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Modified Logarithmic Potential Theory and Applications

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We develop potential theory including a Bernstein–Walsh type estimate for functions of the form $p(g(z))q(f(z))$, where p, q are polynomials and f, g are holomorphic. For $g(z) = z$, such functions arise in the study of certain ensembles of probability measures and in this case we can further extend the theory leading to probabilistic results such as large deviation principles.

1 Introduction

The classical Bernstein–Walsh inequality establishes growth rates for polynomials p outside of a compact set $K \subset \mathbb{C}$ in terms of the supremum norm of p on K and the degree of p:

$$|p(z)| \le \left(\sup_{\zeta\in K} |p(\zeta)|\right) e^{\deg(p)V_K(z)} =: ||p||_K e^{\deg(p)V_K(z)},$$

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where V_K is the extremal function for K (see (4.1)). Given a finite measure μ on K, a Bernstein–Markov type inequality is a comparability between $L^q(\mu)$ norms $(1 \leq q < \infty)$ and supremum norms for polynomials of a given degree:

$$||p||_K \leq M_k ||p||_{L^q(\mu)} \text{ for polynomials of degree } k,$$

where $M_k^{1/k} \to 1$. In a series of articles in various potential-theoretic settings (cf. [3, 4]), the authors have studied analogs of these properties. With these inequalities established, using purely potential-theoretic techniques one can prove probabilistic results such as large deviation principles (LDPs) associated to empirical distributions arising from discretizing the associated potential-theoretic energy minimization problem. An essential ingredient is the weighted version of the problem.

In this article, given $K \subset \mathbb{C}$, we consider the problem of minimizing the weighted energy

$$E_f^Q(\mu) = E^Q(\mu) := \int_K \int_K \log \frac{1}{|x-y||f(x)-f(y)|w(x)w(y)} \mathrm{d}\mu(x)\mathrm{d}\mu(y)$$

over probability measures μ on K, where $w = \mathrm{e}^{-Q}$ is a weight function on K and $f : K \to \mathbb{C}$ is a fixed function. Discretizing the problem, for each $k = 1, 2, \ldots$, we consider maximizing the weighted $f-$Vandermonde of order k:

$$\begin{aligned} &|VDM_k^Q(z_0, \ldots, z_k)| \\ &\quad := |VDM(z_0, \ldots, z_k)| \exp\Big(-k[Q(z_0) + \cdots + Q(z_k)]\Big) |VDM(f(z_0), \ldots, f(z_k))| \end{aligned}$$

over $k+1$ tuples of points $z_0, \ldots, z_k \in K$, where $VDM(z_0, \ldots, z_k) = \prod_{0 \leq i < j \leq k}(z_j - z_i)$ is the classical Vandermonde determinant. After developing the potential-theoretic background for appropriate K, Q, and f in Sections 2 and 3, we obtain Bernstein–Walsh type estimates for the "generalized weighted f-polynomials"

$$z_j \to VDM_k^Q(z_0, \ldots, z_k),$$

where f is holomorphic on a neighborhood of K.

This is a special case of the more general Bernstein–Walsh type estimates (4.9) and (4.10) developed in Section 4 for functions of the form

$$h_k(z) = p_k(g(z))q_k(f(z)), \; p_k, q_k \text{ polynomials of degree } k,$$

where f, g are defined and holomorphic on a neighborhood of K. Section 5 invokes some classical potential theory to verify a (weighted) Bernstein–Markov type estimate (5.1) for such functions h_k. Precisely, for a compact set K which is not thin at each of its points, we prove a quantitative comparability between supremum norms and $L^1(\nu)$–norms for measures ν on K satisfying a mass-density condition (equation (5.2)).

Following standard arguments (cf. [2]), given a measure ν on K satisfying (5.2), it follows that the $k(k+1)/2$ roots of the L^1-averages

$$Z_k := \int_{K^{k+1}} |VDM_k^Q(z_0, \ldots, z_k)| \mathrm{d}\nu(z_0) \cdots \mathrm{d}\nu(z_k)$$

tend to the same limit as the $k(k+1)/2$ roots of the maximal weighted f-Vandermondes $|VDM_k^Q(z_0, \ldots, z_k)|$ over K^{k+1}. This has consequences for the empirical distribution associated to the ensemble of probability measures $Prob_k$ on K^{k+1}, where for a Borel set $A \subset K^{k+1}$,

$$Prob_k(A) := \frac{1}{Z_k} \cdot \int_A |VDM_k^Q(z_0, \ldots, z_k)| \mathrm{d}\nu(z_0) \cdots \mathrm{d}\nu(z_k).$$

These consequences are the main content of Section 6, where we restrict to compact K. The brief Section 7 details the key ingredients needed to make extensions to the unbounded case.

There are numerous articles in the literature where various aspects of the ensembles considered in this article are studied; we simply mention a few. In all these situations the authors restrict to the case of ν being Lebesgue measure on K. For $f(z) = e^z$ and $K = \mathbb{R}$ see Claeys–Wang [8]. For $f(z) = z^\theta$, $\theta > 0$, and $K = \mathbb{R}^+$ they were studied by Borodin [5]. He named them *biorthogonal ensembles*. For $\theta = 2$ they were studied in Leuck *et al.* [14] motivated by physical considerations. For θ a positive integer, a large deviation result was proved by Eichelsbacher *et al.* in [10] under some restrictions on Q; there, for θ even, K is a closed subset of $\mathbb{R}^+$, whereas for θ odd, K is a closed subset of $\mathbb{R}$.

Recent articles of Cheliotis [7] and Forrester–Wang [11] exhibit these ensembles as joint probability distributions of eigenvalues of specific ensembles of random matrices. The case $f(z) = \log z$ also occurs this way.

Work of Muttalib [16] originally provided impetus for studying these ensembles. He had proposed a correction term to the joint probability distribution of the Gaussian unitary ensemble to describe certain physical phenomena. In particular, he proposed to consider $f(z) = \log(\operatorname{arcsinh}^2 z^{1/2})$ on $\mathbb{R}^+$.

An article of Chafai *et al.* [6] establishes a LDP on $\mathbb{R}^d$ under quite general circumstances. Restricted to $\mathbb{R}^2$ or $\mathbb{R}$, there is some overlap with the probabilistic results of this article.

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2 General Potential Theory Results

In this Section we state and prove results, including existence and uniqueness of weighted energy minimizing measures, in a univariate setting generalizing the classical setting in [18] (see also [15] for a particular case). Recall a set $E \subset \mathbb{C}$ is *polar* if there exists $u \not\equiv -\infty$ defined and subharmonic on a neighborhood of E with $E \subset \{u = -\infty\}$ (cf. [18]). We use the terminology that a property holds q.e. (quasi-everywhere) on a set $S \subset \mathbb{C}$ if it holds on $S \setminus P$ where P is a polar set. In [18], given a compact, nonpolar set $K \subset \mathbb{C}$, a real-valued function Q on K is called *admissible* if Q is lower semicontinuous and $\{z \in K : Q(z) < \infty\}$ is not polar. We write $Q \in \mathcal{A}(K)$ and define $w(z) := \mathrm{e}^{-Q(z)}$. If K is closed but unbounded, one requires that

$$\liminf_{|z|\to\infty,\ z\in K} \left[Q(z) - \frac{1}{2}\log(1 + |z|^2) \right] = \infty. \tag{2.1}$$

Suppose now a closed, nonpolar set $K \subset \mathbb{C}$ is given, and $f : K \to \mathbb{C}$ is continuous. From Section 4 onward, we will make more stringent requirements on f. For K compact, the class of admissible weights Q on K suffices for our purposes; for unbounded K, we make the following definition.

Definition 2.1. We call a lower semicontinuous function Q on a closed, unbounded set $K \subset \mathbb{C}$ with $\{z \in K : Q(z) < \infty\}$ not polar *f-admissible* for K if

$$\psi(z) := Q(z) - \frac{1}{2}\log\left[(1 + |z|^2)(1 + |f(z)|^2)\right]$$

satisfies $\lim_{|z|\to\infty,\ z\in K} \psi(z) = \infty$. □

This implies $\psi(z) \geq c = c(Q) > -\infty$ for all $z \in K$; also, since $1 + |f(z)|^2 \geq 1$, we have $\psi(z) \leq Q(z) - \frac{1}{2}\log(1 + |z|^2)$ so that Q is admissible in the usual potential-theoretic sense (2.1) of [18]. The hypothesized growth of Q depends heavily on f. We say Q is *strongly f-admissible* for K if there exists $\delta > 0$ such that $(1 - \delta)Q$ is f-admissible for K.

