PUBLICATIONS IN RANK

(1) T. Bloom, N. Levenberg, F. Piazzon and F. Wielonsky, Bernstein-Markov: a survey, *Dolomites Research Notes on Approximation*, Vol. 8 (special issue), 75-91, 2015.

(2) L. Bos, A. Narayan, N. Levenberg and F. Piazzon, An orthogonality property of Legendre polynomials, *Constructive Approximation*, Vol. 45, 65-81, 2017

(3) T. Bloom, N. Levenberg, V. Totik and F. Wielonsky, Modified logarithmic potential theory and applications, *International Math. Research Notices*, Vol. 2017, No. 4, 1116-1154, 2017

(4) A weighted extremal function and equilibrium measure, L. Bos, N. Levenberg, S. Ma‘u and F. Piazzon, *Math. Scand.*, Vol. 121, No. 2, 243-262, 2017.

(5) A large deviation principle for weighted Riesz interactions, T. Bloom, N. Levenberg and F. Wielonsky, *Constructive Approximation*, Vol. 47, No. 1, 119-140, 2018.

(6) Bernstein-Walsh theory associated to convex bodies and applications to multivariate approximation theory, L. Bos and N. Levenberg, *Computational Methods and Function Theory*, Volume 18, Issue 2, pp. 361-388, 2018.

(7) Pluripotential Theory and Convex Bodies, T. Bayraktar, T. Bloom and N. Levenberg, *Mat. Sbornik*, vol. 209, no. 3, 352-384, 2018.

(8) The extremal function for the complex ball for generalized notions of degree and multivariate polynomial approximation, T. Bloom, L. Bos, N. Levenberg, S. Ma'u and F. Piazzon, *Annales Polonici Math.*, DOI: 10.4064/ap180322-19-11, published online March 28, 2019.

(9) T. Bayraktar, T. Bloom, N. Levenberg and C. H. Lu, Pluripotential theory and convex bodies: large deviation principle, to appear in *Arkiv for Matematik*.

(10) N. Levenberg and M. Perera, A global domination principle for $P-$pluripotential theory, to appear in *CRM Proceedings and Lecture Notes* series, vol. in honor of Tom Ransford.

(11) A. Izzo and N. Levenberg, A Cantor set whose polynomial hull contains no analytic discs, to appear in *Arkiv for Matematik*.

Submitted

(12) N. Levenberg and F. Wielonsky, Zeros of Faber polynomials for Joukowski airfoils, submitted to *Constructive Approximation*.

(13) N. Levenberg and S. Ma'u, Monge-Ampère of Pac-Man, submitted to *Archiv der Mathematik*.

NOTES

(1) This is a survey on Bernstein-Markov measures/properties in various guises. Given a compact set $K \subset \mathbf{C}^d$ and a measure μ with support in K, we say that μ is a Bernstein-Markov measure for K or (K,μ) is a Bernstein-Markov pair if

$$||p||_K := \max_{z\in K} |p(z)| \leq M_n ||p||_{L^2(\mu)}$$

for all polynomials p of degree at most n (we write $p \in \mathcal{P}_n$) and where $M_n^{1/n} \to 1$. Essentially, sup-norms and $L^2(\mu)-$norms of polynomials of a given degree are comparable up to $n-$th root asymptotics. Weighted versions, applied to weighted polynomials pe^{nQ} where $p \in \mathcal{P}_n$ and Q is a fixed function on K, are also considered. Generalizations to "polynomial-like" settings can be formulated (cf., (3), (5), (7)). These have applications to (pluri-)potential theory, multivariate approximation theory and occur, e.g., in statements of large deviation principles. We give sufficient conditions on a measure in order to satisfy an appropriate Bernstein-Markov property (some of these are new) and include some open problems.

(6), (7), (9), (10): Motivated by work of T. Bayraktar on zeros of random sparse polynomials in several complex variables and by N. Trefethen on polynomial approximation, in this series of papers we have developed pluripotential theory associated to a convex body (say C) in $(\mathbf{R}^+)^d$. In brief, C determines both a nested family of finite-dimensional polynomial spaces, $Poly(nC),\ n = 1, 2, ...$ and a plurisubharmonic (psh) growth function H_C in $\mathbf{C}^d$ (indeed writing $z^J := z_1^{j_1} \cdots z_d^{j_d}$,

$$Poly(nC) := \{p(z) = \sum_{J \in nC \cap (\mathbf{Z}^+)^d} c_J z^J : c_J \in \mathbf{C}\}$$

