PAPER

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To cite this article: T. Bayraktar *et al* 2018 *Sb. Math.* **209** 352

Sbornik: Mathematics **209**:3 352–384 *Matematicheskiĭ Sbornik* **209**:3 67–101
DOI: https://doi.org/10.1070/SM8893

Pluripotential theory and convex bodies

T. Bayraktar, T. Bloom and N. Levenberg

Abstract. A seminal paper by Berman and Boucksom exploited ideas from complex geometry to analyze the asymptotics of spaces of holomorphic sections of tensor powers of certain line bundles L over compact, complex manifolds as the power grows. This yielded results on weighted polynomial spaces in weighted pluripotential theory in $\mathbb{C}^d$. Here, motivated by a recent paper by the first author on random sparse polynomials, we work in the setting of weighted pluripotential theory arising from polynomials associated to a convex body in $(\mathbb{R}^+)^d$. These classes of polynomials need not occur as sections of tensor powers of a line bundle L over a compact, complex manifold. We follow the approach of Berman and Boucksom to obtain analogous results.

Bibliography: 16 titles.

Keywords: convex body, P-extremal function.

§ 1. Introduction

Motivated by probabilistic results in [1] as well as some questions in multivariate approximation theory [9], we study pluripotential-theoretic notions associated to closed subsets $K \subset \mathbb{C}^d$ and weight functions Q on K in the following setting. Given a convex body $P \subset (\mathbb{R}^+)^d$ we define finite-dimensional polynomial spaces

$$\mathrm{Poly}(nP) := \biggl\{ p(z) = \sum_{J \in nP \cap (\mathbb{Z}^+)^d} c_J z^J : c_J \in \mathbb{C} \biggr\}, \qquad n = 1, 2, \dots,$$

associated to P. Here $z^J = z_1^{j_1} \cdots z_d^{j_d}$ for $J = (j_1, \dots, j_d)$. The main goal of this work is to give a self-contained presentation of some of Berman, Boucksom and Nyström's results and techniques in [4] and [5], valid in the setting of holomorphic sections of tensor powers of certain line bundles L over compact, complex manifolds, for the spaces $\mathrm{Poly}(nP)$. A key result in [4] relates the asymptotics of ball volume ratios of spaces of holomorphic sections with an Aubin-Mabuchi type energy of appropriate pluripotential-theoretic extremal functions. Our spaces $\mathrm{Poly}(nP)$ do not generally arise as holomorphic sections of tensor powers of a line bundle. However, many of the techniques in [4] and [5] are available and we are able to modify their approach to prove the analogous key result, Theorem 5.1.

N. Levenberg was supported by the Simons Foundation (grant no. 354549).
AMS 2010 *Mathematics Subject Classification.* Primary 32U15, 32U20, 31C15.

For $K \subset \mathbb{C}^d$ closed, we let $Q\colon K \to \mathbb{R}$ be an admissible weight on K. The weighted P-extremal function we will consider, denoted by $V^*_{P,K,Q}$, turns out to be equal to the upper semicontinuous regularization of

$$\lim_{n\to\infty} \frac{1}{n} \log \sup\Big\{|p_n(z)|\colon p_n \in \operatorname{Poly}(nP),\ \sup_{\zeta\in K} |p_n(\zeta)e^{-nQ(\zeta)}| \leqslant 1\Big\}$$

(see Proposition 2.3). Let d_n be the dimension of $\operatorname{Poly}(nP)$, let

$$\mathscr{B}^\infty(K, nQ) := \Big\{p_n \in \operatorname{Poly}(nP)\colon \sup_{\zeta\in K} |p_n(\zeta)e^{-nQ(\zeta)}| \leqslant 1\Big\}$$

be an L^∞-ball in $\operatorname{Poly}(nP)$ and, if μ is a measure on K, let

$$\mathscr{B}^2(K, \mu, nQ) := \Big\{p_n \in \operatorname{Poly}(nP)\colon \int_K |p_n|^2 e^{-2nQ}\, d\mu \leqslant 1\Big\}$$

be an L^2-ball in $\operatorname{Poly}(nP)$. We let 'vol' denote any (Haar) measure on $\operatorname{Poly}(nP)$. The key result is the following.

Theorem 1.1. *Given Q and Q' admissible weights on K and K',*

$$\lim_{n\to\infty} \frac{-(d+1)\gamma_d}{2nd_n} \log \frac{\operatorname{vol}\big(\mathscr{B}^\infty(K, nQ)\big)}{\operatorname{vol}\big(\mathscr{B}^\infty(K', nQ')\big)} = \mathscr{E}(V^*_{P,K,Q}, V^*_{P,K',Q'}).$$

If μ and μ' are measures on K and K' where μ is a Bernstein-Markov measure for (P, K, Q) and μ' is a Bernstein-Markov measure for (P, K', Q'), then

$$\lim_{n\to\infty} \frac{-(d+1)\gamma_d}{2nd_n} \log \frac{\operatorname{vol}\big(\mathscr{B}^2(K, \mu, nQ)\big)}{\operatorname{vol}\big(\mathscr{B}^2(K', \mu', nQ')\big)} = \mathscr{E}(V^*_{P,K,Q}, V^*_{P,K',Q'}).$$

Here

$$\mathscr{E}(u, v) := \int_{\mathbb{C}^d} (u - v) \sum_{j=0}^{d} (dd^c u)^j \wedge (dd^c v)^{d-j}$$

is the energy of u and v (see § 3), and $\gamma_d = \gamma_d(P)$ is a constant depending on d and P. The notions of admissible weight and Bernstein-Markov measure, as well as the definition of γ_d, will be given in the next section.

We obtain consequences of this result similar to those in [4] and [5]; for example, that asymptotically weighted P-Fekete arrays and weighted P-optimal measures distribute asymptotically like the Monge-Ampère measure $(dd^c V^*_{P,K,Q})^d$ of $V^*_{P,K,Q}$ (Corollaries 6.5 and 6.4) as well as a Rumely-type formula [13] relating an energy associated with $V^*_{P,K,Q}$ to a weighted P-transfinite diameter (Theorem 5.3). A difference from [4] and [5] is that here we deduce the existence of a weighted P-transfinite diameter as a consequence of Theorem 5.1 (see Remark 5.2). To explain this last item, write $\operatorname{Poly}(nP) = \operatorname{span}\{e_1, \ldots, e_{d_n}\}$ where $\{e_j\}_{j=1,\ldots,d_n}$ are standard basis monomials. For an admissible weight function Q on K and $\zeta_1, \ldots, \zeta_{d_n} \in K$, let

$$W(\zeta_1, \ldots, \zeta_{d_n}) := \det[e_i(\zeta_j)]_{i,j=1,\ldots,d_n} e^{-nQ(\zeta_1)} \cdots e^{-nQ(\zeta_{d_n})},$$

and let

$$W_n(K) := \max_{\zeta_1,\dots,\zeta_{d_n}\in K} |W(\zeta_1,\dots,\zeta_{d_n})|.$$

An nth weighted P-Fekete set for K and Q is a set of d_n points in K attaining the maximum. Let $l_n := \sum_{j=1}^{d_n} \deg(e_j)$. We will show that the $\lim_{n\to\infty} W_n(K)^{1/l_n}$ exists — this is the weighted P-transfinite diameter — using Theorem 5.1.

In the next section, we give the definitions of the relevant pluripotential-theoretic notions and the results we need. We define the Lelong classes L_P and L_P^+ associated to a convex body $P \subset (\mathbb{R}^+)^d$. For certain $K \subset \mathbb{C}^d$ and $Q\colon K \to \mathbb{R}$ we define a weighted P-extremal function $V_{P,K,Q}$, weighted P-transfinite diameter and weighted P-optimal measures. Ball volume ratios, as defined and utilized in [4], are discussed in § 2.5. In § 3 we discuss the Aubin-Mabuchi type energy $\mathscr{E}(u,v)$ associated to a pair of functions u and v in L_P^+. The differentiability of the composition of $\mathscr{E}$ with a projection operator, proved in § 4, is a key step in verifying the main result, Theorem 5.1. This latter is proved in § 5, as is the Rumely formula, Theorem 5.3. Sections 4 and 5 follow arguments in [4]. The applications described in the previous paragraph are given in § 6, following [5].

§ 2. Background

2.1. *P*-extremal functions. Results from [1]. Let $\mathbb{R}^+ = [0,\infty)$. Fix a convex body $P \subset (\mathbb{R}^+)^d$; that is, P is compact, convex and $P^o \neq \varnothing$. A standard example occurs when P is a non-degenerate convex polytope, that is, the convex hull of a finite subset of $(\mathbb{Z}^+)^d$ in $(\mathbb{R}^+)^d$ with nonempty interior. Associated with P, following [1], we consider the finite-dimensional polynomial spaces

$$\mathrm{Poly}(nP) := \Big\{ p(z) = \sum_{J\in nP\cap(\mathbb{Z}^+)^d} c_J z^J : c_J \in \mathbb{C} \Big\},$$

for $n = 1,2,\dots,$ where $z^J = z_1^{j_1}\cdots z_d^{j_d}$ for $J = (j_1,\dots,j_d)$. We let d_n be the dimension of $\mathrm{Poly}(nP)$. For $P = \Sigma$ where

$$\Sigma := \Big\{ (x_1,\dots,x_d) \in \mathbb{R}^d \colon 0 \leqslant x_i \leqslant 1,\ \sum_{i=1}^d x_i \leqslant 1 \Big\},$$

we have $\mathrm{Poly}(n\Sigma) = \mathscr{P}_n$, the usual space of holomorphic polynomials in $\mathbb{C}^d$ of degree at most n. Given P, there exists a minimal positive integer $A = A(P) \geqslant 1$ such that $P \subset A\Sigma$. Thus

$$\mathrm{Poly}(nP) \subset \mathscr{P}_{An} \quad \text{for all } n.$$

Associated to P we define the logarithmic indicator function on $\mathbb{C}^d$:

$$H_P(z) := \sup_{\widetilde{J}\in P} \log|z^{\widetilde{J}}| := \sup_{\widetilde{J}\in P} \log[|z_1|^{\widetilde{j}_1}\cdots|z_d|^{\widetilde{j}_d}].$$

Note that $\widetilde{j}_k \in \mathbb{R}^+$ need not be an integer. Throughout this paper, we make the assumption on P that

$$\Sigma \subset kP \quad \text{for some } k \in \mathbb{Z}^+. \tag{2.1}$$

In particular, $0 \in P$. Under this hypothesis, we have

$$H_P(z) \geqslant \frac{1}{k} \max_{j=1,\ldots,d} \log^+ |z_j|, \tag{2.2}$$

where $\log^+ |z_j| = \max[0, \log |z_j|]$. We use H_P to define generalizations of the Lelong classes $L(\mathbb{C}^d)$, the set of all plurisubharmonic (psh) functions u on $\mathbb{C}^d$ with the property that $u(z) - \log |z| = O(1)$, $|z| \to \infty$, and

$$L^+(\mathbb{C}^d) = \Big\{u \in L(\mathbb{C}^d) \colon u(z) \geqslant \max_{j=1,\ldots,d} \log^+ |z_j| + C_u\Big\},$$

where C_u is a constant depending on u. Define

$$L_P = L_P(\mathbb{C}^d) := \{u \in \mathrm{PSH}(\mathbb{C}^d) \colon u(z) - H_P(z) = O(1),\ |z| \to \infty\}$$

and

$$L_P^+ = L_P^+(\mathbb{C}^d) = \{u \in L_P(\mathbb{C}^d) \colon u(z) \geqslant H_P(z) + C_u\}.$$

For $p \in \mathrm{Poly}(nP)$, $n \geqslant 1$, we have $\frac{1}{n} \log |p| \in L_P$; also each $u \in L_P^+$ is locally bounded in $\mathbb{C}^d$. We are working on $\mathbb{C}^d$, and not $(\mathbb{C} \setminus 0)^d$ as in [1]. Note that $L_\Sigma = L(\mathbb{C}^d)$ and $L_\Sigma^+ = L^+(\mathbb{C}^d)$.

Given $E \subset \mathbb{C}^d$, the *P-extremal function* of E is $V_{P,E}^*(z) := \limsup_{\zeta \to z} V_{P,E}(\zeta)$ where

$$V_{P,E}(z) := \sup\{u(z) \colon u \in L_P(\mathbb{C}^d),\ u \leqslant 0 \text{ on } E\}.$$

Next, let $K \subset \mathbb{C}^d$ be closed and let $w \colon K \to \mathbb{R}^+$ be an admissible weight function on K: w is a nonnegative, upper semicontinuous function with $\{z \in K \colon w(z) > 0\}$ nonpluripolar. Letting $Q := -\log w$, if K is unbounded, we additionally require that

$$\liminf_{|z| \to \infty,\, z \in K} [Q(z) - H_P(z)] = +\infty.$$

We sometimes refer to Q as the (admissible) weight function. Define the *weighted P-extremal function*

$$V_{P,K,Q}^*(z) := \limsup_{\zeta \to z} V_{P,K,Q}(\zeta),$$

where

$$V_{P,K,Q}(z) := \sup\{u(z) \colon u \in L_P(\mathbb{C}^d),\ u \leqslant Q \text{ on } K\}.$$

If $Q = 0$ we simply write $V_{P,K,Q} = V_{P,K}$, which is consistent with the previous notation. In the case $P = \Sigma$,

$$V_{\Sigma,K,Q}(z) = V_{K,Q}(z) := \sup\{u(z) \colon u \in L(\mathbb{C}^d),\ u \leqslant Q \text{ on } K\} \tag{2.3}$$

is the usual weighed extremal function, for example, as in Appendix B of [14].