The weighted potential theory problem we study is to minimize the weighted energy

$$E_f^Q(\mu) = E^Q(\mu) := \int_K \int_K \log \frac{1}{|x-y||f(x)-f(y)|w(x)w(y)} d\mu(x) d\mu(y) \tag{2.2}$$

over $\mu \in \mathcal{M}(K)$, the set of probability measures on K. Here $w = e^{-Q}$. The double integral in (2.2) is well defined and different from $-\infty$. Indeed, let

$$k(x,y) := -\log\left(|x-y||f(x)-f(y)|w(x)w(y)\right). \tag{2.3}$$

Using the inequality $|u-v| \le \sqrt{1+|u|^2}\sqrt{1+|v|^2}$, we have

$$\begin{aligned} &\log|x-y| + \log|f(x)-f(y)| \\ &\quad \le \frac{1}{2}\log(1+|x|^2) + \frac{1}{2}\log(1+|y|^2) + \frac{1}{2}\log(1+|f(x)|^2) + \frac{1}{2}\log(1+|f(y)|^2). \end{aligned}$$

Hence, by Definition 2.1,

$$k(x,y) \ge \psi(x) + \psi(y) \ge 2c \text{ on } K \times K, \tag{2.4}$$

and the integrand of the double integral is bounded below by $2c$.

We also recall the definition of the logarithmic energy of μ,

$$I(\mu) := \int_K \int_K \log \frac{1}{|x-y|} d\mu(x) d\mu(y) =: \int_K p_\mu(y) d\mu(y)$$

where $p_\mu(y) := \int_K \log \frac{1}{|x-y|} d\mu(x)$ is the logarithmic potential of μ. For $K \subset \mathbb{C}$ compact, the logarithmic capacity of K is

$$cap(K) := \exp\left[-\inf\{I(\mu) : \mu \in \mathcal{M}(K)\}\right]. \tag{2.5}$$

For a Borel set $E \subset \mathbb{C}$, $cap(E)$ may be defined as $\exp\left[-\inf I(\mu)\right]$ where the infimum is taken over all Borel probability measures with compact support in E. The weighted logarithmic energy of μ with respect to Q is

$$I^Q(\mu) := \int_K \int_K \log \frac{1}{|x-y|w(x)w(y)} d\mu(x) d\mu(y). \tag{2.6}$$

Since $1+|f(x)|^2 \geq 1$, the double integral in (2.6) is also well defined and different from $-\infty$. When $I(\mu) \neq -\infty$ or $\int Q\mathrm{d}\mu < \infty$, we can rewrite $I^Q(\mu)$ as

$$I^Q(\mu) = I(\mu) + 2\int_K Q\mathrm{d}\mu.$$

For the push-forward measure $f_*\mu$ of μ on $f(K)$, we have

$$\begin{aligned} I(f_*\mu) &= \int_K\int_K \log\frac{1}{|f(x)-f(y)|}\mathrm{d}\mu(x)\mathrm{d}\mu(y) = \int_{f(K)}\int_{f(K)} \log\frac{1}{|a-b|}\mathrm{d}f_*\mu(a)\mathrm{d}f_*\mu(b) \\ &= \int_{f(K)} p_{f_*\mu}(b)\mathrm{d}f_*\mu(b) = \int_K p_{f_*\mu}(f(z))\mathrm{d}\mu(z). \end{aligned}$$

When $I^Q(\mu) \neq +\infty$ or $I(f_*\mu) \neq -\infty$, the energy $E^Q(\mu)$ can be rewritten as

$$E^Q(\mu) = I^Q(\mu) + I(f_*\mu).$$

Proposition 2.2. Let $K \subset \mathbb{C}$ be closed and let Q be f-admissible for K. Suppose there exists $\nu \in \mathcal{M}(K)$ with $E^Q(\nu) < \infty$. Let $V_w := \inf\{E^Q(\mu),\ \mu \in \mathcal{M}(K)\}$. Then

(1) V_w is finite.

(2) Setting $K_M := \{z : Q(z) \leq M\}$, we have, for sufficiently large $M < \infty$,

$$V_w = \inf\{E^Q(\mu),\ \mu \in \mathcal{M}(K_M)\}.$$

(3) We have existence and uniqueness of $\mu_{K,Q}$ minimizing E^Q. The measure $\mu_{K,Q}$ has compact support and the logarithmic energies $I(\mu_{K,Q})$ and $I(f_*\mu_{K,Q})$ are finite.

(4) The following Frostman-type inequalities hold true:

$$p_{\mu_{K,Q}}(z) + p_{f_*\mu_{K,Q}}(f(z)) + Q(z) \geq F_w \text{ q.e. on } K, \tag{2.7}$$

$$p_{\mu_{K,Q}}(z) + p_{f_*\mu_{K,Q}}(f(z)) + Q(z) \leq F_w \text{ on } \operatorname{supp}(\mu_{K,Q}), \tag{2.8}$$

where $F_w := I(\mu_{K,Q}) + I(f_*\mu_{K,Q}) + \int Q\mathrm{d}\mu_{K,Q} = V_w - \int Q\mathrm{d}\mu_{K,Q}$.

(5) If a measure $\mu \in \mathcal{M}(K)$ with compact support and $E^Q(\mu) < \infty$ satisfies

$$p_\mu(z) + p_{f_*\mu}(f(z)) + Q(z) \geq C \text{ q.e. on } K, \tag{2.9}$$

$$p_\mu(z) + p_{f_*\mu}(f(z)) + Q(z) \leq C \text{ on } \operatorname{supp}(\mu), \tag{2.10}$$

for some constant C, then $\mu = \mu_{K,Q}$. □

Proof. For 1., we have $V_w < \infty$ by assumption. The other inequality $-\infty < V_w$ follows from the fact that the double integral in (2.2) is bounded below by $2c$. The proof of 2. follows the lines of [18, pp. 29–30], namely one first proves that, for M sufficiently large,

$$k(x,y) > V_w + 1 \quad \text{if} \quad (x,y) \notin K_M \times K_M,$$

from which one derives that $E^Q(\mu) = V_w$ is possible only for measures with support in K_M.

We next prove 3. From 2., there is a sequence $\{\mu_n\} \subset \mathcal{M}(K_M)$ with

$$E^Q(\mu_n) \to V_w \quad \text{as} \quad n \to \infty.$$

The set K_M is compact, hence, by Helly's theorem, we get a subsequence of these measures converging weakly to a probability measure μ supported on K_M; and it is easy to see this $\mu := \mu_{K,Q}$ satisfies $E^Q(\mu) = V_w$. For the logarithmic energy of $\mu_{K,Q}$, we have $I(\mu_{K,Q}) > -\infty$ because $\mu_{K,Q}$ has compact support. Since f is continuous and $f_*\mu_{K,Q}$ has its support in $f(K_M)$, we also have $I(f_*\mu_{K,Q}) > -\infty$. Now, recalling that Q is bounded below, we may write $I(\mu_{K,Q})$ as the well-defined expression

$$I(\mu_{K,Q}) = V_w - I(f_*\mu_{K,Q}) - 2\int_K Q \mathrm{d}\mu_{K,Q},$$

from which follows that $I(\mu_{K,Q}) < \infty$ and then also $I(f_*\mu_{K,Q}) < \infty$.

The uniqueness follows from the fact that $\mu \to I(\mu)$ is strictly convex and $\mu \to I(f_*\mu)$ is convex on the subsets of $\mathcal{M}(K)$ where they are finite. To be precise, it is well known that for μ_1 and μ_2 two measures with finite energies, compact supports and $\mu_1(K) = \mu_2(K)$, we have $I(\mu_1 - \mu_2) \geq 0$ and $I(\mu_1 - \mu_2) = 0$ if and only if $\mu_1 = \mu_2$ (cf. Lemma I.1.8 in [18]).

Now if $\bar{\mu} \in \mathcal{M}(K)$ is another measure which minimizes E^Q, we know from the proof of 2. that $\bar{\mu} \in \mathcal{M}(K_M)$. Consequently, $I(\bar{\mu}), I(f_*\bar{\mu}) > -\infty$ and then also $I(\bar{\mu}), I(f_*\bar{\mu}) < \infty$. We have

$$E^Q\left(\frac{1}{2}(\mu_{K,Q} + \bar{\mu})\right) + I\left(\frac{1}{2}(\mu_{K,Q} - \bar{\mu})\right) + I(f_*\left(\frac{1}{2}(\mu_{K,Q} - \bar{\mu})\right) = \frac{1}{2}[E^Q(\mu_{K,Q}) + E^Q(\bar{\mu})] = V_w.$$

The sum $I(\frac{1}{2}(\mu_{K,Q} - \bar{\mu})) + I(f_*(\frac{1}{2}(\mu_{K,Q} - \bar{\mu})) \geq 0$ with equality if and only if $\mu_{K,Q} = \bar{\mu}$; hence the result.

