{Y}$ is the generalized Ornstein-Uhlenbeck process given by*

$$\mathscr{Y}_t = \mathbf{U}_{(t,0)}\mathscr{Y}_0 + \int_0^t \mathbf{U}_{(t,s)}\mathrm{d}M_s. \tag{3.1}$$

Here, M is a continuous square-integrable $\mathscr{H}$-valued Gaussian martingale with independent increments and covariance

$$\langle M(\phi)\rangle_t = \int_0^t \mathscr{E}_r(\phi,\phi)\mathrm{d}r,$$

where $\mathscr{E}_r$ is the bilinear form on $\mathscr{H}$ defined by

$$\mathscr{E}_r(\phi,\psi) := \int_D \nabla\phi(x)\cdot\nabla\psi(x)\, u(r,x)\mathrm{d}x + \int_{\partial D}\phi(x)\psi(x)q(r,x)u(r,x)\mathrm{d}\sigma(x), \tag{3.2}$$

where u is the hydrodynamic limit of the Robin boundary model. $\mathscr{Y}_0$ is a centered Gaussian random variable independent of M. Moreover, $\mathscr{Y}$ is a continuous Gaussian Markov process (*unique in distribution*).

Formally, $\mathscr{Y}$ solves the following stochastic evolution equation (called the *Langevin equation*) in the weak sense:

$$\mathrm{d}Y_t = \mathbf{A}_t Y_t \mathrm{d}t + \mathrm{d}M_t, \quad Y_0 = \mathscr{Y}_0, \tag{3.3}$$

where $\mathbf{A}_t$ is the 'generator' of $\mathbf{U}_{(t,s)}$ in the Hilbert space $\mathscr{H}$.

Theorem 3.1 is called a fluctuation-dissipation theorem in statistical physics, since it quantifies how applied perturbations spread out in space-time. As an application of this theorem, (3.1) implies that for suitable ϕ, we have

$$\mathscr{Y}_t(\phi) = \mathscr{Y}_0(Q_{0,t}\phi) + \int_0^t \sqrt{\mathscr{E}_s(Q_{s,t}\phi)}\,\mathrm{d}B_s. \tag{3.4}$$

where B_t is a standard Brownian motion independent of $\mathscr{Y}_0$. Therefore, we can simulate the evolution of the fluctuations with respect to an observable ϕ by using a *single* Brownian path.

The proof of Theorem 3.1 given in [13] consists of the following 6 steps.

Step 1 $\mathscr{Y}^N$ satisfies the following stochastic integral equation:

$$\mathscr{Y}_t^N = \mathbf{U}_{(t,0)}^N \mathscr{Y}_0^N + \int_0^t \mathbf{U}_{(t,s)}^N \mathrm{d}M_s^N \quad \text{a.s.},$$

where $\mathbf{U}_{(t,s)}^N$ is an evolution system (see [17]) approximating $\mathbf{U}_{(t,s)}$.

Step 2 $M^N \xrightarrow{d} M$ in $D([0,T],\mathscr{H})$.

Step 3 $\mathscr{Y}^N$ is tight in $D([0,T],\mathscr{H})$.

Step 4 $\mathbf{U}_{(t,0)}^N \mathscr{Y}_0^N \xrightarrow{d} \mathbf{U}_{(t,0)}\mathscr{Y}_0$ in $D([0,T],\mathscr{H}_{-\alpha})$.

Step 5

$$\int_0^t \mathbf{U}_{(t,s)}^N \mathrm{d}M_s^N \xrightarrow{d} \int_0^t \mathbf{U}_{(t,s)} \mathrm{d}M_s \quad \text{in } D([0,T],\mathscr{H}_{-\alpha}).$$

Step 6 All the stated properties for the fluctuation limit hold.