We recall some results in [1], modified for our setting of $\mathbb{C}^d$ and $P \subset (\mathbb{R}^+)^d$. Our hypothesis (2.1) implies Lemma 2.2 in [1] holds. This was used to prove a result on total mixed Monge-Ampère masses and a Siciak-Zaharjuta type theorem. Let $\omega := dd^c \max_{j=1,\ldots,d} \log^+ |z_j|$.

Proposition 2.1. *Let $P_i \subset (\mathbb{R}^+)^d$, $i = 1, \ldots, k$, $k \leqslant d$, be convex bodies and let $u_i, v_i \in L_{P_i} \cap L^\infty_{\mathrm{loc}}(\mathbb{C}^d)$, $i = 1, \ldots, k$, with*

$$u_i(z) \leqslant v_i(z) + C_i, \qquad z \in \mathbb{C}^d, \quad i = 1, \ldots, k.$$

Then

$$\int_{\mathbb{C}^d} dd^c u_1 \wedge \cdots \wedge dd^c u_k \wedge \omega^{d-k} \leqslant \int_{\mathbb{C}^d} dd^c v_1 \wedge \cdots \wedge dd^c v_k \wedge \omega^{d-k}.$$

In particular, if $u_i \in L_{P_i,+}$, $i = 1, \ldots, k$, then

$$\int_{\mathbb{C}^d} dd^c u_1 \wedge \cdots \wedge dd^c u_k \wedge \omega^{d-k} = M_k,$$

where M_k is a constant depending only on k, d and $P_1, \ldots, P_k$ (but is independent of $u_i \in L_{P_i,+}$).

Remark 2.2. The constants M_k can be computed; see § 2.1 of [1]. Normalizing so that $\int_{\mathbb{C}^d} \omega^d = 1$, for any $u \in L_P^+$ we have

$$\int_{\mathbb{C}^d} (dd^c u)^d = \int_{\mathbb{C}^d} (dd^c H_P)^d = d!\,\mathrm{Vol}(P) =: \gamma_d, \tag{2.4}$$

where $\mathrm{Vol}(P)$ denotes the Euclidean volume of $P \subset (\mathbb{R}^+)^d$. Proposition 2.1 and (2.4) can also be found in [12], Theorem 3.1 and Proposition 4.2.

Proposition 2.3. *Let $P \subset (\mathbb{R}^+)^d$ be a convex body, let $K \subset \mathbb{C}^d$ be compact, and $w = e^{-Q}$ an admissible weight on K. Then*

$$V_{P,K,Q} = \lim_{n\to\infty} \frac{1}{n} \log \Phi_n = \lim_{n\to\infty} \frac{1}{n} \log \Phi_{n,P,K,Q}$$

pointwise on $\mathbb{C}^d$ where

$$\Phi_n(z) := \sup\Big\{|p_n(z)| \colon p_n \in \mathrm{Poly}(nP),\ \max_{\zeta\in K} |p_n(\zeta) e^{-nQ(\zeta)}| \leqslant 1\Big\}.$$

Moreover, if Q is continuous, that is, $Q \in C(K)$, and $V_{P,K,Q}$ is continuous, the convergence is locally uniform on $\mathbb{C}^d$.

Remark 2.4. Since $P \subset A\Sigma$, we have

$$\Phi_{n,P,K,Q} \leqslant \Phi_{n,A\Sigma,K,Q}.$$

In particular, for $Q = 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log \Phi_{n,P,K,0} = V_{P,K} \leqslant A \cdot \lim_{n\to\infty} \frac{1}{An} \log \Phi_{n,A\Sigma,K,0} = A \cdot V_{\Sigma,K}.$$

Thus for any K,

$$V^*_{\Sigma,K}(z) = 0 \quad \text{implies} \quad V^*_{P,K}(z) = 0. \tag{2.5}$$

A compact set $K \subset \mathbb{C}^d$ is *locally regular* if for all $z \in K$ and all balls $B(z,r) := \{w\colon |w-z| \leqslant r\}$ we have $V^*_{\Sigma,K\cap B(z,r)}(z) = 0$. As examples, the closure of any bounded open set $D \subset \mathbb{C}^d$ with C^1 boundary is locally regular. It is known (see [15], Proposition 2.16) that if K is locally regular and $Q \in C(K)$ then $V_{K,Q}$ in (2.3) is continuous. Using (2.5) for $K \cap B(z,r)$, the same proof shows that for a convex body $P \subset (\mathbb{R}^+)^d$, if K is locally regular and $Q \in C(K)$ then $V_{P,K,Q}$ is continuous.

Following the proofs of Lemma 2.3 and Theorem 2.5 in Appendix B of [14], we have the following.

Proposition 2.5. *Let $P \subset (\mathbb{R}^+)^d$ be a convex body, $K \subset \mathbb{C}^d$ closed, and let $w = e^{-Q}$ be an admissible weight on K. Then $S_w := \operatorname{supp}(dd^cV^*_{P,K,Q})^d$ is compact and*

$$S_w := \operatorname{supp}\big((dd^cV^*_{P,K,Q})^d\big) \subset \{z \in K\colon V^*_{P,K,Q}(z) \geqslant Q(z)\}. \tag{2.6}$$

Moreover, $V^*_{P,K,Q} = Q$ q.e. on $S_w := \operatorname{supp}(dd^cV^*_{P,K,Q})^d$, that is, off of a pluripolar set. In particular, if Q and $V_{P,K,Q}$ are continuous,

$$S_w := \operatorname{supp}\big((dd^cV_{P,K,Q})^d\big) \subset \{z \in K\colon V_{P,K,Q}(z) = Q(z)\}.$$

Remark 2.6. It follows under the hypotheses of Proposition 2.5 that

$$V^*_{P,K,Q} = V^*_{P,S_w,Q|_{S_w}} \in L^+_P.$$

Example 2.7. Let $P \subset (\mathbb{R}^+)^d$ be a convex body and $K = T^d$, the unit d-torus in $\mathbb{C}^d$. Then

$$V_{P,T^d}(z) = H_P(z) = \max_{J\in P} \log|z^J| \in L^+_P. \tag{2.7}$$

This is Example 2.3 in [1].

Remark 2.8. The results (and proofs) of Propositions 2.1, 2.3 and 2.5, as well as Example 2.7, are valid for any convex body $P \subset (\mathbb{R}^+)^d$; some were stated in [1] in the case when $P \subset \mathbb{R}^d$ is a non-degenerate convex polytope, with the appropriate plurisubharmonic functions in $(\mathbb{C} \setminus \{0\})^d$. An alternate proof of (2.7) can be found in [9]. Further explicit examples of weighted P-extremal functions and their Monge-Ampère measures can be found in [1].

Making the appropriate changes in the proof of Theorem 2.6 in Appendix B of [14] we obtain the following result.

Proposition 2.9. *Let $P \subset (\mathbb{R}^+)^d$ be a convex body, let $K \subset \mathbb{C}^d$ be closed, and let $w = e^{-Q}$ be an admissible weight on K. Then for $p_n \in \operatorname{Poly}(nP)$ with $w(z)^n|p_n(z)| \leqslant M$ q.e. $z \in S_w$,*

$$|p_n(z)| \leqslant M\exp(nV^*_{P,K,Q}(z)), \qquad z \in \mathbb{C}^d, \tag{2.8}$$

and

$$w(z)^n|p_n(z)| \leqslant M\exp[n(V^*_{P,K,Q}(z) - Q(z))], \qquad z \in K.$$

Hence $w(z)^n|p_n(z)| \leqslant M$ for q.e. $z \in K$.

For $K \subset \mathbb{C}^d$ compact, $w = e^{-Q}$ an admissible weight function on K and ν a finite measure on K, we say that the triple (K, ν, Q) *satisfies a weighted Bernstein-Markov property* if for all $p_n \in \mathscr{P}_n$,

$$\|w^n p_n\|_K \leqslant M_n \|w^n p_n\|_{L^2(\nu)}, \quad \text{with } \limsup_{n\to\infty} M_n^{1/n} = 1. \tag{2.9}$$

Here, $\|w^n p_n\|_K := \sup_{z\in K} |w(z)^n p_n(z)|$ and

$$\|w^n p_n\|^2_{L^2(\nu)} := \int_K |p_n(z)|^2 w(z)^{2n}\, d\nu(z). \tag{2.10}$$

For K closed but unbounded, we allow ν to be locally finite. In this setting, if $\nu(K) = \infty$ we must assume that the weighted L^2-norms in (2.10) are finite. Next, following [1], given a convex body $P \subset (\mathbb{R}^+)^d$, we say that a finite measure ν with support in a compact set K is a *Bernstein-Markov measure for the triple* (P, K, Q) if (2.9) holds for all $p_n \in \text{Poly}(nP)$. Again, if K is closed but unbounded, if $\nu(K) = \infty$ we must assume the weighted L^2-norms in (2.10) are finite.

Remark 2.10. Since for any P there exists $A = A(P) > 0$ with $\text{Poly}(nP) \subset \mathscr{P}_{An}$ for all n, if (K, ν, Q) satisfies a weighted Bernstein-Markov property, then ν is a Bernstein-Markov measure for the triple $(P, K, \widetilde{Q})$ where $\widetilde{Q} = AQ$. In particular, if ν is a *strong Bernstein-Markov measure* for K, that is, if ν is a weighted Bernstein-Markov measure for any $Q \in C(K)$, then for any such Q, ν is a Bernstein-Markov measure for the triple (P, K, Q).

Remark 2.11. In Example 2.7, the monomials z^J, $J \in nP \cap (\mathbb{Z}^+)^d$, form an orthonormal basis for $\text{Poly}(nP)$ with respect to the normalized Haar measure μ_T on T^d. Moreover, μ_T is a strong Bernstein-Markov measure for T and hence it is a Bernstein-Markov measure for the triple (P, T, Q) for any $Q \in C(T)$.

We refer to [8] for a survey of Bernstein-Markov properties.

2.2. The projection operator. To emphasize the relation between the weight Q and the weighted P-extremal function $V^*_{P,K,Q}$, we may write

$$\Pi(Q) = \Pi_K(Q) := V^*_{P,K,Q}. \tag{2.11}$$

This operator Π is increasing and concave: if $Q_1 \leqslant Q_2$ are admissible weights on K, then $\Pi(Q_1) \leqslant \Pi(Q_2)$; and if $0 \leqslant s \leqslant 1$ and a, a' are admissible weights on K,

$$\Pi(sa + (1-s)a') \geqslant s\Pi(a) + (1-s)\Pi(a'). \tag{2.12}$$

Since $sa + (1-s)a'$ is a convex combination of a and a', it is an admissible weight on K. Then (2.12) follows since the right-hand side is a competitor for the weighted P-extremal function on the left-hand side.

It follows from the definition of Π, Proposition 2.5 and Remark 2.4 that Π is Lipschitz on locally regular compact sets. That is, if $a, b \in C(K)$ and $0 \leqslant t \leqslant 1$ then on $\mathbb{C}^d$,

$$|\Pi(a + t(b-a)) - \Pi(a)| \leqslant Ct \tag{2.13}$$

where $C = C(a,b) = \max[\sup_{D(0)} |b-a|, \sup_{D(t)} |b-a|]$. Here

$$D(t) := \{\Pi(a + t(b-a)) = a + t(b-a)\}.$$

Note by (2.6)

$$\operatorname{supp}(dd^c(\Pi(a + t(b-a)))^d) \subset D(t). \tag{2.14}$$

Similarly, if $u \in C(K)$, for $t \in \mathbb{R}$ we have

$$|\Pi(a + tu) - \Pi(a)| \leqslant C|t|, \tag{2.15}$$

where $C = C(u) = \sup_K |u|$. In the former case, if K is unbounded, in order for $\max[\sup_{D(0)} |b-a|, \sup_{D(t)} |b-a|]$ to be a finite constant which is independent of t we assume that

$$\bigcup_{0\leqslant t\leqslant 1} D(t) \text{ is bounded} \quad \text{and} \quad b - a \in L^\infty\biggl(\bigcup_{0\leqslant t\leqslant 1} D(t)\biggr). \tag{2.16}$$

Then (2.13) holds. This observation will be used in the proof of Theorem 5.1. In both cases, if K is compact, C is finite.

Another result we will need is a comparison principle in L_P^+; we state and prove the version we will use.

Proposition 2.12. *Let $a_1, a_2 \in L_P^+$ and $b_1, b_2 \in L^+(\mathbb{C}^d)$. For $M > 0$, set $u_1 := a_1 + Mb_1$ and $u_2 := a_2 + Mb_2$. Then*

$$\int_{\{u_1<u_2\}} (dd^c u_2)^d \leqslant \int_{\{u_1<u_2\}} (dd^c u_1)^d.$$

Note that each integral is finite by Proposition 2.1.

Proof of Proposition 2.12. By adding a constant to u_1, if necessary, we can assume that $u_1 \geqslant 0$. Then for $\varepsilon > 0$, we have

$$\{(1+\varepsilon)u_1 < u_2\} \subset \{u_1 < u_2\}$$

and $\{(1+\varepsilon)u_1 < u_2\}$ is bounded. By the standard comparison theorem for locally bounded psh functions on bounded domains (see Theorem 3.7.1 in [11]),

$$\int_{\{(1+\varepsilon)u_1<u_2\}} (dd^c u_2)^d \leqslant (1+\varepsilon)^d \int_{\{(1+\varepsilon)u_1<u_2\}} (dd^c u_1)^d. \tag{2.17}$$

Clearly

$$\bigcup_{j=1}^{\infty} \biggl\{\biggl(1 + \frac{1}{j}\biggr)u_1 < u_2\biggr\} = \{u_1 < u_2\},$$

so applying (2.17) with $\varepsilon = 1/j$, the result follows by monotone convergence upon letting $j \to \infty$. The proposition is proved.