We next prove the first inequality in 4. Let $\mu \in \mathcal{M}(K)$ with compact support and consider the measure $\widetilde{\mu} = t\mu + (1-t)\mu_{K,Q}$, $t \in [0,1]$. The inequality $E^Q(\mu_{K,Q}) \leq E^Q(\widetilde{\mu})$ can be rewritten as

$$E^Q(\mu_{K,Q}) \le t^2(I(\mu)+I(f_*\mu)) + (1-t)^2(I(\mu_{K,Q})+I(f_*\mu_{K,Q}))$$
$$+\,2t(1-t)(I(\mu,\mu_{K,Q})+I(f_*\mu,f_*\mu_{K,Q})) + 2\int Q d(t\mu+(1-t)\mu_{K,Q}),$$

where, for two measures μ and ν, we denote by $I(\mu,\nu)$ the mutual logarithmic energy

$$I(\mu,\nu) = -\iint \log|x-y|\mathrm{d}\mu(x)\mathrm{d}\nu(y).$$

The right-hand side of the above inequality is well defined since the assumption that μ has compact support implies that all terms in the sum are larger than $-\infty$. Letting t tend to 0, we obtain

$$F_w = I(\mu_{K,Q}) + I(f_*\mu_{K,Q}) + \int Q\mathrm{d}\mu_{K,Q} \le I(\mu,\mu_{K,Q}) + I(f_*\mu, f_*\mu_{K,Q}) + \int Q\mathrm{d}\mu. \tag{2.11}$$

Now, we proceed by contradiction, assuming that there exists a nonpolar compact subset $\mathcal{K}$ of K such that

$$\forall z \in \mathcal{K}, \quad p_{\mu_{K,Q}}(z) + p_{f_*\mu_{K,Q}}(f(z)) + Q(z) < F_w.$$

Integrating this inequality with respect to a probability measure μ supported on $\mathcal{K}$, we obtain

$$I(\mu,\mu_{K,Q}) + I(f_*\mu, f_*\mu_{K,Q}) + \int Q\mathrm{d}\mu < F_w,$$

which contradicts (2.11).

The proof of the second inequality in 4. is also by contradiction. Assume that

$$\exists x_0 \in \operatorname{supp}(\mu_{K,Q}), \quad p_{\mu_{K,Q}}(x_0) + p_{f_*\mu_{K,Q}}(f(x_0)) + Q(x_0) > F_w.$$

By lower semicontinuity, the inequality is satisfied in a neighborhood V_{x_0} of x_0. Moreover $\mu_{K,Q}(V_{x_0}) > 0$ since $x_0 \in \operatorname{supp}(\mu_{K,Q})$. Using the first inequality (2.7) on $\operatorname{supp}(\mu_{K,Q}) \setminus V_{x_0}$ and the fact that $\mu_{K,Q}(E) = 0$ for E a polar set (since $\mu_{K,Q}$ has finite logarithmic energy $I(\mu_{K,Q})$), we obtain

$$F_w = \int (p_{\mu_{K,Q}}(z) + p_{f_*\mu_{K,Q}}(f(z)) + Q(z))\mathrm{d}\mu_{K,Q}(z)$$
$$> F_w\mu_{K,Q}(V_{x_0}) + F_w\mu_{K,Q}(\operatorname{supp}(\mu_{K,Q}) \setminus V_{x_0}) = F_w,$$

which is a contradiction.

Finally, we prove 5. We write

$$\mu_{K,Q} = \mu + (\mu_{K,Q} - \mu).$$

Then

$$E^Q(\mu) \geq E^Q(\mu_{K,Q}) = E^Q(\mu) + I(\mu_{K,Q} - \mu) + I(f_*(\mu_{K,Q} - \mu)) + 2R$$

with

$$\begin{aligned} R &:= \int_K \left[\int_K -\log|x-y|\mathrm{d}\mu(y) + Q(x)\right] \mathrm{d}(\mu_{K,Q} - \mu)(x) \\ &\quad - \int_K \int_K \log|f(x) - f(y)|\mathrm{d}\mu(y)\mathrm{d}(\mu_{K,Q} - \mu)(x) \\ &= \int_K (p_\mu(x) + Q(x))\mathrm{d}(\mu_{K,Q} - \mu)(x) + \int_K p_{f_*\mu}(f(x))\mathrm{d}(\mu_{K,Q} - \mu)(x) \\ &= \int_K (p_\mu(x) + p_{f_*\mu}(f(x)) + Q(x))\mathrm{d}(\mu_{K,Q} - \mu)(x). \end{aligned}$$

The above computation is justified. Indeed, from the assumptions $E^Q(\mu) < \infty$ and μ has compact support, the quantities $E^Q(\mu)$, $I^Q(\mu)$, $I(f_*\mu)$, $I(\mu)$, $\int Q\mathrm{d}\mu$, and $I(\mu, \mu_{K,Q})$ are all finite. Making use of the inequalities (2.9) and (2.10), we derive

$$R \geq C \int_K \mathrm{d}\mu_{K,Q} - C \int_K \mathrm{d}\mu = 0.$$

Now, recall that $I(\mu_{K,Q} - \mu) + I(f_*(\mu_{K,Q} - \mu)) \geq 0$ with equality if and only if $\mu_{K,Q} = \mu$. Thus

$$E^Q(\mu) \geq E^Q(\mu_{K,Q}) \geq E^Q(\mu)$$

so that equality holds throughout, and $E^Q(\mu) = E^Q(\mu_{K,Q})$, from which follows $\mu = \mu_{K,Q}$. ■

The condition that there exist $\nu \in \mathcal{M}(K)$ with $E^Q(\nu) < \infty$ is not automatic. For example, if f is a constant function, then trivially all measures ν have $I(f_*\nu) = \infty$. We give a sufficient condition on f ensuring the hypothesis of Proposition 2.2.

Proposition 2.3. If $f : K \to \mathbb{C}$ is continuous and

$$\Sigma := \left\{ z \in K : Q(z) < \infty \text{ and } \liminf_{\substack{(z_1,z_2)\to(z,z)\\ z_1,z_2\in K,\ z_1\neq z_2}} \left| \frac{f(z_1)-f(z_2)}{z_1-z_2} \right| > 0 \right\}$$

is not polar, then there exist $\nu \in \mathcal{M}(K)$ with $E^Q(\nu) < \infty$. □

Proof. Let $D := \{(z,z) : z \in K\}$. Define

$$\phi(z_1,z_2) := \left| \frac{f(z_1)-f(z_2)}{z_1-z_2} \right|;$$

this is continuous on $(K \times K) \setminus D$. Extend ϕ to D by defining

$$\phi(z,z) := \liminf_{\substack{(z_1,z_2)\to(z,z)\\ z_1,z_2\in K,\ z_1\neq z_2}} \left| \frac{f(z_1)-f(z_2)}{z_1-z_2} \right|.$$

Then $\phi : K \times K \to \mathbb{C}$ is lower semicontinuous and we can write $\Sigma = \cup_{n=1}^{\infty} \Sigma_n$ where

$$\Sigma_n := \{z \in K : Q(z) < n \text{ and } \phi(z,z) > 1/n\}.$$

This is an increasing union so for all sufficiently large n, Σ_n is not polar. Fix such an n. Since polarity is a local property, see, for example, [12, Remark 4.2.13], there exists $z \in \Sigma_n$ such that, for any neighborhood V_z of z, $\Sigma_n \cap V_z$ is not polar.

Now, the function ϕ is lower semicontinuous on K^2, hence there exists a neighborhood V_z of z such that $\phi(z_1,z_2) > 1/n$ on $(\Sigma_n \cap V_z)^2$ and by the preceding remark, $\Sigma_n \cap V_z$ is not polar. Being not polar, $\Sigma_n \cap V_z$ supports a measure ν of finite logarithmic energy which is also of finite weighted logarithmic energy since $Q(z) < n$ for $z \in \Sigma_n$. It remains to prove that $f_*\nu$ is also of finite logarithmic energy. This follows from

$$\begin{aligned} I(f_*\nu) &= \int_{\Sigma_n\cap V_z}\int_{\Sigma_n\cap V_z} \log \frac{1}{|f(z_1)-f(z_2)|} d\nu(z_1)d\nu(z_2) \\ &\leq \log n + \int_{\Sigma_n\cap V_z}\int_{\Sigma_n\cap V_z} \log \frac{1}{|z_1-z_2|} d\nu(z_1)d\nu(z_2) < \infty. \end{aligned}$$ ■

We mention two specific situations (see Remark 4.2): f is the restriction to K of an entire function and f is the restriction to $K \subset (0,\infty)$ of f holomorphic in the right half plane $H := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. These cases are covered in the following two corollaries.

Corollary 2.4. Assume f is holomorphic on a neighborhood of K and the subset $\{z \in K : f'(z) \neq 0 \text{ and } Q(z) < \infty\}$ is nonpolar. Then there exist $\nu \in \mathcal{M}(K)$ with $E^Q(\nu) < \infty$. □

Corollary 2.5. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function which is differentiable for $x > 0$ and let $K \subset [0, \infty)$. Assume the subset $\{z \in K : f'(z) \neq 0 \text{ and } Q(z) < \infty\}$ is nonpolar. Then there exist $\nu \in \mathcal{M}(K)$ with $E^Q(\nu) < \infty$. □

We state an approximation property that one can use to prove a large deviation result in the unbounded setting. In the next section, we will prove a version for compact sets (Lemma 3.3) which we will need for our LDP in this case.