and H_C is the logarithmic indicator function of C: $H_C(z_1, ..., z_d) := \phi_C(\log |z_1|, ..., \log |z_d|)$ where $\phi_C(x_1, ..., x_d) := \max_{y \in C}[x_1 y_1 + \cdots + x_d y_d]$). The main objects in the theory are then the psh functions u in the Lelong class L_C: u is psh in $\mathbf{C}^d$ and $u(z) \leq H_C(z) + c_u$ for some constant c_u (if, in addition, $u \geq H_C + \tilde{c}_u$, we write $u \in L_{C,+}$). In particular, if $p \in Poly(nC)$ then $u := \frac{1}{n} \log |p| \in L_C$. Given a compact set $K \subset \mathbf{C}^d$,

$$V_{C,K}(z) := \sup\{u(z) : u \in L_C,\ u \leq 0 \text{ on } K\}$$

is the $C-$extremal function of K and it turns out that

$$V_{C,K} = \lim_{n \to \infty} \sup\{\frac{1}{n} \log |p_n(z)| : p_n \in Poly(nC),\ ||p_n||_K := \max_{\zeta \in K} |p_n(\zeta)| \leq 1\}.$$

If $C = \Sigma := \{(x_1, ..., x_d) \in \mathbf{R}^d : 0 \leq x_i \leq 1,\ \sum_{j=1}^d x_i \leq 1\}$, one recovers "classical" pluripotential theory. The associated *complex Monge-Ampère measure* $\mu_{C,K} := (dd^c V_{C,K})^d$ plays an important role. In (6) Bos and I prove a "$C-$version" of a Bernstein-Walsh theorem – a quantitative Runge/Oka-Weil result on uniform approximation of functions f holomorphic in a neighborhood of a (polynomially convex) compact set K by polynomials in $Poly(nC)$. Our results, which involve level sets of the $C-$extremal function $V_{C,K}$, answer questions of Trefethen on the "correct" notion of degree of a multivariate polynomial and provide a more precise formulation and explanation of his queries. (Essentially, Trefethen considered the $(\mathbf{R}^+)^d$ portion of an ℓ^q ball

$$C_q := \{(x_1, ..., x_d) : x_1, ..., x_d \geq 0,\ x_1^q + \cdots + x_d^q \leq 1\}$$

for $q = 1, 2$ and ∞ (note $q = 1$ gives $C_1 = \Sigma$) with the compact set being the "hypercube" $[-1, 1]^d \subset \mathbf{R}^d \subset \mathbf{C}^d$ for specific f).

In (7), we define notions of energy (of pairs of appropriate L_C functions), ball volume ratios, transfinite diameter, etc., in this $C-$setting in the spirit of seminal work of R. Berman and S. Boucksom. Their complex geometric setting covers the $C = \Sigma$ case. As in their work, we necessarily develop *weighted* $C-$pluripotential theory as this is essential to prove the main result. In brief, for an "admissible" weight function Q on a closed subset K of $\mathbf{C}^d$, the weighted $C-$extremal function for K, Q is

$$V_{C,K,Q}(z) := \sup\{u(z) : u \in L_C,\ u \leq Q \text{ on } K\}.$$

There are a few novelties in our work; e.g., the proof of the existence of the limit in the definition of the $C-$transfinite diameter of a compact set is nonstandard. Here, fixing a standard basis of monomials $\{e_1, ..., e_{m_n}\}$ for $Poly(nC)$, given m_n points $\mathbf{a}_n := a_1, ..., a_{m_n} \in \mathbf{C}^d$, define

$$VDM_C(a_1, ..., a_{m_n}) := \det[e_i(a_j)]_{i,j=1,...,m_n},$$

which is a polynomial of degree $l_n := \sum_{j=1}^{m_n} \deg_C e_j$ in $a_1, ..., a_{m_n}$ (for a polynomial p we write $\deg_C(p) := \min\{n \in \mathbf{N} \colon p \in Poly(nC)\}$). For $K \subset \mathbf{C}^d$ compact, we show

$$\delta_C(K) := \lim_{n\to\infty} \big(\max_{\mathbf{a}_n \in K^{m_n}} |VDM_C(\mathbf{a}_n)| \big)^{1/l_n}$$

exists.

With all this background in place, in (9), my collaborators from (7) along with Chinh Lu developed some "Monge-Ampère machinery" in order to prove a large deviation prinicipe for the empirical measure of an associated interacting particle system using weighted $C-$Vandermonde determinants. We used variational techniques to prove existence of solutions u to $(dd^c u)^d = \mu$ for appropriate measures μ where $u \in L_C$ must be in a "finite energy" class. In the $C = \Sigma$ setting, L_Σ functions extend as $\omega-$psh functions on $\mathbf{P}^d$ where ω is the standard Kähler form; this is not the case in our setting and thus the variational approach of Berman, Boucksom, Guedj and Zeriahi is slightly modified. For ν a finite measure on K such that (K, μ) is a Bernstein-Markov pair, we define a probability measure $Prob_n$ on K^{m_n}: for a Borel set $A \subset K^{m_n}$,

$$Prob_n(A) := \frac{1}{Z_n} \cdot \int_A |VDM_C^Q(x_1, ..., x_{m_n})|^2 d\nu(x_1) \cdots d\nu(x_{m_n})$$

where

$$Z_n = \int_{K^{m_n}} |VDM_C(x_1, ..., x_{m_n})|^2 d\nu(x_1) \cdots d\nu(x_{m_n}).$$