The following lemma and corollary will be used in § 4.

Lemma 2.13. *Let a be an admissible weight on a compact set K and let $u \in C^2(K)$. Then*

$$\lim_{t\to 0}\int_{D(0)\setminus D(t)} (dd^c\Pi(a))^d = 0,$$

where $D(t) = \{\Pi(a+tu) = a+tu\}$ for $t \in \mathbb{R}$.

Proof. By (2.14), $\text{supp}(dd^c\Pi(a))^d \subset D(0)$. The hypothesis $u \in C^2(K)$ means that u is the restriction to K of a C^2 function (which we also denote by u) on $\mathbb{C}^d$; clearly we can take this function to have compact support. We prove the result for $t > 0$, that is, as $t \to 0^+$. We can find sufficiently large $M > 0$ depending on u and its support so that $u + M\psi$ is psh where $\psi(z) = \frac{1}{2}\log(1+|z|^2)$. Observing that

$$D(0)\setminus D(t) \subset S \quad \text{and} \quad D(t)\cap S = \varnothing,$$

where

$$S := \{\Pi(a+tu) < \Pi(a)+tu\} = \{\Pi(a+tu)+tM\psi < \Pi(a)+t(u+M\psi)\},$$

we have

$$\begin{aligned}\int_{D(0)\setminus D(t)} (dd^c\Pi(a))^d &\leqslant \int_S (dd^c\Pi(a))^d \\ &\leqslant \int_S [dd^c(\Pi(a)+t(u+M\psi)]^d \leqslant \int_S [dd^c(\Pi(a+tu)+tM\psi)]^d \\ &= \int_S [dd^c(\Pi(a+tu))]^d + O(t) = O(t).\end{aligned}$$

Here, the third inequality comes from Proposition 2.12 (with M replaced by tM). The lemma is proved.

Corollary 2.14. *Let $a, b \in C^2(E)$ be admissible weights on a closed, unbounded set E. If (2.16) holds then*

$$\lim_{t\to 0}\int_{D(0)\setminus D(t)} (dd^c\Pi(a))^d = 0, \tag{2.18}$$

where $D(t) = \{\Pi(a+t(b-a)) = a+t(b-a)\}$ for $0 \leqslant t \leqslant 1$.

Proof. First of all, $(dd^c\Pi(a))^d$ has compact support. Also, by (2.16), the P-extremal functions $\Pi(a+t(b-a))$ for all $0 \leqslant t \leqslant 1$ are independent of the values of a and b outside a large ball. Thus we may assume that $a = b$ outside a fixed ball. In other words, this case is reduced to the case of Lemma 2.13 with $u = b - a$.

Remark 2.15. For the remainder of this paper, K will always denote a compact subset of $\mathbb{C}^d$ while E will be used for a closed but possibly unbounded subset.

2.3. Transfinite diameter. Recall d_n is the dimension of $\text{Poly}(nP)$. We can write

$$\text{Poly}(nP) = \text{span}\{e_1, \ldots, e_{d_n}\},$$

where $\{e_j(z) := z^{\alpha(j)}\}_{j=1,\dots,d_n}$ are the standard basis monomials. The ordering is unimportant. For points $\zeta_1, \dots, \zeta_{d_n} \in \mathbb{C}^d$, let

$$\begin{aligned} \mathrm{VDM}(\zeta_1, \dots, \zeta_{d_n}) &:= \det[e_i(\zeta_j)]_{i,j=1,\dots,d_n} \\ &= \det \begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \dots & e_1(\zeta_{d_n}) \\ \vdots & \vdots & \ddots & \vdots \\ e_{d_n}(\zeta_1) & e_{d_n}(\zeta_2) & \dots & e_{d_n}(\zeta_{d_n}) \end{bmatrix}, \end{aligned} \tag{2.19}$$

and for a compact subset $K \subset \mathbb{C}^d$ let

$$V_n = V_n(K) := \max_{\zeta_1,\dots,\zeta_{d_n} \in K} |\mathrm{VDM}(\zeta_1, \dots, \zeta_{d_n})|.$$

We will show later that the limit

$$\delta(K) := \delta(K, P) := \lim_{n\to\infty} V_n^{1/l_n} \tag{2.20}$$

exists, where

$$l_n := \sum_{j=1}^{d_n} \deg(e_j) = \sum_{j=1}^{d_n} |\alpha(j)|$$

is the sum of the degrees of a set of these basis monomials for $\mathrm{Poly}(nP)$. We call $\delta(K)$ the *P-transfinite diameter* of K. More generally, let w be an admissible weight function on K. Given $\zeta_1, \dots, \zeta_{d_n} \in K$, let

$$\begin{aligned} W(\zeta_1, \dots, \zeta_{d_n}) &:= \mathrm{VDM}(\zeta_1, \dots, \zeta_{d_n}) w(\zeta_1)^n \cdots w(\zeta_{d_n})^n \\ &= \det \begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \dots & e_1(\zeta_{d_n}) \\ \vdots & \vdots & \ddots & \vdots \\ e_{d_n}(\zeta_1) & e_{d_n}(\zeta_2) & \dots & e_{d_n}(\zeta_{d_n}) \end{bmatrix} \cdot w(\zeta_1)^n \cdots w(\zeta_{d_n})^n \end{aligned}$$

be a *weighted Vandermonde determinant*. Let

$$W_n(K) := \max_{\zeta_1,\dots,\zeta_{d_n} \in K} |W(\zeta_1, \dots, \zeta_{d_n})|. \tag{2.21}$$

Definition 2.16. We define an *nth weighted P-Fekete set* for K and w to be a set of d_n points $\zeta_1, \dots, \zeta_{d_n} \in K$ where the maximum in (2.21) is attained, that is, with the property that

$$|W(\zeta_1, \dots, \zeta_{d_n})| = W_n(K).$$

We also write $\delta^{w,n}(K) := W_n(K)^{1/l_n}$ and we will show, more generally, that the *weighted P-transfinite diameter*

$$\delta^w(K) := \delta^w(K, P) := \lim_{n\to\infty} \delta^{w,n}(K) := \lim_{n\to\infty} W_n(K)^{1/l_n} \tag{2.22}$$

exists. For each n, if we take points $f_1^{(n)}, f_2^{(n)}, \dots, f_{d_n}^{(n)} \in K$ for which

$$\lim_{n\to\infty} \left[|\mathrm{VDM}(f_1^{(n)}, \dots, f_{d_n}^{(n)})| w(f_1^{(n)})^n \cdots w(f_{d_n}^{(n)})^n\right]^{1/l_n} = \delta^w(K) \tag{2.23}$$

(these are known as *asymptotically weighted P-Fekete arrays*) and, further, we let $\mu_n := \frac{1}{d_n}\sum_{j=1}^{d_n}\delta_{f_j^{(n)}}$, one of our results, Corollary 6.5, is that

$$\mu_n \to \frac{1}{\gamma_d}(dd^c\Pi(Q))^d \quad \text{weak-}*$$

(recall (2.4)).

Remark 2.17. For $P=\Sigma$, so that $\mathrm{Poly}(n\Sigma)=\mathscr{P}_n$, we have

$$d_n(\Sigma)=\binom{d+n}{d}=O\Big(\frac{n^d}{d!}\Big) \quad \text{and} \quad l_n(\Sigma)=\frac{d}{d+1}nd_n(\Sigma).$$

In particular,

$$\frac{l_n(\Sigma)}{d_n(\Sigma)}=\frac{nd}{d+1}.$$

For a general convex body $P\subset(\mathbb{R}^+)^d$ with $A>0$ so that $P\subset A\Sigma$, we write

$$l_n=f_n(d)\frac{nd}{d+1}d_n=f_n(d)\frac{l_n(\Sigma)}{d_n(\Sigma)}d_n. \tag{2.24}$$

We will need to know in Remark 5.2 that l_n/d_n divided by $l_n(\Sigma)/d_n(\Sigma)$ has a limit, that is, that

$$\lim_{n\to\infty} f_n(d)=:\mathscr{A}=\mathscr{A}(P,d) \tag{2.25}$$

exists.

We first verify (2.25) when $P\subset(\mathbb{R}^+)^d$ is a non-degenerate convex polytope. It follows from Theorem 2 of Lecture 2 in [16]

1) applied to $f(j_1,\dots,j_d)\equiv 1$ that d_n is a polynomial of degree d in n with

$$d_n=\mathrm{Vol}(P)n^d+O(n^{d-1});$$

2) applied to $f(j_1,\dots,j_d)=j_1+\cdots+j_d$ that l_n is a polynomial of degree $d+1$ in n with

$$l_n=C_Pn^{d+1}+O(n^d),$$

where $C_P=\displaystyle\int_P(x_1+\cdots+x_d)\,dx_1\dots dx_d$.

Thus

$$\frac{l_n}{d_n}=\frac{C_Pn^{d+1}+O(n^d)}{\mathrm{Vol}(P)n^d+O(n^{d-1})}=\frac{nC_P}{\mathrm{Vol}(P)}+O(1),$$

which proves (2.25):

$$f_n(d)=\frac{(d+1)l_n}{ndd_n}=\frac{(d+1)}{d}\frac{l_n}{nd_n}\to\frac{(d+1)}{d}\frac{C_P}{\mathrm{Vol}(P)}.$$

For a convex body we still have the leading term asymptotics in 1) and 2), that is,

$$d_n\asymp\mathrm{Vol}(P)n^d \quad \text{and} \quad l_n\asymp C_Pn^{d+1}.$$

Indeed,

$$d_n = \sum_{J\in nP\cap(\mathbb{Z}^+)^d} 1 = \sum_{J\in P\cap\frac{1}{n}(\mathbb{Z}^+)^d} 1 = n^d\cdot\frac{1}{n^d}\sum_{J\in P\cap\frac{1}{n}(\mathbb{Z}^+)^d} 1$$

and $\frac{1}{n^d}\sum_{J\in P\cap\frac{1}{n}(\mathbb{Z}^+)^d} 1$ are Riemann sums approximating the integral for $\mathrm{Vol}(P) = \int_P 1\,dx_1\ldots dx_d$. Similarly,

$$\begin{aligned} l_n &= \sum_{J\in nP\cap(\mathbb{Z}^+)^d} (j_1+\cdots+j_d) = n\sum_{J\in P\cap\frac{1}{n}(\mathbb{Z}^+)^d} (j_1+\cdots+j_d) \\ &= n^{d+1}\cdot\frac{1}{n^d}\sum_{J\in P\cap\frac{1}{n}(\mathbb{Z}^+)^d} (j_1+\cdots+j_d) \end{aligned}$$

and $\frac{1}{n^d}\sum_{J\in P\cap\frac{1}{n}(\mathbb{Z}^+)^d}(j_1+\cdots+j_d)$ are Riemann sums approximating $C_P = \int_P (x_1+\cdots+x_d)\,dx_1\ldots dx_d$. We thank Alexander Barvinok for this observation.

2.4. Gram matrices and *P*-optimal measures. Let $E\subset\mathbb{C}^d$ be closed and let w be an admissible weight on E. We take μ to be a locally finite measure on E and for each n we define a weighted inner product on $\mathrm{Poly}(nP)$:

$$\langle f,g\rangle_{\mu,w} := \int_E f(z)\overline{g(z)}w(z)^{2n}\,d\mu. \tag{2.26}$$

We assume that $\|f\|^2_{L^2(w^{2n}d\mu)} = \langle f,f\rangle_{\mu,w} < \infty$ for all $f\in\mathrm{Poly}(nP)$ and that (2.26) is nondegenerate in the sense that $\|f\|_{L^2(w^{2n}d\mu)} = 0$ implies $f\equiv 0$. Fixing a basis $\beta_n = \{p_1,p_2,\ldots,p_{d_n}\}$ of $\mathrm{Poly}(nP)$ we form the Gram matrix

$$G_n^{\mu,w} = G_n^{\mu,w}(\beta_n) := [\langle p_i,p_j\rangle_{\mu,w}] \in \mathbb{C}^{d_n\times d_n},$$

and the associated nth Bergman function

$$B_n^{\mu,w}(z) := \sum_{j=1}^{d_n}|q_j(z)|^2 w(z)^{2n}, \tag{2.27}$$

where $Q_n = \{q_1,q_2,\ldots,q_{d_n}\}$ is an orthonormal basis for $\mathrm{Poly}(nP)$ with respect to the inner-product (2.26).

We make an observation which will be used in Lemma 2.18 below.