Lemma 2.6. Let K be a closed and nonpolar subset of $\mathbb{C}$ and let Q be f-admissible on K. Given $\mu \in \mathcal{M}(K)$, there exist an increasing sequence of compact sets K_m in K and a sequence of measures $\mu_m \in \mathcal{M}(K_m)$ such that

(1) the measures μ_m tend weakly to μ as $m \to \infty$;
(2) the energies $E^{Q_m}(\mu_m)$ tend to $E^Q(\mu)$ as $m \to \infty$, where $Q_m := Q|_{K_m}$. □

Proof. Since the measure μ has finite mass, there exist an increasing sequence of compact subsets K_m of K with $\mu(K \backslash K_m) \leq 1/m$. Then, the measures $\widetilde{\mu}_m := \mu|_{K_m}$ are increasing and tend weakly to μ. Denoting as usual by $k^+(x, y)$ and $k^-(x, y)$ the positive and negative parts of the function $k(x, y)$ that was defined in (2.3), we have, as $m \to \infty$,

$$\chi_m(x, y)k^+(x, y) \uparrow k^+(x, y) \quad \text{and} \quad \chi_m(x, y)k^-(x, y) \uparrow k^-(x, y),$$

$(\mu \times \mu)$-almost everywhere on $K \times K$, where $\chi_m(x, y)$ is the characteristic function of $K_m \times K_m$ and we agree that the left-hand sides vanish when $x = y \notin K_m$. By monotone convergence, we deduce that $E^{Q_m}(\widetilde{\mu}_m)$ tend to $E^Q(\mu)$ (possibly equal to $+\infty$) as $m \to \infty$, where we recall that the energy $E^Q(\mu)$, given by the double integral in (2.2), is always well defined since Q is f-admissible (recall (2.4)). Setting $\mu_m := \widetilde{\mu}_m/\mu(K_m)$ gives the result. ■

3 Discretization and Additional Results for K Compact

In this section, we restrict to the case where K is *compact*. Let $Q \in \mathcal{A}(K)$ and $w := e^{-Q}$. Note in this compact setting, the class $\mathcal{A}(K)$ is universal; that is, the same for all f. Here we naturally assume f is such that there exists $\nu \in \mathcal{M}(K)$ with $E^Q(\nu) < \infty$ and we discretize the weighted energy problem (2.2). Let

$$|VDM_k^Q(z_0,\dots,z_k)| = \textit{weighted Vandermonde of order } k$$
$$:= |VDM(z_0,\dots,z_k)| \exp(-k[Q(z_0)+\cdots+Q(z_k)])|VDM(f(z_0),\dots,f(z_k))|, \tag{3.1}$$

where $VDM(z_0,\dots,z_k) = \prod_{0\le i<j\le k}(z_j - z_i)$ and

$$\big(\delta_k^Q(f)\big)(K) = \delta_k^Q(K) := \max_{z_0,\dots,z_k\in K} |VDM_k^Q(z_0,\dots,z_k)|^{2/k(k+1)}.$$

Note the existence of maximizing points follows from compactness of K, continuity of f, and lower semicontinuity of Q; precisely, these yield upper semicontinuity of the mapping

$$(z_0,\dots,z_k) \to |VDM_k^Q(z_0,\dots,z_k)|$$

on K^{k+1}. We will use terminology such as *weighted Fekete points*, for notions defined relative to weighted Vandermondes as defined in (3.1). The proofs of Propositions 3.1–3.3 of [3] carry over in this setting.

Theorem 3.1. Given $K\subset\mathbb{C}$ compact and not polar, and $Q\in\mathcal{A}(K)$,

(1) if $\{z_j^{(k)}\}_{j=0,\dots,k;\ k=2,3,\dots}\subset K$ and $\{\mu_k = \frac{1}{k+1}\sum_{j=0}^k \delta_{z_j^{(k)}}\}\subset\mathcal{M}(K)$ converge weakly to $\mu\in\mathcal{M}(K)$, then

$$\limsup_{k\to\infty} |VDM_k^Q(z_0^{(k)},\dots,z_k^{(k)})|^{2/k(k+1)} \le \exp\left(-E^Q(\mu)\right); \tag{3.2}$$

(2) we have

$$\delta^Q(K) := \lim_{k\to\infty}\delta_k^Q(K) = \exp\left(-E^Q(\mu_{K,Q})\right); \tag{3.3}$$

(3) if $\{z_j^{(k)}\}_{j=0,\dots,k;\ k=2,3,\dots}\subset K$ and

$$\lim_{k\to\infty} |VDM_k^Q(z_0^{(k)},\dots,z_k^{(k)})|^{2/k(k+1)} = \exp\left(-E^Q(\mu_{K,Q})\right) \tag{3.4}$$

then

$$\mu_k = \frac{1}{k+1}\sum_{j=0}^k \delta_{z_j^{(k)}} \to \mu_{K,Q} \text{ weakly.}$$

□

Proof. We indicate the main ingredients. To prove the analog of Proposition 3.1 of [3], which is 1. above, we first observe that for any M,

$$h_M(x,y) := \min(M, -\log|x-y| - \log|f(x)-f(y)|)$$
$$\leq -\log|x-y| - \log|f(x)-f(y)| := h(x,y)$$

and $h(x,y)$ is lower semicontinuous if f is continuous. Then one can follow the proof of Proposition 3.1 of [3], which is similar in spirit to the proof of the principle of descent, Theorem I.6.8 of [18]. For 2., the analog of Proposition 3.2 of [3], for any points $a_0^{(k)},\ldots,a_k^{(k)} \in K$, integrating the inequality

$$\frac{-k(k+1)}{2}\log\delta_k^Q(K) \leq -\log|VDM_k^Q(a_0^{(k)},\ldots,a_k^{(k)})|$$

with respect to $\mathrm{d}\sigma(a_0^{(k)})\cdots\mathrm{d}\sigma(a_k^{(k)})$ where $\sigma \in \mathcal{M}(K)$ gives

$$\mathrm{e}^{-E^Q(\sigma)} \leq \liminf_{k\to\infty}\delta_k^Q(K). \tag{3.5}$$

Then, as observed above, maximizing $(k+1)-$tuples $z_0^{(k)},\ldots,z_k^{(k)} \in K$ for $\delta_k^Q(K)$ (*weighted Fekete points*) exist; taking μ to be any weak-* limit of $\{\mu_k = \frac{1}{k+1}\sum_{j=0}^k \delta_{z_j^{(k)}}\}$, (3.2) and (3.5) yield

$$E^Q(\mu) \leq \liminf_{k\to\infty}[-\log\delta_k^Q(K)] \leq \limsup_{k\to\infty}[-\log\delta_k^Q(K)] \leq E^Q(\sigma)$$

for any $\sigma \in \mathcal{M}(K)$ and any such μ. By uniqueness of the weighted energy minimizing measure $\mu_{K,Q}$, we obtain (3.3). The proof of 3. follows along the same lines. ■

Remark 3.2. Arrays $\{z_j^{(k)}\}_{j=0,\ldots,k;\ k=2,3,\ldots} \subset K$ satisfying (3.4) will be called *asymptotic weighted Fekete arrays* for K, Q, f. □

As a last result in this section, we give a refined version of Lemma 2.6 when K is a compact subset of $\mathbb{C}$. This is an analog of results in [4, Section 5] and will be used in a similar fashion to prove our large deviation result in the compact case. Here $C(K)$ denotes the class of continuous, real-valued functions on K.

Lemma 3.3. Let $K \subset \mathbb{C}$ be compact and nonpolar and let $\mu \in \mathcal{M}(K)$ with $E^Q(\mu) < \infty$. There exist an increasing sequence of compact sets K_m in K, a sequence of functions $\{Q_m\} \subset C(K)$, and a sequence of measures $\mu_m \in \mathcal{M}(K_m)$ satisfying

(1) the measures μ_m tend weakly to μ, as $m \to \infty$;

(2) the energies $I(\mu_m)$ tend to $I(\mu)$ as $m \to \infty$;

(3) the energies $I(f_*\mu_m)$ tend to $I(f_*\mu)$ as $m \to \infty$;

(4) the measures μ_m are equal to the weighted equilibrium measures μ_{K,Q_m}. □

Proof. By Lusin's continuity theorem applied in K and $f(K)$, it is easy to verify that, for every integer $m \geq 1$, there exists a compact subset K_m of K such that $\mu(K \setminus K_m) \leq 1/m$, p_μ is continuous on K_m, and $p_{f_*\mu}$ is continuous on $f(K_m)$, (respectively, considered as functions on K_m and $f(K_m)$ only). We may assume that K_m is increasing as m tends to infinity. Then, the measures $\widetilde{\mu}_m := \mu_{|K_m}$ are increasing and tend weakly to μ; similarly the measures $f_*\widetilde{\mu}_m = f_*(\mu_{|K_m})$ are increasing and tend weakly to $f_*\mu$. As in the proof of Lemma 2.6, we have

$$\chi_m(z,t)\log^+|z-t| \uparrow \log^+|z-t| \quad \text{and} \quad \chi_m(z,t)\log^+|f(z)-f(t)| \uparrow \log^+|f(z)-f(t)|,$$

as $m \to \infty$, $(\mu \times \mu)$-almost everywhere on $K \times K$ where $\chi_m(z,t)$ is the characteristic function of $K_m \times K_m$ and we agree that the left-hand sides vanish when $z = t \notin K_m$. Similar pointwise convergence holds true for the negative parts of the log functions. Hence, by monotone convergence we have

$$I(\widetilde{\mu}_m) \to I(\mu), \quad I(f_*\widetilde{\mu}_m) \to I(f_*\mu), \quad \text{as } m \to \infty,$$

where we observe that the compactness of K implies that the energies $I(\mu)$ and $I(f_*\mu)$ are well defined. Indeed, because of the assumption $E^Q(\mu) < \infty$, the energies $I(\mu)$ and $I(f_*\mu)$ are finite but this is not used here.