For arrays of points $\{\mathbf{a}^{(n)} := (a_1^{(n)}, ..., a_{m_n}^{(n)})\}_{n=1,2,...}$ in K, we form the associated sequence of discrete probability measures $\mu_n = \frac{1}{m_n} \sum_{j=1}^{m_n} \delta_{a_j^{(n)}}$. With probability one in $\prod_{n=1}^{\infty}(K^{m_n}, Prob_n)$, we have $\mu_n \to c\mu_{C,K}$ weak-* (c a normalizing constant). More to the point, using our Monge-Ampère existence results, we prove a large deviation principle (LDP) on the sequence of push-foward measures

$$\{\sigma_n := j_{n*}(Prob_n)\} \subset \mathcal{M}(\mathcal{M}(K))$$

where $j_n : K^{m_n} \to \mathcal{M}(K)$ via $j_n(\mathbf{a}^{(n)}) = \mu_n$. Indeed, we do all this in the more general weighted setting.

Finally, in (10), together with a current PhD student, we prove a so-called global domination principle (GDP) in this C setting. Precisely, if $u \in L_C,\ v \in L_{C,+}$ with $u \leq v$ a.e. $(dd^c v)^d$, then $u \leq v$ in $\mathbf{C}^d$. We do this by completely elementary (univariate) methods. The GDP is the most important tool in $(C-)$pluripotential theory for verifying that a candidate function for a $(C-)$-extremal function is, indeed, a $(C-)$-extremal function. As a corollary, we prove a formula for the $C-$extremal function of a product of planar compact sets. An earlier (weaker) version of this product property was proved in (6).

(4) and (8): One item severely lacking in pluripotential theory are explicit examples of extremal functions and their associated Monge-Ampère measures. In (4), we combine a variety of techniques (e.g., extremal functions of compact subsets of algebraic sets; weighted extremal functions; properties of the complex Lie norm) to compute the weighted extremal function and associated Monge-Ampère measure of $\mathbf{R}^d \subset \mathbf{C}^d$ with weight function $Q(x) = \frac{1}{2}\log(1+|x|^2)$ (the Kähler potential restricted to $\mathbf{R}^d$). In (8), in our search in the $C-$setting to find explicit examples of $C-$extremal functions for sets other than product sets, we take a natural compact set K, the (closed) Euclidean unit ball in $\mathbf{C}^2$, and attempt to find $C-$extremal functions for K when $C = C_q$ with $q > 1$. We were successful with an explicit calculation only when $q = \infty$; however, we were also able to compute the Monge-Ampère measure and found its support lies in a two-dimensional torus in the boundary of K. For $q = 1$ (the "classical" case), the Monge-Ampère measure is supported on the whole topological boundary of the ball.

(3) and (5): In these papers our main goal, as in (9), is a large deviation principle (LDP). Unlike (9), in (3) and (5) we are essentially in the realm of linear potential theory. We simply describe the potential-theoretic settings.

In (3), given $K \subset \mathbf{C}$, we consider the weighted energy minimization problem (Q is a weight function on K)

$$E_f^Q(\mu) := \int_K \int_K \log \frac{1}{|x-y||f(x)-f(y)|w(x)w(y)} d\mu(x)d\mu(y)$$

for μ a probability measure on K where $w = e^{-Q}$ and $f : K \to \mathbf{C}$ is a fixed function. Our motivation comes from certain *biorthogonal ensembles* studied by, e.g., Borodin and Claeys-Wang. The discretization of this energy leads to consideration of maximizing the weighted $f-$Vandermonde of order k:

$$|VDM_k^Q(z_0,...,z_k)| :=$$

$$|VDM(z_0,...,z_k)| \exp\Big(-k[Q(z_0)+\cdots+Q(z_k)]\Big)|VDM(f(z_0),...,f(z_k))|$$

over $k+1$ tuples of points $z_0,...,z_k \in K$ where $VDM(z_0,...,z_k) = \prod_{0\leq i<j\leq k}(z_j - z_i)$. Along the way, we prove a Bernstein-Walsh type estimate for functions of the form $p(g(z))q(f(z))$ where p, q are polynomials and f, g are holomorphic. In proving our results, we utilize some nice tools from logarithmic potential theory in the plane.