With this basis β_n, if we write

$$P(z) = \begin{bmatrix} p_1(z)\\ p_2(z)\\ \vdots\\ p_{d_n}(z)\end{bmatrix} \in \mathbb{C}^{d_n}. \tag{2.28}$$

then

$$w(z)^{2n}P(z)^*\big(G_n^{\mu,w}(\beta_n)\big)^{-1}P(z) = B_n^{\mu,w}(z), \tag{2.29}$$

where P^* denotes the conjugate transpose. To see this, $G := G_n^{\mu,w}(\beta_n)$ is a positive definite Hermitian matrix; hence $G^{1/2}$ and $G^{-1/2} := (G^{-1})^{1/2}$ exist; writing $P := P(z)$, we have

$$P^*G^{-1}P = P^*G^{-1/2}G^{-1/2}P = (G^{-1/2}P)^*G^{-1/2}P.$$

To verify that $w(z)^{2n}$ times the right-hand side yields $B_n^{\mu,w}(z)$, note that since $G = \int_E PP^*w^{2n}\,d\mu$, the polynomials $\{\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_{d_n}\}$ defined by

$$G^{-1/2}P := \begin{bmatrix} \widetilde{p}_1(z) \\ \widetilde{p}_2(z) \\ \vdots \\ \widetilde{p}_{d_n}(z) \end{bmatrix} \in \mathbb{C}^{d_n} \tag{2.30}$$

form an orthonormal basis for $\mathrm{Poly}(nP)$ in $L^2(w^{2n}\mu)$, since

$$\int_E G^{-1/2}P \cdot (G^{-1/2}P)^*w^{2n}\,d\mu = G^{-1/2}\left[\int_E PP^*w^{2n}\,d\mu\right]G^{-1/2} = G^{-1/2}GG^{-1/2} = I$$

(I is the identity $(d_n \times d_n)$-matrix). Thus

$$B_n^{\mu,w}(z) = \sum_{j=1}^{d_n} |\widetilde{p}_j(z)|^2 w(z)^{2n} = w^{2n}(G^{-1/2}P)^*G^{-1/2}P.$$

Given E, and w on E, for a function $u \in C(E)$, we consider the weight $w_t(z) := w(z)\exp(-tu(z))$, $t \in \mathbb{R}$. A priori, w_t need not be admissible. Let $\{\mu_n\}$ be a sequence of measures on E. Fixing a basis $\beta_n := \{p_1, \ldots, p_{d_n}\}$ of $\mathrm{Poly}(nP)$, we set

$$f_n(t) := -\frac{1}{2l_n}\log\det(G_n^{\mu_n,w_t}), \tag{2.31}$$

where $G_n^{\mu_n,w_t} = G_n^{\mu_n,w_t}(\beta_n)$. We have the following result (Lemma 5.1 in [4] or Lemma 3.5 in [7]) which will be used to prove Theorem 5.1.

Lemma 2.18. *Suppose u has compact support, so that w_t is admissible for t in an interval containing* 0*. For such t, we have*

$$f_n'(t) = \frac{n}{l_n}\int_E u(z)B_n^{\mu_n,w_t}(z)\,d\mu_n. \tag{2.32}$$

Proof. Recall that $G_n^{\mu_n,w_t}$ is a positive definite Hermitian matrix; hence we can define $\log(G_n^{\mu_n,w_t})$. Using $\log\det(G_n^{\mu_n,w_t}) = \mathrm{trace}\log(G_n^{\mu_n,w_t})$, we obtain

$$\begin{aligned} 2l_nf_n'(t) &= -\frac{d}{dt}\,\mathrm{trace}(\log(G_n^{\mu_n,w_t})) = -\,\mathrm{trace}\left(\frac{d}{dt}\log(G_n^{\mu_n,w_t})\right) \\ &= -\,\mathrm{trace}\left((G_n^{\mu_n,w_t})^{-1}\frac{d}{dt}G_n^{\mu_n,w_t}\right) \\ &= 2n\,\mathrm{trace}\left((G_n^{\mu_n,w_t})^{-1}\left[\int_E p_i(z)\overline{p_j(z)}u(z)w(z)^{2n}\exp(-2ntu(z))\,d\mu_n\right]\right). \end{aligned}$$

For a column vector C, a row vector B and a square matrix A, we use

$$\mathrm{trace}(ABC) = \mathrm{trace}(CAB) = CAB$$

to write the previous line as

$$2n\int_E P^*(z)(G_n^{\mu_n,w_t})^{-1}P(z)u(z)w(z)^{2n}\exp(-2ntu(z))\,d\mu_n$$
$$= 2n\int_E u(z)P^*(z)(G_n^{\mu_n,w_t})^{-1}P(z)w_t(z)^{2n}\,d\mu_n = 2n\int_E u(z)B_n^{\mu_n,w_t}(z)\,d\mu_n;$$

where the last equality follows from (2.29):

$$w_t^{2n}P^*(G_n^{\mu_n,w_t})^{-1}P = B_n^{\mu_n,w_t}.$$

The lemma is proved.

Note that since u has compact support, we can differentiate under the integral sign in (2.32); then similar, but more involved calculations, give the following result (cf. Lemma 3.6 of [7]).

Lemma 2.19. *The functions $f_n(t)$ in (2.31) are concave, that is, f_n is twice differentiable and $f_n''(t) \leqslant 0$.*

Now we restrict ourselves to the case when $K \subset \mathbb{C}^d$ is compact and non-pluripolar. Fix a probability measure μ on K and an admissible weight w on K. If μ has the property that

$$\det(G_n^{\mu',w}) \leqslant \det(G_n^{\mu,w}) \tag{2.33}$$

for all other probability measures μ' on K then μ is said to be a *P-optimal measure of degree n* for K and w. This property is independent of the basis used for $\mathrm{Poly}(nP)$. An equivalent characterization is that

$$\max_{z\in K} B_n^{\mu,w}(z) \leqslant \max_{z\in K} B_n^{\mu',w}(z)$$

for all other probability measures μ' on K. Note that for any probability measure μ', $\int_K B_n^{\mu',w}(z)\,d\mu' = d_n$, so that

$$\max_{z\in K} B_n^{\mu',w}(z) \geqslant d_n.$$

For a P-optimal measure we have equality.

Proposition 2.20. *Let w be an admissible weight on K. A probability measure μ is a P-optimal measure of degree n for K and w if and only if*

$$\max_{z\in K} B_n^{\mu,w}(z) = d_n.$$

It follows that if μ is P-optimal for K and w then

$$B_n^{\mu,w}(z) = d_n \quad \mu\text{-a.e.} \tag{2.34}$$

We omit the proof; see [10] or Proposition 3.1 of [7].

2.5. Ball volume ratios. Given a (complex) M-dimensional vector space V, and two subsets A and B in V, we write

$$[A : B] := \log \frac{\mathrm{vol}(A)}{\mathrm{vol}(B)},$$

where 'vol' denotes any (Haar) measure on V (taking the ratio makes $[A : B]$ independent of the choice of measure). In particular, if V is equipped with two Hermitian inner products h and h', and B and B' are the corresponding unit balls, then an exercise in linear algebra shows that

$$[B : B'] = \log \det[h'(e_i, e_j)]_{i,j=1,\dots,M}, \tag{2.35}$$

where $e_1, \dots, e_M$ is an h-orthonormal basis for V. In other words, $[B : B']$ is a Gram determinant with respect to the h' inner product relative to the h-orthonormal basis. Indeed, $[B : B']$ is independent of the h-orthonormal basis chosen for V.

We will generally take $V = \mathrm{Poly}(nP)$ and our subsets to be unit balls with respect to norms on $\mathrm{Poly}(nP)$; in this case we call (2.35) a *ball volume ratio*. In particular, given P, let μ be a locally finite measure on a closed set $E \subset \mathbb{C}^d$, and let w be an admissible weight on E such that (2.26) is non-degenerate and $\|f\|^2_{L^2(w^{2n}d\mu)} < \infty$ for all $f \in \mathrm{Poly}(nP)$. We noted in Remark 2.11 that for the unit torus T^d, the standard basis monomials $\beta_n = \{z^J,\ J \in nP \cap (\mathbb{Z}^+)^d\}$ form an orthonormal basis for $\mathrm{Poly}(nP)$ with respect to the standard Haar measure μ_T on T^d. Letting

$$B_n = \{p_n \in \mathrm{Poly}(nP)\colon \|p_n w^n\|_{L^2(\mu)} = \|p_n\|_{L^2(w^{2n}\mu)} \leqslant 1\}$$

and

$$B'_n = \{p_n \in \mathrm{Poly}(nP)\colon \|p_n\|_{L^2(\mu_T)} \leqslant 1\}$$

be L^2-balls in $\mathrm{Poly}(nP)$, we have

$$[B_n : B'_n] = \log \det G_n^{\mu,w}(\beta_n). \tag{2.36}$$

We will also use L^∞-balls in $\mathrm{Poly}(nP)$.

Taking $E = K$ compact and μ finite, replacing the standard basis monomials $\{z^J,\ J \in nP \cap (\mathbb{Z}^+)^d\}$ by orthogonal polynomials $\{r_J(z) = z^J + \cdots\}$ using the Gram-Schmidt process in $L^2(w^{2n}\mu)$, the Gram determinants $\det(G_n^{\mu,w}) = \prod_J \|r_J\|^2_{L^2(w^{2n}\mu)}$ are unchanged and we have

$$\det(G_n^{\mu,w}) = \frac{1}{d_n!}\, Z_n := \frac{1}{d_n!}\, Z_n(\mu, w), \tag{2.37}$$

where

$$Z_n := \int_{K^{d_n}} |\mathrm{VDM}(z_1, \dots, z_{d_n})|^2 w(z_1)^{2n} \cdots w(z_{d_n})^{2n}\, d\mu(z_1) \dots d\mu(z_{d_n}).$$

It is easy to see that if μ is a Bernstein-Markov measure for the triple (P, K, Q) where $w = e^{-Q}$, that is, (2.9) holds for μ, then

$$Z_n \leqslant \delta^{w,n}(K)^{2l_n} \mu(K)^{d_n} \leqslant \mu(K)^{d_n} M_n^{2d_n} Z_n. \tag{2.38}$$

Conjecture 2.21. *Let $K \subset \mathbb{C}^d$ be compact and let $w = e^{-Q}$ be an admissible weight on K. If μ is a Bernstein-Markov measure for the triple (P, K, Q), then*

$$\lim_{n\to\infty} Z_n^{1/(2l_n)} = \lim_{n\to\infty} \det(G_n^{\mu,w})^{1/(2l_n)} =: \mathscr{F}_P(K, Q) \tag{2.39}$$

exists.

Equation (2.37) shows the limits in (2.39) coincide. We verify the existence of this limit and hence Conjecture 2.21 in Remark 5.2. It then follows from (2.38) that $\lim_{n\to\infty} \delta^{w,n}(K)$ exists and equals $\mathscr{F}_P(K, Q)$. This gives the existence of the limit in the definition of the P-transfinite diameter (2.20) and the weighted P-transfinite diameter (2.22). We also have the following.

Proposition 2.22. *Let K be compact and w an admissible weight function on K. Assume that (2.39) holds. For $n = 1, 2, \dots,$ let μ_n be a P-optimal measure of order n for K and w. Then*

$$\lim_{n\to\infty} \det(G_n^{\mu_n,w})^{1/(2l_n)} = \mathscr{F}_P(K, Q).$$

Proof. We will use

$$\int_{K^{d_n}} |\mathrm{VDM}(z_1, \dots, z_{d_n})|^2 w(z_1)^{2n} \cdots w(z_{d_n})^{2n}\, d\mu_n(z_1) \dots d\mu_n(z_{d_n}) \\ = d_n! \det(G_n^{\mu_n,w}).$$

It follows, since μ_n is a probability measure, that

$$\det(G_n^{\mu_n,w}) \leqslant \frac{1}{d_n!}\, (\delta^{w,n}(K))^{2l_n}.$$

Now if $f_1, f_2, \dots, f_{d_n} \in K$ are weighted P-Fekete points of order n for K, that is, points in K for which

$$|W(z_1, \dots, z_{d_n})| = |\mathrm{VDM}(z_1, \dots, z_{d_n})| w(z_1)^n \cdots w(z_{d_n})^n$$

is maximal, then the discrete measure

$$\nu_n = \frac{1}{d_n} \sum_{k=1}^{d_n} \delta_{f_k} \tag{2.40}$$

is a candidate for a P-optimal measure of order n; hence

$$\det(G_n^{\nu_n,w}) \leqslant \det(G_n^{\mu_n,w}).$$

But

$$\det(G_n^{\nu_n,w}) = \frac{1}{d_n^{d_n}} |\mathrm{VDM}(f_1, \dots, f_{d_n})|^2 w(f_1)^{2n} \cdots w(f_{d_n})^{2n} = \frac{1}{d_n^{d_n}} (\delta^{w,n}(K))^{2l_n}$$

so that

$$\frac{1}{d_n^{d_n}} (\delta^{w,n}(K))^{2l_n} \leqslant \det(G_n^{\mu_n,w}).$$

The result follows since (2.39) implies $\lim_{n\to\infty} \delta^{w,n}(K)$ exists and equals $\mathscr{F}_P(K, Q)$. The proposition is proved.

For future use we note that the ball volume ratios satisfy $[A : B] = -[B : A]$ and the cocycle condition:

$$[A : B] + [B : C] + [C : A] = 0;$$

and they are 'monotone' in the first argument: for any $B \subset \mathrm{Poly}(nP)$, if $E \subset \mathbb{C}^d$ is closed with admissible weights $Q_1 \leqslant Q_2$ and

$$\mathscr{B}^\infty(E, nQ_i) := \{p_n \in \mathrm{Poly}(nP) \colon \|p_n e^{-nQ_i}\|_E \leqslant 1\}, \qquad i = 1, 2,$$

then

$$[\mathscr{B}^\infty(E, nQ_1) : B] \leqslant [\mathscr{B}^\infty(E, nQ_2) : B] \tag{2.41}$$

(with a similar statement for L^2-balls for μ a measure on E). Analogous properties will hold for the energy functional, which we discuss next.