Next, define $\mu_m := \widetilde{\mu}_m/\mu(K_m)$ and for $z \in K$,

$$Q_m(z) := -p_{\mu_m}(z) - p_{f_*\mu_m}(f(z)).$$

To show Q_m is continuous on K_m, since p_{μ_m} and $p_{f_*\mu_m}$ are lower semicontinuous, it suffices to show they are upper semicontinuous. For p_{μ_m} this follows since $p_{\mu-\mu_m} = p_\mu - p_{\mu_m}$ is upper semicontinuous and $p_\mu(z)$ is continuous on K_m. Similarly, $p_{f_*\mu_m}$ is upper semicontinuous since $p_{f_*\mu} - p_{f_*\mu_m}$ is upper semicontinuous and $p_{f_*\mu}(z)$ is continuous on K_m.

Item 4. follows from the fact that μ_m has compact support with $E^{Q_m}(\mu_m) < \infty$ (because $E^Q(\mu) < \infty$), and it clearly satisfies the Frostman-type inequalities of Proposition 2.2 for K and the weight Q_m; hence we have $\mu_m = \mu_{K,Q_m}$. We note that the assumption $E^Q(\mu) < \infty$ has only been used to prove 4. ■

4 Bernstein–Walsh Inequality

Observe that if we fix all the variables in $VDM_k^Q(z_0, \ldots, z_k)$ in (3.1) except one, say z_j, the function $z_j \to VDM_k^Q(z_0, \ldots, z_j, \ldots, z_k)$ is of the form $p_k(z_j)q_k(f(z_j))e^{-kQ(z_j)}$, where p_k, q_k are polynomials of degree at most k (we write $p_k, q_k \in \mathcal{P}_k$). Let $K \subset \mathbb{C}$ be compact and nonpolar. In this section, we prove a Bernstein–Walsh type inequality for functions of the slightly more general form

$$h_k(z) = p_k(g(z))q_k(f(z)) \text{ where } p_k, q_k \in \mathcal{P}_k$$

as well as a weighted Bernstein–Walsh type inequality for functions of the form $h_k(z)e^{-kQ(z)}$ (Theorem 4.1) where we assume f, g are *holomorphic* functions on a neighborhood U of K. We will utilize this in the next section in conjunction with a mass density assumption on a finite measure ν on K to obtain (weighted) Bernstein–Markov properties (Theorem 5.6).

The extremal function of logarithmic growth of K is defined as

$$V_K(z) := \sup\{u(z) : \ u \leq 0 \text{ on } K \text{ and } u \in \mathcal{L}\},$$

where

$$\mathcal{L} := \{u(z) : \ u \text{ is subharmonic on } \mathbb{C} \ \text{ and } u(z) \leq \log^+|z| + C, \text{ for some } C = C(u)\}.$$

For K compact, we have

$$V_K(z) := \max[0, \sup\{\frac{1}{\deg(p)} \log|p(z)| : p \in \cup_{k\geq 1}\mathcal{P}_k, \ ||p||_K \leq 1\}] \tag{4.1}$$

We let $V_K^*(z) := \limsup_{\zeta\to z} V_K(\zeta)$ denote the upper semicontinuous regularization of V_K; thus if K is not polar, V_K^* is subharmonic on $\mathbb{C}$, harmonic on $\mathbb{C} \setminus K$, and is, in fact, the Green function with a logarithmic pole at ∞ for $\mathbb{C} \setminus K$. We say K is *regular* if V_K^* is continuous; equivalently, $V_K = V_K^*$. This is a property of the outer boundary of K; that is, the boundary of the unbounded component of the complement of K. We denote by $\widehat{K}$ the complement of the unbounded component of the complement of K (the *polynomial hull* of K):

$$\widehat{K} = \{z \in \mathbb{C} : |p(z)| \leq ||p||_K \text{ for all } p \in \bigcup_k \mathcal{P}_k\}.$$

Then $V_K = V_{\widehat{K}}$. The logarithmic capacity of K defined in (2.5) can be recovered from V_K^*:

$$cap(K) = \exp\left(-\lim_{|z|\to\infty}[V_K^*(z) - \log|z|]\right).$$

The classical Bernstein–Walsh inequality, coming from (4.1), is

$$|p_k(z)| \leq ||p_k||_K e^{kV_K(z)} \tag{4.2}$$

for polynomials $p_k \in \mathcal{P}_k$.

Let K be compact and nonpolar. Given an admissible weight Q on K, the weighted extremal function for the pair K, Q is $V_{K,Q}^*(z) = \limsup_{\zeta\to z} V_{K,Q}(\zeta)$ where

$$\begin{aligned} V_{K,Q}(z) &:= \max[0, \sup\{\frac{1}{\deg(p)}\log|p(z)| : p \in \cup_{k\geq 1}\mathcal{P}_k,\ ||pe^{-\deg(p)Q}||_K \leq 1\}] \\ &= \sup\{u(z) : u \in \mathcal{L},\ u \leq Q \text{ on } K\}. \end{aligned} \tag{4.3}$$

We have $V_K^*, V_{K,Q}^* \in \mathcal{L}^+$, where

$$\mathcal{L}^+ := \{u \text{ subharmonic in } \mathbb{C} : \exists C_1, C_2 \text{ with } C_1 + \log^+|z| \leq u(z) \leq C_2 + \log^+|z|\}.$$

We refer the reader to Appendix B of [18], particularly Theorem 2.8, for proofs of (4.1) and (4.3) (which work in $\mathbb{C}^N$ for $N \geq 1$).

Returning to our situation where f, g are holomorphic on a neighborhood U of K, we denote by $\mathcal{F}_k$ the collection of functions of the form

$$h_k(z) = p_k(g(z))q_k(f(z)),\ z \in U,\ p_k, q_k \in \mathcal{P}_k. \tag{4.4}$$

The precise statement of our Bernstein–Walsh estimate involves the upper semicontinuous regularizations of extremal-like functions

$$W_K(z) := \max\left[0, \sup_{\cup_{k\geq 1}\mathcal{F}_k}\left\{\frac{1}{k}\log|h_k(z)| :\ ||h_k||_K \leq 1\right\}\right] \tag{4.5}$$

and for $Q \in \mathcal{A}(K)$,

$$W_{K,Q}(z) := \max\left[0, \sup_{\cup_{k\geq 1}\mathcal{F}_k}\left\{\frac{1}{k}\log|h_k(z)| :\ ||h_k e^{-kQ}||_K \leq 1\right\}\right]. \tag{4.6}$$

Observe by definition of W_K in (4.5) and $W_{K,Q}$ in (4.6),

$$|h_k(z)| \le ||h_k||_K e^{kW_K(z)} \text{ and } |h_k(z)| \le ||h_k e^{-kQ}||_K e^{kW_{K,Q}(z)} \text{ for } z \in U.$$

Our main goal is to prove W_K^* and $W_{K,Q}^*$ are subharmonic on U.

Theorem 4.1. Let D_A be the closure of a bounded domain in $\mathbb{C}$, where D_A is regular and has logarithmic capacity A. Let f, g be holomorphic and nonconstant on a neighborhood U of D_A and let $\tau > 0$ be given. Let K be a compact subset of D_A such that

$$K \subset D_A \subset U \text{ with } f(K), g(K) \subset D_A \tag{4.7}$$

and $cap(K) > \tau$. Then W_K^* is subharmonic on U and there is a positive constant $M = M(D_A, \tau)$ such that

$$W_K^*(z) \le 2M + V_{D_A}(g(z)) + V_{D_A}(f(z)),\ z \in U. \tag{4.8}$$

Furthermore, if $Q \in \mathcal{A}(K)$, then $W_{K,Q}^*$ is subharmonic on U. □

It follows that for all $h_k \in \mathcal{F}_k$, we have the Bernstein–Walsh estimates

$$|h_k(z)| \le ||h_k||_K e^{kW_K^*(z)},\ z \in U, \text{ and} \tag{4.9}$$

$$|h_k(z)| \le ||h_k e^{-kQ}||_K e^{kW_{K,Q}^*(z)},\ z \in U, \tag{4.10}$$

where $W_K^*(z)$ and $W_{K,Q}^*(z)$ are finite valued for $z \in U$.

Remark 4.2. Note if $g(z) = z$, condition (4.7) reduces to

$$K \subset D_A \text{ with } f(K) \subset D_A \text{ and } D_A \subset U. \tag{4.11}$$

If $f, g : \mathbb{C} \to \mathbb{C}$ are entire, then for any $K \subset \mathbb{C}$ one can find D_A so that the condition (4.7) holds; thus the Bernstein–Walsh estimates (4.9) and (4.10) hold on all of $\mathbb{C}$. Another interesting situation arises taking f and/or g to be branches of power functions $z \to z^\theta$ where $\theta > 0$. Taking, for example, f to be a branch defined and holomorphic on $\mathbb{C}\setminus(-\infty, 0]$ with $f(z) = |z|^\theta$ for $z = |z| > 0$, for any $K \subset (0, \infty)$ one can find $D_A \subset H := \{z \in \mathbb{C} : \text{Re}\, z > 0\}$ so that the condition (4.11) holds. Thus (4.9) and (4.10) hold on all of H. □

In the next four potential theoretic lemmas, we fix D_A as in Theorem 4.1. Then $V_{D_A}^* = V_{D_A}$ and $\lim_{|z|\to\infty}[V_{D_A}(z) - \log|z|] = \log A$.