In (5), we consider a weighted Riesz energy minimization problem. Given a compact set $K \subset \mathbf{R}^d$ of positive Riesz capacity, and a lower semicontinuous function Q on K with $\{x \in K : Q(x) < \infty\}$ of positive Riesz capacity we define $w(x) := e^{-Q(x)}$ and minimize

$$I^Q(\mu) := \int_K \int_K \frac{1}{|x-y|^\alpha} d\mu(x) d\mu(y) + 2 \int_K Q(x) d\mu(x).$$

over probability measures μ on K. Discretizing, we are led to functions on $n-$tuples of points in K that have properties analogous to (weighted) Vandermondes but lead to "polynomial-like" objects for which it is necessary to develop the appropriate Bernstein-Markov measure machinery as in (1).

(2): We give a second orthogonality property of the classical Legendre polynomials on the real interval $[-1,1]$. Polynomials up to degree n from this family are mutually orthogonal under the arcsine measure weighted by the degree n normalized Christoffel function; i.e., if $K_n(x) = \sum_{j=0}^n |P_j^*(x)|^2$ where $\{P_j^*\}$ is an orthonormal basis in $L^2(dx)$ for $\mathcal{P}_n$, then

$$\int_{-1}^1 P_i^*(x) P_j^*(x) \frac{1}{K_n(x)} \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} dx = \delta_{ij}, \ i,j = 0,...,n.$$

This was observed numerically by Narayan, motivating our work.

(11): The polynomial hull $\hat{X}$ of a compact set $X \subset \mathbf{C}^d$ is defined as

$$\hat{X} = \{z \in \mathbf{C}^d : |p(z)| \leq ||p||_X, \text{ all polynomials } p\}.$$

A standard way a point $z_0 \notin X$ can be shown to belong to $\hat{X}$ is if z_0 lies in an analytic disk whose boundary is contained in X. Examples going back to Stolzenberg and Wermer show that a compact set X can have nontrivial hull even if $\hat{X} \setminus X$ contains no analytic disks. On the other hand, a Cantor set, being totally disconnected, is often polynomially convex; but B. Joricke constructed a Cantor set in the unit sphere whose polynomial hull was more than nontrivial: it contained an open set. In our paper, we modify a construction of Wermer to construct a Cantor set $X \subset \mathbf{C}^3$ which has nontrivial hull with the property that $\hat{X} \setminus X$ contains no analytic disks.

(12): This work was motivated by some numerical work and a question related to Chebyshev quadrature. Faber polynomials are a classical object from one complex variable. Let $K \subset \mathbf{C}$ be compact (and have more than one point) and suppose the unbounded component Ω of $\bar{\mathbf{C}} \setminus K$ is simply connected. Let Φ be the conformal map from Ω to the complement of the (closed) unit disk with

$$\Phi(\infty) = \infty, \qquad \Phi'(\infty) > 0.$$

If Ψ is the inverse map of Φ, then the *Faber polynomials* $\{F_n\}$ *for* K can be defined from the following:

$$\frac{\Psi'(w)}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}.$$

We study the case where K is the closure of a bounded region with simply connected complement whose boundary is a piecewise analytic curve with at least one outward cusp. In this setting, the asymptotics of zeros of Faber polynomials for K are not well understood. To be precise, we study Joukowski airfoils: $\Psi : \{z : |z| > 1\} \to \mathbf{C} \setminus K$ is the composition $\Psi = J \circ T$ where

$$J(\zeta) = \frac{1}{2}(\zeta + 1/\zeta)$$

is the Joukowski map and $\zeta = T(z) = az + b$ with $a, b \in \mathbf{C}$ chosen so that -1 lies in the interior of K and 1 lies on ∂K. We determine the (unique) weak-* limit of the full sequence of normalized counting measures of the Faber polynomials and show it is *never* equal to the potential-theoretic equilibrium measure of K.

(13): One case where (classical) extremal functions and their associated Monge-Ampère measures are fairly well understood is the case when K is a convex body in $\mathbf{R}^d \subset \mathbf{C}^d$. It is known that the Monge-Ampère measure $\mu_K = \mu_{\Sigma,K}$ is absolutely continuous with respect to Lebesgue measure on $\mathbf{R}^d$ with density $\rho_K(x)$ which is comparable to $[dist(x, \partial K)]^{-1/2}$. In particular, this density blows up as x approaches the boundary of K. R. Berman asked what happens near a non-convex boundary point of a non-convex body. We show that the Monge-Ampère density $\rho_P(x)$ of the extremal function V_P for a non-convex Pac-Man set $P \subset \mathbf{R}^2$ – a disk minus a centrally symmetric wedge of opening less than π – tends to a finite limit as we approach the vertex p of P linearly but with a value that may vary with the line; while along a tangential approach to p the Monge-Ampère density becomes unbounded.