§ 3. Energy

For $u, v \in L_P^+$, we define the energy

$$\mathscr{E}(u, v) := \int_{\mathbb{C}^d} (u - v) \sum_{j=0}^{d} (dd^c u)^j \wedge (dd^c v)^{d-j}. \tag{3.1}$$

A reason for this definition will appear in Proposition 3.1, and Theorem 5.1 will relate the asymptotics of certain ball volume ratios to the energy of appropriate u and v. Note that $\mathscr{E}(u, v) = -\mathscr{E}(v, u)$. For any functions $A, B \in L_P^+$, the difference $A - B$ is uniformly bounded on $\mathbb{C}^d$. We will need an integration by parts formula in this setting. Using results from Bedford-Taylor [2], we can show the following.

Given $A, B, C, D \in L_P^+$, let $u_1, \ldots, u_{d-1} \in L_P^+$. Then

$$\begin{aligned} &\int_{\mathbb{C}^d} (A - B)(dd^c C - dd^c D) \wedge dd^c u_1 \wedge \cdots \wedge dd^c u_{d-1} \\ &\quad = \int_{\mathbb{C}^d} (C - D)(dd^c A - dd^c B) \wedge dd^c u_1 \wedge \cdots \wedge dd^c u_{d-1} \\ &\quad = -\int_{\mathbb{C}^d} d(A - B) \wedge d^c(C - D) \wedge dd^c u_1 \wedge \cdots \wedge dd^c u_{d-1}. \end{aligned} \tag{3.2}$$

The proof of the following fundamental differentiability property of the energy is exactly the same as that in Proposition 4.1 of [4].

Proposition 3.1. *Let $u, u', v \in L_P^+$. For $0 \leqslant t \leqslant 1$, let*

$$f(t) := \mathscr{E}(u + t(u' - u), v).$$

Then $f'(t)$ exists for $0 \leqslant t \leqslant 1$ and

$$f'(t) = (d + 1) \int_{\mathbb{C}^d} (u' - u)(dd^c(u + t(u' - u)))^d. \tag{3.3}$$

Remark 3.2. Here we mean the appropriate one-sided derivatives at $t = 0$ and $t = 1$; for instance,

$$f'(0) := \lim_{t\to 0^+} \frac{f(t) - f(0)}{t} = (d+1)\int_{\mathbb{C}^d} (u' - u)(dd^c u)^d. \tag{3.4}$$

This last statement implies (3.3). For if s is fixed,

$$g(t) := f(s+t) = \mathscr{E}(u + (s+t)(u' - u), v) = \mathscr{E}(u + s(u' - u) + t(u' - u), v)$$

and applying (3.4) to g (so $u \to u + s(u' - u)$) we get

$$g'(0) = f'(s) = (d+1)\int_{\mathbb{C}^d} (u' - u)(dd^c(u + s(u' - u)))^d.$$

We sometimes write (3.4) in 'directional derivative' notation as

$$\langle \mathscr{E}'(u), u' - u\rangle = (d+1)\int (u' - u)(dd^c u)^d. \tag{3.5}$$

Note that the differentiation formula (3.3) is independent of v. This also follows from the following cocycle property.

Proposition 3.3. *Let $u, v, w \in L_P^+$. Then*

$$\mathscr{E}(u, v) + \mathscr{E}(v, w) + \mathscr{E}(w, u) = 0.$$

Proof. Let

$$f(t) := \mathscr{E}(u + t(w - u), v) + \mathscr{E}(v, u)$$

and

$$g(t) := \mathscr{E}(u + t(w - u), w) + \mathscr{E}(w, u).$$

Then $f(0) = g(0) = 0$ as $\mathscr{E}$ is antisymmetric. From (3.3),

$$f'(t) = (d+1)\int_{\mathbb{C}^d} (w - u)(dd^c(u + t(w - u)))^d = g'(t)$$

for all t. Thus $f(1) = g(1)$, that is,

$$\mathscr{E}(w, v) + \mathscr{E}(v, u) = \mathscr{E}(w, w) + \mathscr{E}(w, u) = \mathscr{E}(w, u).$$

The proposition is proved.

It follows from Proposition 3.3 that if $v, v' \in L_P^+$ then

$$\mathscr{E}(u + t(u' - u), v) - \mathscr{E}(u, v) = \mathscr{E}(u + t(u' - u), v') - \mathscr{E}(u, v').$$

This implies the independence of $f'(t)$ on v as in (3.3).

We often consider $\mathscr{E}$ as a functional on the first argument with the second fixed. As such, it is increasing and concave; the proof is exactly as for Proposition 4.4 in [4] and requires formula (3.2).

Proposition 3.4. *Let $u, v, w \in L_P^+$. Then*

$$u \geqslant v \quad \textit{implies that} \quad \mathscr{E}(u,w) \geqslant \mathscr{E}(v,w),$$

and for $0 \leqslant t \leqslant 1$

$$\mathscr{E}(tu+(1-t)v,w) \geqslant t\mathscr{E}(u,w)+(1-t)\mathscr{E}(v,w),$$

that is, $g(t) := \mathscr{E}(tu+(1-t)v,w)$ *satisfies* $g''(t) \leqslant 0$.

A consequence of concavity is the following. Let $u_1, u_2, v \in L_P^+$. Letting

$$g(s) := \mathscr{E}(u_1 + s(u_2-u_1), v)$$

for $0 \leqslant s \leqslant 1$, we have $g(s) \leqslant g(0) + g'(0)s$. In particular, at $s = 1$ we have $g(1) \leqslant g(0) + g'(0)$, that is,

$$\mathscr{E}(u_2,v) \leqslant \mathscr{E}(u_1,v) + (d+1)\int_{\mathbb{C}^d}(u_2-u_1)(dd^c u_1)^d. \tag{3.6}$$

For future use, we record the following.

Lemma 3.5. *Let $\{w_j\}, \{v_j\} \subset L_P^+$ with $w_j \uparrow w \in L_P^+$ a.e. and $v_j \uparrow v \in L_P^+$ a.e. Then*

$$\mathscr{E}(w_j,v) \to \mathscr{E}(w,v) \quad \textit{and} \quad \mathscr{E}(w_j,v_j) \to \mathscr{E}(w,v).$$

Proof. By Proposition 3.3, it suffices to prove the first statement. This follows directly from the proof of Lemma 6.3 in [2]: given

$$w, \{v_j\}, v, \{u_{1,j}\}, u_1, \dots, \{u_{d,j}\}, u_d \quad \text{in } L_P^+$$

with $v_j \uparrow v$, $u_{1,j} \uparrow u_1$, $\dots$, $u_{d,j} \uparrow u_d$ a.e.,

$$\lim_{j\to\infty}\int_{\mathbb{C}^d}(w-v_j)\,dd^c u_{1,j}\wedge\cdots\wedge dd^c u_{d,j} = \int_{\mathbb{C}^d}(w-v)\,dd^c u_1\wedge\cdots\wedge dd^c u_d.$$

The lemma is proved.

We remark that if $w_j \downarrow w \in L_P^+$ and $v_j \downarrow v \in L_P^+$ then we also have

$$\mathscr{E}(w_j,v) \to \mathscr{E}(w,v) \quad \text{and} \quad \mathscr{E}(w_j,v_j) \to \mathscr{E}(w,v). \tag{3.7}$$

The first result is standard and the second follows from the first by Proposition 3.3.

§ 4. Differentiability of $\mathscr{E} \circ \Pi$

We turn to the main differentiability result. Our exposition mimics Lemmas 4.10 and 4.11 of [4]; since this is the key ingredient in proving Theorem 5.1 we include all the details. Generally we will fix a function $v \in L_P^+$ which will be in the second argument of all the energy terms and, for any $\widetilde{v} \in L_P^+$, we simply write

$$\mathscr{E}(\widetilde{v}) := \mathscr{E}(\widetilde{v}, v).$$

If we need to emphasize a specific v, we revert to the notation on the right-hand side of this equation. Recall that for $E \subset \mathbb{C}^d$ closed and an admissible weight a on E, we write $\Pi(a)$ (sometimes $\Pi_E(a)$) to denote the regularized weighted P-extremal function $V^*_{P,E,a}$.

We state two versions of differentiability for $\mathscr{E} \circ \Pi$. One version, Proposition 4.1, is for a second admissible weight b on E where we consider the perturbed weight $a + t(b - a)$ and the associated weighted P-extremal function $\Pi(a + t(b - a))$ and we prove that

$$F(t) := \mathscr{E}(\Pi(a + t(b - a)))$$

is differentiable. Taking $v = \Pi(a)$, as we will in Propositions 4.1, 4.2 and Lemma 4.3,

$$F(0) = \mathscr{E}(\Pi(a)) = \mathscr{E}(\Pi(a), \Pi(a)) = 0. \tag{4.1}$$

If E is unbounded, we will need to make an additional assumption on $u := b - a$ so that (2.13) holds; also, in this case, we restrict ourselves to $0 \leqslant t \leqslant 1$ so that $a+t(b-a) = tb+(1-t)a$, being a convex combination of a and b, is admissible on E. Proposition 4.2, the second version of differentiability for $\mathscr{E} \circ \Pi$, is for a compact set K and an arbitrary real t. We take a function $u \in C(K)$, consider the perturbed weight $a + tu$, and show that

$$F(t) := \mathscr{E}(\Pi(a + tu))$$

is differentiable. A priori, since $t \in \mathbb{R}$, we must assume u is continuous so that $a + tu$ is an admissible (lower semicontinuous) weight. The following results utilize Lemma 2.13 and Corollary 2.14; hence we assume that a and b, and/or u are C^2-regular.

Proposition 4.1. *Let $v \in L^+_P$. For admissible weights $a, b \in C^2(E)$ on a closed set $E \subset \mathbb{C}^d$, let $u := b - a$ and let*

$$F(t) := \mathscr{E}(\Pi(a + tu), v))$$

for $t \in \mathbb{R}$. If E is unbounded, assume that (2.16) holds and that $0 \leqslant t \leqslant 1$. Then F is differentiable and

$$F'(t) = (d + 1) \int_{\mathbb{C}^d} u(dd^c\Pi(a + tu))^d. \tag{4.2}$$

Proposition 4.2. *Let $v \in L^+_P$. For an admissible weight a on a compact set $K \subset \mathbb{C}^d$ and $u \in C^2(K)$, let*

$$F(t) := \mathscr{E}(\Pi(a + tu), v),$$

for $t \in \mathbb{R}$. Then F is differentiable and

$$F'(t) = (d + 1) \int_{\mathbb{C}^d} u(dd^c\Pi(a + tu))^d. \tag{4.3}$$

We prove Propositions 4.1 and 4.2 simultaneously.

Proof of Propositions 4.1 and 4.2. We take $v = \Pi(a)$ and we prove the one-sided limit as $t \to 0^+$:

$$F'(0) := \lim_{t\to 0^+} \frac{F(t)-F(0)}{t} = (d+1)\int_{\mathbb{C}^d} u(dd^c\Pi(a))^d. \tag{4.4}$$

As in Remark 3.2, this implies (4.2), for if s is fixed,

$$G(t) := F(s+t) = \mathscr{E}(\Pi(a+(s+t)u), v) = \mathscr{E}(\Pi(a+su+tu), v),$$

and applying (4.4) to G (replacing a by $a+su$) we get

$$G'(0) = F'(s) = (d+1)\int_{\mathbb{C}^d} u(dd^c\Pi(a+su))^d.$$

Note that $F(0) = 0$ (see (4.1)) and to verify (4.4) it suffices to prove that

$$\mathscr{E}(\Pi(a+tu), \Pi(a)) = (d+1)t\int_{\mathbb{C}^d} u(dd^c\Pi(a))^d + o(t). \tag{4.5}$$

We need two ingredients for (4.5):

$$\mathscr{E}(\Pi(a+tu), \Pi(a)) = (d+1)\int_{\mathbb{C}^d} [\Pi(a+tu)-\Pi(a)](dd^c\Pi(a))^d + o(t) \tag{4.6}$$

and

$$\lim_{t\to 0}\int_{D(0)\setminus D(t)} (dd^c\Pi(a))^d = 0, \tag{4.7}$$

where we recall that

$$D(t) := \{z \in \mathbb{C}^d \colon \Pi(a+tu)(z) = (a+tu)(z)\}.$$

We have proved (4.7) in Lemma 2.13. We state and prove (4.6) in a separate lemma. Given (4.6) and (4.7), and observing from (2.6) that

$$\operatorname{supp}(dd^c\Pi(a))^d \subset D(0), \tag{4.8}$$

(4.5) follows as in [4]:

$$\begin{aligned}
\mathscr{E}(\Pi(a+tu), \Pi(a)) &= (d+1)\int_{\mathbb{C}^d} [\Pi(a+tu)-\Pi(a)](dd^c\Pi(a))^d + o(t)\\
&= (d+1)\int_{D(0)\setminus D(t)} [\Pi(a+tu)-\Pi(a)](dd^c\Pi(a))^d\\
&\qquad + (d+1)\int_{D(0)\cap D(t)} [\Pi(a+tu)-\Pi(a)](dd^c\Pi(a))^d + o(t)\\
&= (d+1)\int_{D(0)\setminus D(t)} [\Pi(a+tu)-\Pi(a)](dd^c\Pi(a))^d\\
&\qquad + (d+1)t\int_{D(0)\cap D(t)} u(dd^c\Pi(a))^d + o(t)\\
&= (d+1)\int_{D(0)\setminus D(t)} [\Pi(a+tu)-\Pi(a)-tu](dd^c\Pi(a))^d\\
&\qquad + (d+1)t\int_{D(0)} u(dd^c\Pi(a))^d + o(t),
\end{aligned}$$

since $\Pi(a+tu)-\Pi(a)=tu$ on $D(0)\cap D(t)$. Now (2.13) or (2.15) implies that

$$|\Pi(a+tu)-\Pi(a)-tu|=O(t)$$

on the bounded set $D(0)\setminus D(t)$ (recall if E is unbounded we assume (2.16) holds in the setting of Proposition 4.1) and this fact, combined with (4.7) and (4.8), completes the proof.