Lemma 4.3. Let D_A and $\tau > 0$ be given. There is a positive constant $L = L(D_A, \tau)$ such that for all compact subsets $K \subset D_A$ with $cap(K) > \tau$, all $k = 1, 2, \ldots$, and all polynomials p_k of degree k we have

$$||p_k||_{D_A} \leq e^{kL(D_A,\tau)}||p_k||_K.$$

□

Proof. Consider the function $V_K^*(z) - V_{D_A}(z)$. This function is nonnegative and harmonic on $\overline{\mathbb{C}} \setminus D_A$ and has value

$$\log A - \log cap(K) \leq \log A - \log \tau$$

at ∞. By Harnack's inequality (cf. [18] Lemma 0.4.9) we have, for z with $V_{D_A}(z) = \log 2$,

$$V_K^*(z) - V_{D_A}(z) \leq C \log\left(\frac{A}{\tau}\right)$$

where C is a constant independent of K. Thus,

$$V_K^*(z) \leq C \log\left(\frac{A}{\tau}\right) + \log 2$$

on $\{z \in \mathbb{C} : V_{D_A}(z) = \log 2\}$.

Since V_K^* is subharmonic on $\mathbb{C}$, by the maximum principle the above bound holds on ∂D_A. Now, by the usual Bernstein–Walsh inequality (4.2),

$$||p_k||_{D_A} \leq ||p_k||_K e^{kL(D_A,\tau)},$$

where

$$L(D_A, \tau) = C \log\left(\frac{A}{\tau}\right) + \log 2.$$

■

Lemma 4.4. Let D_A and $\tau > 0$ be given. Let K be a compact subset of D_A with $cap(K) > \tau$. Suppose that

$$K = \cup_{i=1}^s B_i,$$

where the B_i are Borel sets. Then there is a constant $\sigma = \sigma(D_A, \tau, s) > 0$ such that at least one of the sets B_i is of capacity at least σ. □

Proof. The proof follows from Theorem 5.1.4(a) of [17]. The constant σ depends on the diameter of the bounded set D_A. ∎

Lemma 4.5. Let f be holomorphic and nonconstant on a neighborhood of D_A. Given $\tau > 0$, let K be a subset of D_A such that $cap(K) > \tau$. Then there is a constant $\beta = \beta(D_A, \tau, f) > 0$ such that

$$cap(f(K)) \geq \beta. \qquad \square$$

Proof. For each point $z_0 \in D_A$, there is a neighborhood V of z_0 such that the restriction of f to $V, f|_V = h^m$, where h is a biholomorphism and $m \in \mathbb{Z}^+$. Namely if $f'(z_0) \neq 0$ then $f|_V$ is a biholomorphism and $m = 1$, otherwise m is the least integer such that $f^{(j)}(z_0) \neq 0$. We may cover D_A by a finite collection of such sets, say V_i for $i = 1, 2, \ldots, s$ with corresponding positive integers m_i. Then we can shrink each set V_i to obtain sets W_i which still cover D_A and such that each W_i has compact closure in V_i.

For J a compact subset of a $\overline{W}_i$ we have

$$cap(f|_{W_i}(J)) \geq C(cap(J))^{m_i},$$

where for $m_i = 1$ we use [17], Theorem 5.3.1 applied to $(f|_{W_i})^{-1}$ and if $m_i \geq 2$ we use the cited theorem and the fact that under the power map $e_m : z \to z^m$, Theorem 5.2.5 of [17] gives

$$[cap(e_m(J))]^{1/m} = cap(e_m^{-1}(e_m(J))) \geq cap(J).$$

Now $K = \cup_{i=1}^{s}(K \cap \overline{W}_i)$ so by Lemma 4.4 for one of the sets in the union, say $K \cap \overline{W}_{i_0}$ we have $cap(K \cap \overline{W}_{i_0}) \geq \sigma(D_A, \tau, s)$ so

$$cap(f(K)) \geq cap(f(\overline{W}_{i_0} \cap K)) \geq Ccap(\overline{W}_{i_0} \cap K)^{m_{i_0}} \geq C\sigma(D_A, \tau, s)^{m_{i_0}}.$$

The constants C which appear above depend only on f and the sets V_i, W_i and not on K so the proof is complete. ∎

Remark 4.6. Note we do not require $f(K) \subset D_A$, but this assumption will be needed in the next result. □

In the upper envelope (4.5) defining W_K, given $h_k \in \mathcal{F}_k$ as $h_k(z) = p_k(g(z))q_k(f(z))$, we may multiply p_k by a non-zero scalar c and q_k by $1/c$ without changing h_k. We use the following normalization: for $h_k \in \mathcal{F}_k$ and $||h_k||_K = 1$ choose a point $z_0 \in K$ such that $|h_k(z_0)| = 1$. Then multiply p_k and q_k by scalars as above so that $|p_k(g(z_0))| = 1$ and $|q_k(f(z_0))| = 1$. The key estimate in this setting is the next result.

Lemma 4.7. Let f, g be holomorphic and nonconstant on a neighborhood of D_A and let $\tau > 0$ be given. Let K be a compact subset of D_A such that $f(K), g(K) \subset D_A$, and $cap(K) > \tau$. Then there is a constant $M = M(D_A, \tau) > 0$ such that for all $k = 1, 2, \ldots$ and all $h_k \in \mathcal{F}_k$ normalized as above,

$$||p_k||_{D_A} \leq e^{Mk} \quad and \quad ||q_k||_{D_A} \leq e^{Mk}.$$

□

Proof. We give the argument for q_k; the one for p_k is similar. We have $cap(f(K)) \geq \beta(D_A, \tau, f) > 0$ by Lemma 4.5. Let $\tau' = \sigma(D_A, \beta(D_A, \tau, f), 2)$ from Lemma 4.4 and let

$$F_k := \{t \in f(K) : \ q_k(t) \leq e^{M_1 k}\}$$

where M_1 is to be chosen. If

$$cap(F_k) \geq \tau' \tag{4.12}$$

then by Lemma 4.3 we have the required estimate on q_k:

$$||q_k||_{D_A} \leq ||q_k||_{F_k} e^{kL(D_A, \tau')} \leq e^{k(M_1 + L(D_A, \tau'))}.$$

Note here we have used the hypothesis that $f(K) \subset D_A$ to ensure that $F_k \subset D_A$.

We will show by contradiction that if M_1 is sufficiently large then (4.12) must hold. If (4.12) does not hold then by Lemma 4.4

$$cap(G_k) \geq \tau'$$

where

$$G_k := \{t \in f(K) : \ q_k(t) \geq e^{M_1 k}\}.$$

Now $|p_k(g(z))| \leq e^{-M_1 k}$ on $f^{-1}(G_k) \cap K = \{z \in K : \ f(z) \in G_k\}$ since $||h_k||_K = 1$ and by [17], Theorem 5.3.1

$$cap(f^{-1}(G_k) \cap K) = cap(\{z \in K : \ f(z) \in G_k\}) \geq \frac{1}{C} cap(G_k) \geq \tau'/C$$

where $C = \sup ||f'||_{D_A}$. But

$$|p_k(w)| \le e^{-M_1 k} \text{ for } w \in g\big(f^{-1}(G_k) \cap K\big)$$

and Lemma 4.5 gives

$$cap(g\big(f^{-1}(G_k) \cap K\big)) \ge \beta(D_A, \tau'/C, g) > 0.$$

Thus, by Lemma 4.3,

$$||p_k||_{D_A} \le e^{kL(D_A, \beta(D_A, \tau'/C, g))} ||p_k||_{g\left(f^{-1}(G_k) \cap K\right)} \le e^{kL(D_A, \beta(D_A, \tau'/C, g))} e^{-M_1 k}.$$

Here we have used $g(K) \subset D_A$ to insure $g\big(f^{-1}(G_k) \cap K\big) \subset D_A$. For M_1 sufficiently large this contradicts $|p_k(g(z_0))| = 1$. ■

We may now give the proof of Theorem 4.1.

Proof. We combine the bounds in Lemma 4.7 with the Bernstein–Walsh estimates (4.2) for polynomials and the set D_A: for f, g holomorphic on $U \supset D_A$ and p_k, q_k as in Lemma 4.7, that is, with $h_k \in \mathcal{F}_k$ normalized so $||h_k||_K = 1$,

$$\frac{1}{k} \log |p_k(g(z))| \le V_{D_A}(g(z)) + M$$

and

$$\frac{1}{k} \log |q_k(f(z))| \le V_{D_A}(f(z)) + M$$

provided $z \in U$. We obtain the estimate

$$\frac{1}{k} \log |h_k(z)| \le 2M + V_{D_A}(g(z)) + V_{D_A}(f(z)), \ z \in U$$

for some constant M for $h_k \in \mathcal{F}_k$ normalized so $||h_k||_K = 1$. Thus the family of subharmonic functions

$$\bigcup_{k \ge 1} \{ \frac{1}{k} \log |h_k(z)| : \ h_k \in \mathcal{F}_k \quad \text{and} \quad ||h_k||_K \le 1 \}$$

is locally bounded above in U. This implies that W_K^* is subharmonic on U (see [17], Theorem 3.4.2) and we have the bound (4.8). This gives the Bernstein–Walsh estimate (4.9) for functions $h_k \in \mathcal{F}_k$. The right-hand estimate in (4.8) depends on D_A but the estimate is valid at all $z \in U$ (i.e., at points where $f(z)$ is holomorphic).