Note that

$$t \mapsto \int_0^t \mathbf{U}_{(t,s)} \mathrm{d}M_s$$

is not a martingale, even though

$$\theta \mapsto \int_s^\theta \mathbf{U}_{(t,r)} \mathrm{d}M_r$$

is a martingale for $\theta \in [s,t]$. The standard method based on Kotelenez's submartingale inequality [31] does not seem to work. This is because in our case, $\mathbf{U}_{(t,s)}$ is not exponentially bounded; that is, there is no $\beta > 0$ so that the operator norm (see [31])

$$\|\mathbf{U}_{(t,s)}\| \leqslant \mathrm{e}^{\beta(t-s)}, \quad t \geqslant s.$$

In fact, we suspect that it is not even a bounded operator on $\mathscr{H}_{-\alpha}$ due to the singular interaction near the boundary. Our approach is based on suitably extending the functional analytic framework of [32] and a direct analysis that uses heat kernel estimates and Dirichlet form method.

3.2 Fluctuations for annihilating diffusion model

The techniques developed in the above Robin boundary model allow us to overcome some (but not all) challenges for the fluctuation of the annihilating diffusion model. We need two new ingredients, namely, the *asymptotic expansion of the correlation functions* and the *Boltzmann-Gibbs principle*. More precisely, by considering an approximating BBGKY hierarchy and extending the approach of [19], we show that the correlation functions have the structure

$$F_t^{N,(n,m)}(\boldsymbol{x},\boldsymbol{y}) = \prod_{i=1}^{n} u_+(t,x_i) \prod_{j=1}^{m} u_-(t,y_j) + \frac{B_t^{N,(n,m)}(\boldsymbol{x},\boldsymbol{y})}{N} + \frac{C_N}{N}, \tag{3.5}$$

where (u_+, u_-) is the hydrodynamic limit, $B_t^{N,(n,m)}$ is an explicit function, and C_N is a real number that tends to zero as $N \to \infty$ uniformly for small time t and for $(\boldsymbol{x},\boldsymbol{y}) \in D_+^n \times D_-^m$. This implies propagation of chaos and allows explicit calculations of the covariance of the fluctuation process. The proof of (3.5) is based on a comparison of the BBGKY hierarchy satisfied by the correlation functions with two other approximating hierarchies.

On the other hand, the Boltzmann-Gibbs principle, first formulated mathematically and proven for some zero range processes in equilibrium in [4], says that the fluctuation fields of non-conserved quantities change on a time scale much faster than the conserved ones, hence in a time integral only the component along those fields of conserved quantities survives. Although this principle has been established for a few non-equilibrium situations (see [3] and the references therein), it is not known whether it holds in general. The validity of the principle for our annihilating diffusion model is far from obvious, since there is no conserved quantity.

We define $\mathscr{H}^+$ and $\langle \cdot, \cdot \rangle_\pm$ just as in the Robin boundary model for the domain $D_\pm$. Consider

$$\mathscr{Z}_t^N := \mathscr{Y}_t^{N,+} \oplus \mathscr{Y}_t^{N,-} \in \mathbf{H},$$

where

$$\langle \mu^+ \oplus \mu^-, (\phi_+, \phi_-) \rangle := \langle \mu^+, \phi_+ \rangle_+ + \langle \mu^-, \phi_- \rangle_-, \quad \mathbf{H} := \mathscr{H}^+ \oplus \mathscr{H}^-.$$

The 'transportation part' of the limit is now governed by the evolution semigroup $\{Q_{(s,t)}\}_{s\leqslant t}$ generated on $C(\overline{D}_+)\times C(\overline{D}_-)$ by the coupled backward PDEs

$$\begin{cases} \dfrac{\partial v^{\pm}}{\partial s} = -\dfrac{1}{2}\Delta v^{\pm} & \text{in } (0,t)\times D_{\pm}, \\ \dfrac{\partial v^{\pm}}{\partial \boldsymbol{n}_{\pm}} = (v^+ + v^-)u_{\mp}\mathbf{1}_I & \text{on } (0,t)\times \partial D_{\pm}, \end{cases} \tag{3.6}$$

which can be viewed as a linearization of the hydrodynamic limit. As before, we define

$$\langle \mathbf{U}_{(t,s)}\mu, (\phi_+,\phi_-)\rangle := \langle \mu, Q_{s,t}(\phi_+,\phi_-)\rangle$$

and state our fluctuation result from [14].