In (4.6), since $(dd^c\Pi(a))^d$ is supported in $D(0)$,

$$\int_{\mathbb{C}^d}[\Pi(a+tu)-\Pi(a)](dd^c\Pi(a))^d=\int_{D(0)}[\Pi(a+tu)-\Pi(a)](dd^c\Pi(a))^d,$$

and, on $D(t)\cap D(0)$, we have $\Pi(a+tu)-\Pi(a)=tu$. The meaning of (4.7) is that the contribution to this integral on $D(0)\setminus D(t)$ is negligible. The meaning of (4.6), Lemma 4.3 below, is that the contribution of each of the $d+1$ terms in the energy $\mathscr{E}(\Pi(a+tu),\Pi(a))$ is the same, up to $o(t)$, as that involving the term $(dd^c\Pi(a))^d$. Again we write

$$\begin{aligned}F(t)&:=\mathscr{E}(\Pi(a+tu))=\mathscr{E}(\Pi(a+tu),\Pi(a))\\&=\int[\Pi(a+tu)-\Pi(a)][(dd^c\Pi(a+tu))^d+\cdots+(dd^c\Pi(a))^d].\end{aligned}$$

Another interpretation of (4.6) is that to prove the differentiability of $\mathscr{E}\circ\Pi$, we can replace $\mathscr{E}$ by its 'linearization' at $\Pi(a)$. As in previous arguments, we only give the proof at $t=0$ and for the one-sided limit in (4.3) as $t\to 0^+$. The next result does not require u to be smooth.

Lemma 4.3. *For an admissible weight a on E and $u\in C(E)$, let*

$$\begin{aligned}F(t)&=\mathscr{E}(\Pi(a+tu))\\&=\int[\Pi(a+tu)-\Pi(a)][(dd^c\Pi(a+tu))^d+\cdots+(dd^c\Pi(a))^d]\end{aligned}$$

and

$$G(t):=(d+1)\int[\Pi(a+tu)-\Pi(a)](dd^c\Pi(a))^d.$$

Then

$$\lim_{t\to 0^+}\frac{F(t)-F(0)}{t}=\lim_{t\to 0^+}\frac{G(t)-G(0)}{t}.$$

Proof. Note that $F(0)=\mathscr{E}(\Pi(a))=0$ and $G(0)=0$. By the concavity of Π (recall (2.12)) and the linearity of the map $f\to\int f(dd^c\Pi(a))^d$, the function $G(t)$ is concave so that

$$A:=\lim_{t\to 0^+}\frac{G(t)-G(0)}{t}\tag{4.9}$$

exists. By the concavity of $\mathscr{E}$, we have (recall (3.5))

$$\mathscr{E}(\Pi(a+tu))\leqslant\mathscr{E}(\Pi(a))+\langle\mathscr{E}'(\Pi(a)),\Pi(a+tu)-\Pi(a)\rangle,$$

that is, from (3.6) with $u_1 = \Pi(a)$, $u_2 = \Pi(a+tu)$ and $v = \Pi(a)$,

$$\mathscr{E}(\Pi(a+tu)) \leqslant \mathscr{E}(\Pi(a)) + (d+1)\int [\Pi(a+tu) - \Pi(a)](dd^c\Pi(a))^d.$$

Thus

$$\limsup_{t\to 0^+} \frac{F(t)-F(0)}{t} \leqslant A.$$

We will prove that

$$\liminf_{t\to 0^+} \frac{F(t)-F(0)}{t} \geqslant A.$$

Since $A := \lim_{t\to 0^+} \frac{G(t)-G(0)}{t}$ exists, given $\varepsilon > 0$ we can choose $\delta > 0$ sufficiently small so that

$$\frac{G(\delta)-G(0)}{\delta} = \frac{d+1}{\delta}\int [\Pi(a+\delta u) - \Pi(a)](dd^c\Pi(a))^d \geqslant A - \varepsilon,$$

that is,

$$(d+1)\int [\Pi(a+\delta u) - \Pi(a)](dd^c\Pi(a))^d \geqslant \delta(A-\varepsilon).$$

From Proposition 3.1, for $t > 0$ sufficiently small we have

$$\begin{aligned}\frac{\mathscr{E}(\Pi(a) + t[\Pi(a+\delta u) - \Pi(a)]) - \mathscr{E}(\Pi(a))}{t}& \\ \geqslant (d+1)\int [\Pi(a+\delta u) - \Pi(a)](dd^c\Pi(a))^d - \delta\varepsilon,&\end{aligned}$$

that is,

$$\begin{aligned}\mathscr{E}((1-t)\Pi(a) + t\Pi(a+\delta u)) &= \mathscr{E}(\Pi(a) + t[\Pi(a+\delta u) - \Pi(a)]) \\ &\geqslant \mathscr{E}(\Pi(a)) + t(d+1)\int [\Pi(a+\delta u) - \Pi(a)](dd^c\Pi(a))^d - t\delta\varepsilon.\end{aligned}$$

Combining these last two inequalities, we have

$$\mathscr{E}((1-t)\Pi(a) + t\Pi(a+\delta u)) \geqslant \mathscr{E}(\Pi(a)) + t\delta A - 2t\delta\varepsilon.$$

As Π is concave,

$$\Pi(a+t\delta u) = \Pi((1-t)a + t(a+\delta u)) \geqslant (1-t)\Pi(a) + t\Pi(a+\delta u),$$

so that, since $\mathscr{E}$ is monotonic,

$$\mathscr{E}(\Pi(a+t\delta u)) \geqslant \mathscr{E}((1-t)\Pi(a) + t\Pi(a+\delta u)) \geqslant \mathscr{E}(\Pi(a)) + t\delta A - 2t\delta\varepsilon$$

for $t > 0$ sufficiently small. Thus,

$$\liminf_{t\to 0^+} \frac{F(t)-F(0)}{t} \geqslant A - 2\varepsilon$$

for all $\varepsilon > 0$, yielding the result. The lemma is proved.

We now finish the proof of Proposition 4.1 and Proposition 4.2 by finding A in (4.9). The proof that $A = \int u(dd^c\Pi(a))^d$ was essentially given in the verification of (4.5) assuming (4.6) and (4.7) hold; for the reader's convenience, we give the details. We write $S_a := \operatorname{supp}(dd^c\Pi(a))^d$. For each t, $D(t) = \{z \in \mathbb{C}^d : \Pi(a+tu)(z) = a(z) + tu(z)\}$ is a bounded set. From Proposition 2.5, $\Pi(a) = a$ $(dd^c\Pi(a))^d$-a.e. on $S_a \subset D(0)$; thus

$$\begin{aligned}
\int [\Pi(a+tu) - \Pi(a)](dd^c\Pi(a))^d &= \int_{S_a} [\Pi(a+tu) - \Pi(a)](dd^c\Pi(a))^d \\
&= \int_{D(t)\cap S_a} [\Pi(a+tu) - \Pi(a)](dd^c\Pi(a))^d \\
&\quad + \int_{S_a\setminus D(t)} [\Pi(a+tu) - \Pi(a)](dd^c\Pi(a))^d \\
&= \int_{D(t)\cap S_a} [a+tu-a](dd^c\Pi(a))^d + \int_{S_a\setminus D(t)} [\Pi(a+tu) - \Pi(a)](dd^c\Pi(a))^d \\
&= \int_{D(t)\cap S_a} tu(dd^c\Pi(a))^d + \int_{S_a\setminus D(t)} [\Pi(a+tu) - \Pi(a)](dd^c\Pi(a))^d \\
&= \int_{S_a} tu(dd^c\Pi(a))^d + \int_{S_a\setminus D(t)} [\Pi(a+tu) - \Pi(a) - tu](dd^c\Pi(a))^d.
\end{aligned}$$

Now we use the observation (2.13) (or (2.15)) to see that

$$|\Pi(a+tu) - \Pi(a) - tu| = O(t)$$

on the bounded set $S_a \setminus D(t)$; the conclusion follows from Lemma 2.13.

Propositions 4.1 and 4.2 are proved.

We give integrated versions of Proposition 4.1 and Proposition 4.2 which we will use.

Proposition 4.4. *For admissible weights $a, b \in C^2(E)$ on an unbounded closed set E satisfying (2.16),*

$$\mathscr{E}(\Pi(b), \Pi(a)) = (d+1) \int_{t=0}^{1} dt \int_{\mathbb{C}^d} (b-a)(dd^c\Pi(a+t(b-a)))^d, \tag{4.10}$$

and for a compact set K with admissible weight a and $u \in C^2(K)$,

$$\mathscr{E}(\Pi(a+u), \Pi(a)) = (d+1) \int_{t=0}^{1} dt \int_{\mathbb{C}^d} u(dd^c\Pi(a+tu))^d. \tag{4.11}$$

Proof. We will only prove (4.10) as (4.11) is similar. We begin with Proposition 4.1 using $v = \Pi(a)$ so that $F(t) = \mathscr{E}(\Pi(a+t(b-a)), \Pi(a))$ and (4.2) becomes

$$F'(t) = (d+1) \int_{\mathbb{C}^d} (b-a)(dd^c\Pi(a+t(b-a)))^d.$$

Integrating this expression from $t = 0$ to $t = 1$ gives (4.10) since $F(1) - F(0) = \mathscr{E}(\Pi(b), \Pi(a))$. The proposition is proved.

§ 5. The main theorem

In this section, we state and prove the main result which relates the asymptotics of certain ball-volume ratios with the energies associated with P-extremal functions. For $E \subset \mathbb{C}^d$ closed, following the notation in [4], we let φ be an admissible weight on E. Let

$$\mathscr{B}^\infty(E, n\varphi) := \{p_n \in \mathrm{Poly}(nP)\colon |p_n(z)^2 e^{-2n\varphi(z)}| \leqslant 1 \text{ on } E\}$$

be an L^∞-ball and, if μ is a measure on E, let

$$\mathscr{B}^2(E, \mu, n\varphi) := \left\{p_n \in \mathrm{Poly}(nP)\colon \int_E |p_n|^2 e^{-2n\varphi} d\mu \leqslant 1\right\}$$

be an L^2-ball in $\mathrm{Poly}(nP)$. The key result is the following.

Theorem 5.1. *Given admissible weights φ and φ' on E and E',*

$$\lim_{n\to\infty} \frac{-(d+1)\gamma_d}{2nd_n} [\mathscr{B}^\infty(E, n\varphi) : \mathscr{B}^\infty(E', n\varphi')] = \mathscr{E}(V^*_{P,E,\varphi}, V^*_{P,E',\varphi'}).$$

If μ and μ' are measures on E and E' where μ is a Bernstein-Markov measure for (P, E, φ) and μ' is a Bernstein-Markov measure for (P, E', φ'), then

$$\lim_{n\to\infty} \frac{-(d+1)\gamma_d}{2nd_n} [\mathscr{B}^2(E, \mu, n\varphi) : \mathscr{B}^2(E', \mu', n\varphi')] = \mathscr{E}(V^*_{P,E,\varphi}, V^*_{P,E',\varphi'}).$$

Remark 5.2. We will verify Conjecture 2.21 using Theorem 5.1. Taking $E' = T$ and $\varphi' = 0$, from (2.7) we have $V^*_{P,E',\varphi'} = H_P$. Setting $\mu' = \mu_T$ and taking (K, μ, Q) for the triple (E, μ, φ) where K is compact and μ is a Bernstein-Markov measure for (P, K, Q), we use (2.36) and (2.25) to obtain (2.39), and so the limit

$$\lim_{n\to\infty} \frac{1}{2l_n} \log\det(G_n^{\mu,w}) = \frac{-1}{\gamma_d d\mathscr{A}} \mathscr{E}(V^*_{P,K,Q}, H_P) = \log \mathscr{F}_P(K, Q) \tag{5.1}$$

exists. Thus we obtain the asymptotics of weighted Gram determinants associated to (K, μ, Q) as well as the other results mentioned in § 2, namely, the existence of the limit of the scaled maximal weighted Vandermonde determinants

$$\delta^w(K) := \lim_{n\to\infty} \delta^{w,n}(K) = \mathscr{F}_P(K, Q)$$

in (2.22) and Proposition 2.22 on P-optimal measures.

Before proving Theorem 5.1, we discuss an immediate application in the next subsection.

5.1. A Rumely formula. We prove an 'energy version' of a remarkable formula due to Rumely [13] relating the weighted transfinite diameter and the energy of the weighted extremal function.

Theorem 5.3. *Let $K \subset \mathbb{C}^d$ be compact and $w = e^{-Q}$ with $Q \in C(K)$. Then*

$$\log \delta^w(K) = \frac{-1}{\gamma_d d\mathscr{A}} \mathscr{E}(V^*_{P,K,Q}, H_P). \tag{5.2}$$

Proof. We first recall from Remark 2.11 that the monomials $\beta_n = \{z^J,\ J \in nP \cap (\mathbb{Z}^+)^d\}$ form an orthonormal basis for $\text{Poly}(nP)$ with respect to the normalized Haar measure μ_T on T^d. Since μ_T is a strong Bernstein-Markov measure for T it is a Bernstein-Markov measure for the triple (P, T, q) for any $q \in C(T)$.