Finally, to verify that $W^*_{K,Q}(z)$ is subharmonic on U, giving the weighted Bernstein–Walsh estimate (4.10), first observe by definition of $W_{K,Q}$ in (4.6) we have $W_{K,Q}(z) \leq Q(z)$ for $z \in K$. Since $\{z \in K : Q(z) < \infty\}$ is not polar, for sufficiently large C the compact set $F := \{z \in K : Q(z) \leq C\}$ is not polar. Then for $h_k \in \mathcal{F}_k$ with $||h_k e^{-kQ}||_K \leq 1$ we have $||h_k e^{-kQ}||_F \leq 1$ and

$$||h_k||_F \leq e^{kC}.$$

From definitions (4.5) and (4.6), $W_{K,Q}(z) \leq W^*_F(z) + C$ for all $z \in F$. Applying (4.8) with F instead of K (and $M = M(F)$), the family of subharmonic functions defining W^*_F and hence $W_{K,Q}$ is locally bounded above on U and $W^*_{K,Q}(z)$ is subharmonic on U. ■

For future use, we generalize the weighted Bernstein–Walsh estimate (4.10) to the unbounded setting in the case where $g(z) = z$. Here, for $K \subset \mathbb{C}$ closed and Q an f-admissible weight on K where f is a holomorphic function on a neighborhood U of K, the functions in $\mathcal{F}_k$ are of the form $h_k(z) = p_k(z)q_k(f(z))$ where $p_k, q_k \in \mathcal{P}_k$. We define $W_{K,Q}$ on U as in (4.6).

Proposition 4.8. Let $K \subset \mathbb{C}$ be closed and let f be holomorphic on a neighborhood U of K. Suppose Q is an f-admissible weight on K. Let

$$S = \{z \in K : W^*_{K,Q}(z) \geq Q(z)\}. \tag{4.13}$$

For all $h_k \in \mathcal{F}_k$, we have

$$|h_k(z)e^{-kQ(z)}| \leq ||h_k e^{-kQ}||_S \cdot e^{k[W^*_{K,Q}(z) - Q(z)]} \text{ for } z \in K. \tag{4.14}$$

□

Proof. Since by definition

$$|h_k(z)| \leq e^{kW^*_{K,Q}(z)},\ z \in U$$

for $h_k \in \mathcal{F}_k$ with $||h_k e^{-kQ}||_K = 1$, for such h_k,

$$|h_k(z)e^{-kQ(z)}| \leq e^{k[W^*_{K,Q}(z) - Q(z)]},\ z \in K. \tag{4.15}$$

For $z \in K \setminus S$, from (4.15) we have $|h_k(z)e^{-kQ(z)}| < 1$ for such h_k; hence

$$||h_k e^{-kQ}||_K = ||h_k e^{-kQ}||_S = 1.$$

Inserting this into the right-hand-side of (4.15) we have (4.14) for $h_k \in \mathcal{F}_k$ normalized so that $||h_k e^{-kQ}||_K = 1$. Then (4.14) follows for all $h_k \in \mathcal{F}_k$ by normalizing h_k. ■

Remark 4.9. Letting $K_R := K \cap \{|z| \leq R\}$, if we assume for some R sufficiently large that we have both $\{z \in K_R : Q(z) < \infty\}$ is nonpolar and $f(K_R) \subset U$, then for such R, taking a bounded neighborhood $D_A \subset U$ of the compact set $K_R \cup f(K_R)$, we conclude by the compact case that there is a $M > 0$ such that

$$W^*_{K,Q}(z) \leq W^*_{K_R,Q|_{K_R}}(z) \leq 2M + V_{D_A}(z) + V_{D_A}(f(z)),\ z \in U. \tag{4.16}$$

Since $K \subset U$, this estimate together with f-admissibility of Q imply that the set S in (4.13) is compact. In particular, this holds for f as in the two cases described in Remark 4.2. □

5 Bernstein–Markov Property

Our goal in this section is to give a sufficient mass-density condition on a finite measure ν on K to obtain weighted Bernstein–Markov properties for functions $h_k \in \mathcal{F}_k$.

Definition 5.1. Given $Q \in \mathcal{A}(K)$, a Borel measure μ on K satisfies a weighted Bernstein–Markov inequality for $\mathcal{F}_k$, if given $\epsilon > 0$, there is a constant C such that for all $k = 1, 2, \ldots$ and all $h_k \in \mathcal{F}_k$ we have

$$||h_k e^{-kQ}||_K \leq C e^{\epsilon k} \int_K |h_k(z)| e^{-kQ(z)} d\mu(z). \tag{5.1}$$

If μ satisfies a weighted Bernstein–Markov inequality for all continuous Q on K, we say μ satisfies a strong Bernstein–Markov inequality for $\mathcal{F}_k$ on K. □

We consider the following mass-density condition for positive Borel measures μ on K: there exist constants $T, r_0 > 0$ such that for all $z \in K$,

$$\mu(D(z, r)) \geq r^T \text{ for } 0 < r \leq r_0. \tag{5.2}$$

Here $D(z, r) := \{w \in \mathbb{C} : |w - z| < r\}$.

We will work with the following class of compact sets:

Definition 5.2. We call a compact set K *strongly regular* if every connected component of $\mathbb{C} \setminus K$ is regular with respect to the Dirichlet problem. □

A strongly regular compact set is, indeed, regular; for K is regular precisely when the unbounded component of $\mathbb{C} \setminus K$ is regular with respect to the Dirichlet problem. Thus any regular compact set K with connected complement, that is, $K = \widehat{K}$, is strongly regular. In particular, any regular compact subset of the real line is strongly regular, as is the closure of a bounded domain with C^1 boundary. The union of the unit circle with a non-regular compact subset of the open unit disk is regular but not strongly regular. The reason we consider the class of sets in Definition 5.2 is that regularity of a compact set is a property of its outer boundary while, when one considers weighted situations, other points in K can be of influence. An alternate characterization of a strongly regular set K, which we prove in Proposition 5.8, is that K *is not thin at each of its points* (see 2. in Lemma 5.4 and Corollary 5.5). Recall that a set S is not thin at a point $\zeta \in \bar{S}$ if $\limsup_{z\to\zeta,\ z\in S\setminus\{\zeta\}} u(z) = u(\zeta)$ for all functions u that are subharmonic in a neighborhood of ζ; otherwise we say S is thin at ζ. We return to a general discussion of strongly regular K after proving the sufficiency of (5.2) for μ on such a set to satisfy (5.1) (Theorem 5.6). The following result, whose proof we defer to the end of this section, will be needed.

Lemma 5.3. Let K be a strongly regular compact subset of $\mathbb{C}$. For any $z \in K$ and $r > 0$, there is a regular compact set $L \subset K \cap D(z, r)$ which contains $K \cap D(z, r/2)$. □

We next prove a type of regularity of W_K^* (Corollary 5.5) which will also be used in the proof of Theorem 5.6. We begin with a lemma.

Lemma 5.4. Let $K \subset \mathbb{C}$ be compact and let u be a subharmonic function on a neighborhood of K with $u \leq 0$ q.e. on K.

(1) If K is regular and u is subharmonic on a neighborhood of $\widehat{K}$, then $u \leq 0$ on $\widehat{K}$.
(2) If K is not thin at each of its points, then $u \leq 0$ on K. □

Proof. Since u is upper semicontinuous, the set $F = \{z \in K : \ u(z) > 0\}$ is an F_σ set. Since F is a polar set it is thin at all points of $\mathbb{C}$ (see [17], Theorem 3.8.2). In Case 1., K is not thin at any of its outer boundary points ([17], Theorem 4.2.4) so $K \setminus F$ is not thin at any outer boundary point of K. This implies that for ξ an outer boundary point, $u(\xi) = \limsup_{z\in K\setminus F,\ z\to\xi} u(z) \le 0$. Then since $u \le 0$ on the outer boundary of K, by the maximum principle $u \le 0$ on $\widehat{K}$. In Case 2., $K \setminus F$ is not thin at any point of K; thus for any point $\xi \in K$, we have $u(\xi) = \limsup_{z\in K\setminus F,\ z\to\xi} u(z) \le 0$. ■

Corollary 5.5. Let $K \subset \mathbb{C}$ be a compact, regular set satisfying (4.7):

$$K \subset D_A \subset U \text{ with } f(K), g(K) \subset D_A.$$

(1) If $U \supset \widehat{K}$, then $W_K^* = 0$ on $\widehat{K}$.