Theorem 3.2 *Suppose that $\delta_N \to 0$ with $\liminf_{N\to\infty} N\delta_N^{2d} > 0$, and the initial position of particles in each of $\overline{D}_{\pm}$ are i.i.d. with distribution $u_0^{\pm} \in C(\overline{D}_{\pm})$. Then $\mathscr{Z}^N \xrightarrow{d} \mathscr{Z}$ in $D([0,T_0],\mathbf{H})$, where $\mathscr{Z}$ is the generalized Ornstein-Uhlenbeck process*

$$\mathscr{Z}_t = \mathbf{U}_{(t,0)}\mathscr{Z}_0 + \int_0^t \mathbf{U}_{(t,s)}\mathrm{d}M_s. \tag{3.7}$$

Here,

$$T_0 := (\|u_0^+\| \vee \|u_0^-\|)^{-2}C$$

with $\|\cdot\|$ being the sup-norm and $C>0$ a constant that depends only on D, and M is a (unique in law) continuous square-integrable $\mathbf{H}$-valued Gaussian martingale with independent increments and covariance functional characterized by

$$\begin{aligned}\langle M(\phi_+,\phi_-)\rangle_t = \int_0^t \bigg(&\int_{D_+} |\nabla\phi_+|^2 u_+(s)\mathrm{d}x + \int_{D_-} |\nabla\phi_-|^2 u_-(s)\mathrm{d}y \\ &+ \int_I (\phi_+ + \phi_-)^2 u_+(s)u_-(s)\mathrm{d}\sigma\bigg)\mathrm{d}s.\end{aligned}$$

In the above, (u_+,u_-) is the hydrodynamic limit of the annihilating diffusion system given in Theorem 2.1 and

$$\mathscr{Z}_0 := \mathscr{Y}_0^+ \oplus \mathscr{Y}_0^-,$$

where $\mathscr{Y}_0^{\pm}$ is a centered Gaussian random variable in $\mathscr{H}^{\pm}$ such that $\{M, \mathscr{Y}_0^+, \mathscr{Y}_0^-\}$ are independent. In particular, $\mathscr{Z}$ is a continuous Gaussian Markov process and is unique in distribution. Moreover, $\mathscr{Z}$ has a version in $C^{\gamma}([0,T_0],\mathbf{H})$ for any $\gamma \in (0,1/2)$.

This result implies the co-existence of two effects at the level of fluctuation: the effect of diffusive transport and that of the interaction between two types of particles. In other words, none of these two effects dominate the other in the limit. The basic outline for the proof of Theorem 3.2 is the same as that of Theorem 3.1 but with an additional step. Roughly speaking, Step 1 is now replaced by

$$\mathscr{Z}_t^N = \mathbf{U}_{(t,0)}^N \mathscr{Z}_0^N + \int_0^t \mathbf{U}_{(t,s)}^N \mathrm{d}M_s^N + E_t^N,$$

where E_t^N is an extra error term. The additional step is to show that this error goes to zero in a suitable sense. This is true due to the choice of $\mathbf{U}_{(t,s)}^N$ which is facilitated by the Boltzmann-Gibbs principle. An intuitive explanation for the validity of the Boltzmann-Gibbs principle for our annihilating diffusion model is as follows: the high density assumption $\liminf_{N\to\infty} N\delta_N^{2d} > 0$ guarantees that the interaction near I changes the occupation number of the particles at a slow rate with respect to diffusion (which conserves the particle number). In other words, the particle number is conserved on the time scale that is relevant for the principle.

Remark 3.3 Note that (3.7) is only established for $t \in [0, T_0]$. We do not know if T_0 can be taken to be an arbitrary positive number. Such phenomenon also appears in [19] and [33]. We need this property to guarantee (3.5). It is interesting to investigate whether the theorem holds only on finite time interval or not.

This article surveys the recent results we have obtained for the annihilating random walk model and annihilating diffusion model. There are many other aspects of these two models that are worth exploring, such as large deviation principle, properties of the Gaussian field M, and the solutions $\mathscr{Y}_t$ and $\mathscr{Z}_t$ of the SPDEs that arose in Theorems 3.1 and 3.2. We plan to investigate these questions in future.

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