Next, from Remark 2.10, if ν is a strong Bernstein-Markov measure for K, then for any $Q \in C(K)$, ν is a Bernstein-Markov measure for the triple (P, K, Q). Fixing such a ν, from (2.36),

$$\log\det G_n^{\nu,w}(\beta_n) = [\mathscr{B}^2(K,\nu,nQ) : \mathscr{B}^2(T,\mu_T,n\cdot 0)].$$

Thus, on the one hand, by Remark 5.2,

$$\lim_{n\to\infty} \frac{1}{2l_n}\cdot \log\det G_n^{\nu,w}(\beta_n) = \log\delta^w(K),$$

while on the other hand, by Theorem 5.1,

$$\lim_{n\to\infty} \frac{(d+1)\gamma_d}{2nd_n}[\mathscr{B}^2(T,\mu_T,n\cdot 0) : \mathscr{B}^2(K,\nu,nQ)] = \mathscr{E}(V^*_{P,K,Q}, H_P),$$

yielding the result. The theorem is proved.

The rest of this section is devoted to the proof of Theorem 5.1. The first step, § 5.2, is a preliminary result on a version of Bergman asymptotics due to Berman [3]. We will use this in § 5.3.

5.2. The asymptotics of weighted Bergman functions in $\mathbb{C}^d$. We state a result on the asymptotics of Bergman functions given in [3]. The setting is this: $\varphi \in C^{1,1}(\mathbb{C}^d)$ with

$$\varphi(z) \geqslant (1+\varepsilon)H_P(z) \quad \text{for } |z| >> 1 \text{ for some } \varepsilon > 0. \tag{5.3}$$

We will call a global admissible weight φ satisfying (5.3) *strongly admissible*. For $p_n \in \text{Poly}(nP)$, we write

$$\|p_n\|^2_{n\varphi} := \|p_n\|^2_{\omega_d,n\varphi} = \int_{\mathbb{C}^d} |p_n(z)|^2 e^{-2n\varphi(z)}\omega_d(z),$$

where ω_d is Lebesgue measure on $\mathbb{C}^d$. Using (2.2), under the growth assumption on φ, for each polynomial $p_n \in \text{Poly}(nP)$, $\|p_n\|_{n\varphi} < +\infty$.

Given an orthonormal basis $\{q_1, \ldots, q_{d_n}\}$ of $\text{Poly}(nP)$, in this section we use the notation

$$B_{n,\varphi}(z) := \left[\sum_{j=1}^{d_n} |q_j(z)|^2\right] e^{-2n\varphi(z)}$$

for the nth Bergman function; and we recall that

$$B_{n,\varphi}(z) = \sup_{p_n\in\text{Poly}(nP)\setminus\{0\}} \frac{|p_n(z)|^2 e^{-2n\varphi(z)}}{\|p_n\|^2_{n\varphi}}.$$

Finally, let

$$S := \{z \in \mathbb{C}^d \colon dd^c\varphi(z) \text{ exists and } dd^c\varphi(z) > 0\},$$

and if u is a $C^{1,1}$ function such that $(dd^c u)^d$ is absolutely continuous with respect to Lebesgue measure, we write

$$\det(dd^c u)\omega_d := (dd^c u)^d.$$

Theorem 5.4. *Given $\varphi \in C^{1,1}(\mathbb{C}^d)$ satisfying (5.3), the following results hold: $V_{P,\mathbb{C}^d,\varphi} \in C^{1,1}(\mathbb{C}^d)$; the Monge-Ampère measure $(dd^c V_{P,\mathbb{C}^d,\varphi})^d$ has compact support and is absolutely continuous with respect to Lebesgue measure;*

$$(dd^c V_{P,\mathbb{C}^d,\varphi})^d = \det(dd^c V_{P,\mathbb{C}^d,\varphi})\omega_d$$

as (d,d)-forms with $L^\infty_{\mathrm{loc}}(\mathbb{C}^d)$ coefficients; and a.e. on the set

$$D := \{V_{P,\mathbb{C}^d,\varphi} = \varphi\}$$

we have $\det(dd^c\varphi) = \det(dd^c V_{P,\mathbb{C}^d,\varphi})$. Moreover,

$$\frac{\gamma_d}{d_n} B_{n,\varphi} \to \chi_{D\cap S} \det(dd^c\varphi) \quad \text{in } L^1(\mathbb{C}^d),$$

and the measures

$$\frac{\gamma_d}{d_n} B_{n,\varphi}\omega_d \to (dd^c V_{P,\mathbb{C}^d,\varphi})^d \text{ weakly.}$$

Recall the (strong) admissibility of φ implies, by Proposition 2.5, that $(dd^c V_{P,\mathbb{C}^d,\varphi})^d$ has compact support.

Remark 5.5. From [6], $(D, \omega_d|_D, \varphi|_D)$ satisfies a weighted Bernstein-Markov property for $\mathscr{P}_n$ or $A\mathscr{P}_n$; from Remark 2.10, $\omega_d|_D$ is a Bernstein-Markov measure for the triple (P, D, φ). Using Proposition 2.9,

$$\sup_{\mathbb{C}^d} |p_n e^{-n\varphi}| = \sup_D |p_n e^{-n\varphi}|$$

for $p_n \in \mathrm{Poly}(nP)$. Hence, from (2.9),

$$\sup_{\mathbb{C}^d} |p_n e^{-n\varphi}| \leqslant M_n \left[\int_D |p_n|^2 e^{-2n\varphi}\omega_d\right]^{1/2} \leqslant M_n \left[\int_{\mathbb{C}^d} |p_n|^2 e^{-2n\varphi}\omega_d\right]^{1/2},$$

where $M_n^{1/n} \to 1$. This last integral is finite by (2.2).

5.3. Proof of Theorem 5.1. We consider several cases.

Case 1: *$E = E' = \mathbb{C}^d$ and $\varphi, \varphi' \in C^2(\mathbb{C}^d)$ are strongly admissible with $\varphi' = \varphi$ outside a ball $\mathscr{B}_R$ for some R; $d\mu = d\mu' = \omega_d$.* We first consider Case 1 in L^2. Note that (2.16) holds, as in this case all of the weights $\varphi + t(\varphi' - \varphi)$ are strongly admissible with a uniform ε (recall (5.3)). Let $u := \varphi' - \varphi$; then u is continuous with compact support. For $0 \leqslant t \leqslant 1$ let

$$\varphi_t := \varphi + tu = \varphi + t(\varphi' - \varphi) = (1-t)\varphi + t\varphi',$$

so that $\varphi_0 = \varphi$ and $\varphi_1 = \varphi'$; equivalently, $w_t(z) := w(z)\exp(-tu(z))$ (note $w_0 = w = e^{-\varphi}$ and $w_1 = w' = e^{-\varphi'}$). Then from Theorem 5.4, for each t,

$$\frac{\gamma_d}{d_n} B_{n,\varphi+tu}\omega_d \to (dd^c\Pi(\varphi + tu))^d \text{ weakly.}$$

Now set

$$f_n(t) := -\frac{1}{2l_n}\log\det(G_n^{\mu,w_t}(\beta_n)),$$

where $\mu = \mu_n := \omega_d$ for all n and the basis $\beta_n := \{p_1, \ldots, p_{d_n}\}$ of $\mathrm{Poly}(nP)$ is chosen to be an orthonormal basis with respect to the weighted L^2-norm $p \to \|w^n p\|_{L^2(\mu)}$. Then $G_n^{\mu,w}(\beta_n)$ is the identity $(d_n \times d_n)$-matrix so that we have $f_n(0) = 0$; and, using Lemma 2.18 and the fact that u has compact support (thus all the weights w_t are admissible),

$$\lim_{n\to\infty}\frac{l_n}{nd_n}f_n'(t) = \lim_{n\to\infty}\frac{1}{d_n}\int uB_{n,\varphi+tu}\omega_d = \frac{1}{\gamma_d}\int u(dd^c\Pi(\varphi+tu))^d.$$

We now integrate $\frac{l_n}{nd_n}f_n'(t)$ from $t=0$ to $t=1$:

$$\begin{aligned}\frac{l_n}{nd_n}[f_n(1)-f_n(0)] &= \frac{l_n}{nd_n}[f_n(1)] = \frac{-1}{2nd_n}\log\det(G_n^{\mu,w'}(\beta_n)) \\ &\overset{(2.35)}{=} \frac{-1}{2nd_n}[\mathscr{B}^2(\mathbb{C}^d,\mu,n\varphi):\mathscr{B}^2(\mathbb{C}^d,\mu,n\varphi')] \\ &\overset{\text{Lemma 2.18}}{=} \frac{1}{d_n}\int_{t=0}^1 dt\int B_{n,\varphi+tu}(\varphi-\varphi')\omega_d \\ &\to \frac{1}{\gamma_d}\int_{t=0}^1 dt\int(\varphi-\varphi')(dd^c\Pi(\varphi+tu))^d.\end{aligned}$$

But by (4.10), since (2.16) holds,

$$(d+1)\int_{t=0}^1 dt\int(\varphi-\varphi')(dd^c\Pi(\varphi+tu))^d = \mathscr{E}(\Pi(\varphi'),\Pi(\varphi)),$$

which proves Theorem 5.1 in Case 1 for L^2. By Remark 5.5 this also proves Case 1 for L^∞.

Case 2: $E = E' = \mathbb{C}^d$ *and* $\varphi, \varphi' \in C^2(\mathbb{C}^d)$ *are strongly admissible*; $d\mu = d\mu' = \omega_d$. We first do Case 2 for L^∞. Remark 2.6 and Proposition 2.3 imply that

$$\Pi(\varphi) = \Pi_{S_w}(\varphi|_{S_w}),$$

where $S_w = \mathrm{supp}(dd^c\Pi(\varphi))^d$ is compact; moreover, for $p_n \in \mathrm{Poly}(nP)$, from Proposition 2.9, $\|p_n e^{-n\varphi}\|_{S_w} = \|p_n e^{-n\varphi}\|_{\mathbb{C}^d}$ so that

$$\mathscr{B}^\infty(S_w, n\varphi|_{S_w}) = \mathscr{B}^\infty(\mathbb{C}^d, n\varphi).$$

Thus modifying φ and φ' outside a large ball in such a way as to make them equal outside a possibly larger ball, we neither change the L^∞-ball volume ratios nor the P-extremal functions $\Pi(\varphi)$ and $\Pi(\varphi')$. Hence Case 2 for L^∞ follows from Case 1 for L^∞. By Remark 5.5 this also proves Case 2 for L^2.

Case 3 (general): $E, E' \subset \mathbb{C}^d$ *closed with admissible weights* φ *and* φ'; μ *and* μ' *are Bernstein-Markov measures for* $(P,E,\varphi), (P,E',\varphi')$. We only consider Case 3 for L^∞; Case 3 for L^2 follows from the definition of Bernstein-Markov measure for (P,E,φ) and (P,E',φ'). We claim that by the cocycle property for the ball volume ratios $[A : B]$ and energies $\mathscr{E}(u_1, u_2)$, we can assume that one of the sets is $\mathbb{C}^d$ with a strongly admissible $C^2(\mathbb{C}^d)$ weight $\widehat{\varphi}$. For, using the notation $\Pi_E(\varphi) := V^*_{P,E,\varphi}$, we have

$$\mathscr{E}(\Pi_E(\varphi), \Pi_{E'}(\varphi')) = -\mathscr{E}(\Pi_{E'}(\varphi'), \Pi_{\mathbb{C}^d}(\widehat{\varphi})) + \mathscr{E}(\Pi_E(\varphi), \Pi_{\mathbb{C}^d}(\widehat{\varphi})).$$

The second argument in both terms on the right is $\Pi_{\mathbb{C}^d}(\widehat{\varphi})$. Similarly, with respect to the ball volume ratios, for each n we have

$$\begin{aligned}&[\mathscr{B}^\infty(E, n\varphi) : \mathscr{B}^\infty(E', n\varphi')] \\ &\quad = -[\mathscr{B}^\infty(E', n\varphi') : \mathscr{B}^\infty(\mathbb{C}^d, n\widehat{\varphi})] + [\mathscr{B}^\infty(E, n\varphi) : \mathscr{B}^\infty(\mathbb{C}^d, n\widehat{\varphi})].\end{aligned}$$

Now to deduce the case where one of the sets is $\mathbb{C}^d$ with a strongly admissible $C^2(\mathbb{C}^d)$ weight $\widehat{\varphi}$ and the other is a general closed set E with admissible weight φ from Case 2 where both sets are $\mathbb{C}^d$ with strongly admissible $C^2(\mathbb{C}^d)$ weights $\widehat{\varphi}$ and ψ, we first observe that we can assume that E is compact (that is, bounded). This follows because, again by Proposition 2.3, if $w = e^{-\varphi}$, then $\Pi_E(\varphi) = \Pi_{S_w}(\varphi|_{S_w})$ where $S_w = \operatorname{supp}(dd^c\Pi_E(\varphi))^d$ is compact, and for $p_n \in \operatorname{Poly}(nP)$, $\|p_n e^{-n\varphi}\|_{S_w} = \|p_n e^{-n\varphi}\|_E$ so that

$$\mathscr{B}^\infty(S_w, n\varphi|_{S_w}) = \mathscr{B}^\infty(E, n\varphi).$$

Thus we can assume that E is compact; since $V^*_{P,E,\varphi} \in L^+_P$, we can also assume φ is bounded above on E. We take a large sublevel set $B_R := \{z \in \mathbb{C}^d\colon H_P(z) < \log R\}$ containing E and extend φ from E to $\widehat{\psi}$ on $\mathbb{C}^d$:

$$\widehat{\psi} := \varphi \quad \text{on } E; \qquad \widehat{\psi} = 2\log R \quad \text{on } B_R \setminus E; \qquad \widehat{\psi} = 2kH_P(z) \quad \text{on } \mathbb{C}^d \setminus B_R.$$

Now, $\widehat{\psi}$ is lower semicontinuous (recall (2.2)) and, taking R sufficiently large, $\Pi_{\mathbb{C}^d}(\widehat{\psi}) = \Pi_E(\varphi)$; then we take a sequence of strongly admissible $C^2(\mathbb{C}^d)$ weights $\{\varphi_j\}$ with $\varphi_j \uparrow \widehat{\psi}$. We can apply Case 2 to $(\mathbb{C}^d, \varphi_j)$ and $(\mathbb{C}^d, \widehat{\varphi})$ to conclude that

$$\lim_{n\to\infty} \frac{-(d+1)\gamma_d}{2nd_n}[\mathscr{B}^\infty(\mathbb{C}^d, n\varphi_j) : \mathscr{B}^\infty(\mathbb{C}^d, n\widehat{\varphi})] = \mathscr{E}(\Pi_{\mathbb{C}^d}(\varphi_j), \Pi_{\mathbb{C}^d}(\widehat{\varphi})).$$

But $\varphi_j \uparrow \widehat{\psi}$ implies that $\Pi_{\mathbb{C}^d}(\varphi_j) \uparrow \Pi_{\mathbb{C}^d}(\widehat{\psi}) = \Pi_E(\varphi)$ a.e., and hence

$$\mathscr{E}(\Pi_{\mathbb{C}^d}(\varphi_j), \Pi_{\mathbb{C}^d}(\widehat{\varphi})) \quad \text{converges to} \quad \mathscr{E}(\Pi_E(\varphi), \Pi_{\mathbb{C}^d}(\widehat{\varphi})) \tag{5.4}$$

as $j \to \infty$ by Lemma 3.5.

We want to conclude that

$$\lim_{n\to\infty} \frac{-(d+1)\gamma_d}{2nd_n}[\mathscr{B}^\infty(E, n\varphi) : \mathscr{B}^\infty(\mathbb{C}^d, n\widehat{\varphi})] = \mathscr{E}(\Pi_E(\varphi), \Pi_{\mathbb{C}^d}(\widehat{\varphi})). \tag{5.5}$$

To this end, first observe that

$$\begin{aligned}-\mathscr{E}(\Pi_{\mathbb{C}^d}(\varphi_j), \Pi_{\mathbb{C}^d}(\widehat{\varphi})) &= \lim_{n\to\infty} \frac{(d+1)\gamma_d}{2nd_n}[\mathscr{B}^\infty(\mathbb{C}^d, n\varphi_j) : \mathscr{B}^\infty(\mathbb{C}^d, n\widehat{\varphi})] \\ &\leqslant \liminf_{n\to\infty} \frac{(d+1)\gamma_d}{2nd_n}[\mathscr{B}^\infty(E, n\varphi) : \mathscr{B}^\infty(\mathbb{C}^d, n\widehat{\varphi})] \\ &\leqslant \limsup_{n\to\infty} \frac{(d+1)\gamma_d}{2nd_n}[\mathscr{B}^\infty(E, n\varphi) : \mathscr{B}^\infty(\mathbb{C}^d, n\widehat{\varphi})],\end{aligned}$$

since $\Pi_{\mathbb{C}^d}(\varphi_j) \uparrow \Pi_E(\varphi)$ implies from (2.41) that

$$[\mathscr{B}^\infty(\mathbb{C}^d, n\varphi_j) : \mathscr{B}^\infty(\mathbb{C}^d, n\widehat{\varphi})] \leqslant [\mathscr{B}^\infty(E, n\varphi) : \mathscr{B}^\infty(\mathbb{C}^d, n\widehat{\varphi})].$$

Now we take a sequence of smooth, strongly admissible weights $\{\psi_j\}$ on $\mathbb{C}^d$ with $\psi_j \downarrow \Pi_E(\varphi)$; for instance, we can take $\psi_j = (1+\varepsilon_j)[(\Pi_E(\varphi))_{\varepsilon_j}]$ where $(\Pi_E(\varphi))_{\varepsilon_j}$ is a smoothing of $\Pi_E(\varphi)$. Then $\Pi_{\mathbb{C}^d}(\psi_j) \downarrow \Pi_E(\varphi)$ and

$$\begin{aligned}\limsup_{n\to\infty} \frac{(d+1)\gamma_d}{2nd_n}&[\mathscr{B}^\infty(E, n\varphi) : \mathscr{B}^\infty(\mathbb{C}^d, n\widehat{\varphi})] \\ &\leqslant \lim_{n\to\infty} \frac{(d+1)\gamma_d}{2nd_n}[\mathscr{B}^\infty(\mathbb{C}^d, n\psi_j) : \mathscr{B}^\infty(\mathbb{C}^d, n\widehat{\varphi})]\end{aligned}$$

again by (2.41); this limit equals

$$-\mathscr{E}(\Pi_{\mathbb{C}^d}(\psi_j), \Pi_{\mathbb{C}^d}(\widehat{\varphi}))$$

by applying Case 2, this time to $(\mathbb{C}^d, \psi_j)$ and $(\mathbb{C}^d, \widehat{\varphi})$. Now

$$\mathscr{E}(\Pi_{\mathbb{C}^d}(\psi_j), \Pi_{\mathbb{C}^d}(\widehat{\varphi})) \quad \text{converges to} \quad \mathscr{E}(\Pi_E(\varphi), \Pi_{\mathbb{C}^d}(\widehat{\varphi})) \tag{5.6}$$

as $j\to\infty$ by (3.7). Then (5.4) and (5.6) imply (5.5) which completes the proof of Theorem 5.1.

§ 6. Asymptotic weighted *P*-Fekete measures, weighted *P*-optimal measures and the asymptotics of Bergman functions

As in [5], we will apply the following calculus lemma (cf. Lemma 7.6 in [4] or Lemma 3.1 in [5]) to an appropriate sequence of real-valued functions $\{f_n\}$ in order to prove a general result, Proposition 6.2, on convergence to the Monge-Ampère measure of a weighted P-extremal function. This proposition utilizes the differentiability result, Proposition 4.2, and yields immediate corollaries on the items in the title of this section.

Lemma 6.1. *Let f_n be a sequence of real-valued, concave functions on $\mathbb{R}$ and let g be a function on $\mathbb{R}$. Suppose*

$$\liminf_{n\to\infty} f_n(t) \geqslant g(t) \textit{ for all } t \quad \textit{and} \quad \lim_{n\to\infty} f_n(0) = g(0),$$

and that f_n and g are differentiable at 0. Then $\lim_{n\to\infty} f_n'(0) = g'(0)$.

Here 'differentiable at the origin' means that the usual (two-sided) limit of the difference quotients exists; the conclusion is not true with one-sided limits.

As in Lemma 2.18 in § 2.4, given a closed set E, an admissible weight $w = e^{-Q}$ on E, and a function $u \in C(E)$, we consider the weight $w_t(z) := w(z)\exp(-tu(z))$, $t\in\mathbb{R}$, and we let $\{\mu_n\}$ be a sequence of measures on E.

For the rest of this section, we take $E = K$, a compact set, so each w_t is admissible. In addition, in computing Gram matrices, we fix the standard monomial basis $\beta_n = \{e_1, \dots, e_{d_n}\}$ of $\operatorname{Poly}(nP)$; and we fix $v = H_P$ in the second argument of $\mathscr{E}(u, v)$.

Now let μ be a probability measure on K and let $u \in C^2(K)$. Recalling (2.25), define

$$g(t) := -\log \delta^{w_t}(K) = \frac{1}{\gamma_d d\mathscr{A}} \mathscr{E}(\Pi(Q + tu)).$$

Then

$$g(0) = -\log \delta^{w}(K) = \frac{1}{\gamma_d d\mathscr{A}} \mathscr{E}(\Pi(Q)).$$

From Proposition 4.2

$$g'(0) = \frac{d+1}{\gamma_d d\mathscr{A}} \int_K u(z)(dd^c \Pi(Q))^d.$$

Note that for each n, μ_n is a candidate to be a P-optimal measure of order n for K and w_t. Thus, if μ_n^t is a P-optimal measure of order n for K and w_t, we have

$$\det G_n^{\mu_n, w_t} \leqslant \det G_n^{\mu_n^t, w_t}$$

and, from Proposition 2.22 (see Remark 5.2),

$$\lim_{n\to\infty} \frac{1}{2l_n} \log\det G_n^{\mu_n^t, w_t} = \log \delta^{w_t}(K) = -g(t).$$

Thus with

$$f_n(t) := -\frac{1}{2l_n} \log\det(G_n^{\mu_n, w_t})$$

as in (2.31), we have

$$f_n(0) = -\frac{1}{2l_n} \log\det(G_n^{\mu_n, w})$$

and $\liminf_{n\to\infty} f_n(t) \geqslant g(t)$ for all t. From Lemma 2.18, we have

$$f_n'(0) = \frac{n}{l_n} \int_K u(z) B_n^{\mu_n, w}(z)\, d\mu_n,$$

and from Lemma 2.19, the functions $f_n(t)$ are concave, that is, $f_n''(t) \leqslant 0$.

Using Lemma 6.1 and (2.25), we have the following general result.

Proposition 6.2. *Let $K \subset \mathbb{C}^d$ be compact with admissible weight w. Let $\{\mu_n\}$ be a sequence of probability measures on K with the property that*

$$\lim_{n\to\infty} \frac{1}{2l_n} \log\det(G_n^{\mu_n, w}) = \log \mathscr{F}_P(K, Q), \tag{6.1}$$

that is, $\lim_{n\to\infty} f_n(0) = g(0)$. Then

$$\frac{n}{l_n} B_n^{\mu_n, w} d\mu_n \to \frac{d+1}{\gamma_d d\mathscr{A}} (dd^c \Pi(Q))^d \quad \textit{weak-}*,$$

that is,

$$\frac{\gamma_d}{d_n} B_n^{\mu_n, w} d\mu_n \to (dd^c \Pi(Q))^d \quad \textit{weak-}*. \tag{6.2}$$

Note that since all the μ_n are probability measures on K, to verify weak-$*$ convergence, it suffices to test with C^2-functions on K.

From Theorem 5.1 (more precisely, Remark 5.2 and equation (5.1)) we have the general result on the asymptotics of Bergman functions.

Corollary 6.3 (asymptotics of Bergman functions). *If μ is a Bernstein-Markov measure for the triple (P, K, Q), then*

$$\frac{\gamma_d}{d_n} B_n^{\mu,w} d\mu \to (dd^c \Pi(Q))^d \quad \textit{weak-}*.$$

Next, suppose μ_n is a P-optimal measure of order n for K and w.

Corollary 6.4 (weighted P-optimal measures). *Let $K \subset \mathbb{C}^d$ be compact with admissible weight w. Let $\{\mu_n\}$ be a sequence of P-optimal measures for K and w. Then*

$$\mu_n \to \frac{1}{\gamma_d}(dd^c \Pi(Q))^d \quad \textit{weak-}*.$$

Proof. We have $B_n^{\mu_n,w} = d_n$ μ_n-a.e. on K from 2.34 so that the result follows immediately from Proposition 2.22 and Proposition 6.2, specifically, equation (6.2).

Finally, we prove the result promised in § 2.3.

Corollary 6.5 (asymptotic weighted P-Fekete measures). *Let $K \subset \mathbb{C}^d$ be compact with admissible weight w. For each n, take points $z_1^{(n)}, z_2^{(n)}, \dots, z_{d_n}^{(n)} \in K$ for which*

$$\lim_{n\to\infty} \left[\left| \mathrm{VDM}(z_1^{(n)}, \dots, z_{d_n}^{(n)}) \right| w(z_1^{(n)})^n \cdots w(z_{d_n}^{(n)})^n \right]^{1/l_n} = \mathscr{F}_P(K, Q) \tag{6.3}$$

(asymptotically weighted P-Fekete arrays) and let $\mu_n := \frac{1}{d_n}\sum_{j=1}^{d_n} \delta_{z_j^{(n)}}$. Then

$$\mu_n \to \frac{1}{\gamma_d}(dd^c \Pi(Q))^d \quad \textit{weak-}*.$$

Proof. By direct calculation, we have $B_n^{\mu_n,w}(z_j^{(n)}) = d_n$ for $j = 1, \dots, d_n$, and hence μ_n-a.e. on K. Indeed, this property holds for any discrete, equally weighted measure $\mu_n := \frac{1}{d_n}\sum_{j=1}^{d_n} \delta_{z_j^{(n)}}$ with

$$\left| \mathrm{VDM}(z_1^{(n)}, \dots, z_{d_n}^{(n)}) \right| w(z_1^{(n)})^n \cdots w(z_{d_n}^{(n)})^n \neq 0.$$

Using

$$\det(G_n^{\mu_n,w}) = \frac{1}{d_n^{d_n}} \left| \mathrm{VDM}(z_1^{(n)}, \dots, z_{d_n}^{(n)}) \right|^2 w(z_1^{(n)})^{2n} \cdots w(z_{d_n}^{(n)})^{2n}$$

the result follows from Proposition 6.2, specifically, equation (6.2).

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Received 25/DEC/16 and 21/MAR/17

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