(2) If K is not thin at each of its points, then $W_K^* = 0$ on K. □

Proof. From Theorem 4.1, W_K^* is subharmonic on $U \supset K$. Since $W_K^* = W_K$ q.e., and by definition (4.5) $W_K = 0$ on K, Case 2. of Lemma 5.4 gives 2. of the corollary: $W_K^* = 0$ on K. If $U \supset \widehat{K}$, from the definition of W_K and the fact that for $z \in \widehat{K}$ we have $u(z) \le \sup_K u$ for all subharmonic functions u on U, it follows that $W_K(z) = 0$ on $\widehat{K}$. Thus $W_K^* = 0$ on $\widehat{K}$ follows from Case 1. of Lemma 5.4. ■

Theorem 5.6. Suppose $K \subset \mathbb{C}$ is a compact, strongly regular set satisfying (4.7) and μ is a Borel measure on K satisfying the mass-density condition (5.2). Then μ is a strong Bernstein–Markov measure for $\mathcal{F}_k$ on K. □

Proof. Fix $Q \in C(K)$. Given $\epsilon > 0$ choose $\delta > 0$ such that $|Q(z_1) - Q(z_2)| \le \epsilon$ for $z_1, z_2 \in K$ with $|z_1 - z_2| \le \delta$ and so that $\{z \in \mathbb{C} : d(z,K) \le \delta\} \Subset U$ (d being the Euclidean distance). Take a finite collection of disks $\{D(z_j, \delta/4)\}_{j=1,\ldots,m}$ with centers $z_j \in K$ that cover K. Since K is strongly regular, by Lemma 5.3, for each j we can find a regular compact set $L_j \subset K \cap \overline{D(z_j,\delta/2)}$ with $K \cap \overline{D(z_j,\delta/4)} \subset L_j$. By Corollary 5.5 $W_{L_j}^*$ is continuous on L_j and $W_{L_j}^* = 0$ on L_j. Thus we can find $\sigma = \sigma(\epsilon) > 0$ with $W_{L_j}^* \le \epsilon$ for all ζ with $d(\zeta, L_j) \le \sigma$ for $j = 1, \ldots, m$.

Now fix $h_k \in \mathcal{F}_k$ and let $w \in K$ be a point where the function $|h_k(z)|e^{-kQ(z)}$ assumes its maximum on K. Then $w \in L_j$ for some $j \in \{1, \ldots, m\}$. For $\zeta \in \overline{D(w,\sigma)}$,

$$|h_k(\zeta)| \le ||h_k||_{L_j} e^{k\epsilon} \tag{5.3}$$

by the Bernstein–Walsh estimate (4.9) for L_j. On the other hand, by choice of w, for any $z \in L_j$,

$$|h_k(z)|e^{-kQ(z)} \le |h_k(w)|e^{-kQ(w)}$$

(since $L_j \subset K$) and since $|z-w| \le \delta$ (for $z, w \in \overline{D(z_j, \delta/2)}$), we have $|Q(z) - Q(w)| \le \epsilon$ so that

$$|h_k(z)| \le |h_k(w)| e^{k\epsilon}$$

for all $z \in L_j$; that is

$$||h_k||_{L_j} \le |h_k(w)| e^{k\epsilon}. \tag{5.4}$$

Combining (5.3) and (5.4), we have

$$|h_k(\zeta)| \le |h_k(w)| e^{2k\epsilon}$$

for all $\zeta \in \overline{D(w, \sigma)}$.

Consider the function

$$U(t) := h_k \left(w + t \frac{z-w}{|z-w|} \right). \tag{5.5}$$

Then $t \to U(t)$ is holomorphic and $U(0) = h_k(w)$ while $U(|z-w|) = h_k(z)$. Also

$$||U||_{|t| \le \sigma} \le |h_k(w)| e^{2k\epsilon} \tag{5.6}$$

and

$$h_k(z) - h_k(w) = U(|z-w|) - U(0) = \int_0^{|z-w|} U'(t) \mathrm{d}t.$$

For $z \in D(w, \frac{\sigma}{2})$ we have

$$|h_k(z) - h_k(w)| \le |z-w| ||U'||_{|t| \le \frac{\sigma}{2}}.$$

Using the Cauchy estimate on U' and (5.6) we have

$$|h_k(z) - h_k(w)| \le |z-w| \frac{2}{\sigma} |h_k(w)| e^{2k\epsilon}.$$

Now let $r_k := e^{-3k\epsilon}$, so $r_k \le \frac{\sigma}{4} e^{-2k\epsilon}$ for k large. For $z \in D(w, r_k)$, we have

$$|h_k(z) - h_k(w)| \le \frac{\sigma}{4} e^{-2k\epsilon} \frac{2}{\sigma} |h_k(w)| e^{2k\epsilon} = \frac{1}{2} |h_k(w)|.$$

So

$$|h_k(z)| \ge \frac{1}{2} |h_k(w)|$$

for $z \in D(w, r_k)$ and

$$\begin{aligned} ||h_k e^{-kQ}||_{L^1(\mu)} &\geq \int_{K\cap D(w,r_k)} |h_k| e^{-kQ} d\mu \\ &\geq \frac{1}{2} |h_k(w)| e^{-kQ(w)} e^{-k\epsilon} \mu(D(w, r_k)) \\ &\geq C e^{-k\epsilon_1} ||h_k e^{-kQ}||_K \end{aligned}$$

where $\epsilon_1 = \epsilon(1 + 3T)$, since for k sufficiently large $\mu(D(w, r_k)) \geq r_k^T \geq e^{-3k\epsilon T}$. ■

Example 5.7. Some cases where condition (5.2) is satisfied are the following:

(1) $K \subset \mathbb{R}$ is a finite union of compact intervals and $d\mu = dx$, Lebesgue measure;
(2) $K = [0, 1] \subset \mathbb{R}$ and $d\mu = x^\alpha dx$ where $\alpha > 0$;
(3) $K \subset \mathbb{C}$ is a fat ($K = \overline{K^o}$) compact set with C^1 boundary and μ is planar Lebesgue measure. □

We return to the notion of strongly regular compact sets. Recall for a point $z \in K$, we say that Wiener's criterion holds at z if

$$\sum_n \frac{n}{\log 1/cap(K \cap S_n)} = \infty \tag{5.7}$$

where $S_n = D(z, 2^{-n}) \setminus D(z, 2^{-n-1})$. Wiener's theorem (cf. [17], Theorem 5.4.1) states that K is not thin at z precisely when (5.7) holds. In particular, if z is a boundary point of a connected component G of $\mathbb{C} \setminus K$, then z is a regular boundary point of G with respect to the Dirichlet problem if and only if Wiener's criterion holds at z (cf. [18] Theorem 2.1 of Appendix A). Thus G is regular with respect to the Dirichlet problem if and only if Wiener's criterion holds at every boundary point of G. This observation proves the "if" direction of the following proposition.

Proposition 5.8. Let $K \subset \mathbb{C}$ be compact. Then K is strongly regular if and only if K is not thin at each of its points. □

Proof. For the converse, suppose K is strongly regular. Fix $z \in K$; without loss of generality we may assume $z = 0$. Since the capacity of the annulus $S_n = D(0, 2^{-n}) \setminus D(0, 2^{-n-1})$ is 2^{-n}, Wiener's criterion is certainly true at 0 if 0 is an interior point. Also, by strong regularity and Wiener's theorem this criterion holds provided 0 is a boundary point of a connected component of $\mathbb{C} \setminus K$. Thus it is left to verify Wiener's criterion when 0 is a boundary point of K, but 0 does not belong to the boundary of any of the components of $\mathbb{C} \setminus K$.

There are two cases:

1. There are infinitely many n such that for every $r \in [2^{-n-1}, 2^n)$ the circle $C(0,r) = \{w : |w| = r\}$ intersects K. We consider such an n and let $w_r \in C(0,r) \cap K$. The mapping $w \to |w|$ is a contraction mapping of $K \cap (D(0,2^{-n}) \setminus D(0,2^{-n-1}))$ to the interval $[2^{-n-1}, 2^{-n})$, which, by assumption, maps onto $[2^{-n-1}, 2^{-n})$. Since the logarithmic capacity does not increase under a contraction mapping, and the capacity of $[2^{-n-1}, 2^{-n})$ is $2^{-n-1}/4 = 2^{-n-3}$, we obtain in this case that $cap(K \cap S_n) \geq 2^{-n-3}$, and hence for this particular n we have

$$\frac{n}{\log 1/cap(K \cap S_n)} \geq \frac{n}{(n+3)\log 2} \geq \frac{1}{8}.$$

Since this is true for infinitely many n, (5.7) holds.

2. For all sufficiently large n there is an $r_n \in [2^{-n-1}, 2^{-n})$ such that $C(0, r_n)$ is disjoint from K, that is, it lies in a component G_{r_n} of $\mathbb{C} \setminus K$. This G_{r_n} cannot be the same for infinitely many n, for then 0 would be a boundary point of that component. Thus, there are infinitely many n such that G_{r_n} and $G_{r_{n+1}}$ are different. But then every radial segment $\{re^{it} : r_{n+1} \leq r \leq r_n\}$ must intersect K, hence the mapping $\{re^{it} \to 2^{-n-2}e^{it}\}$ is a contraction mapping from $K \cap (S_n \cup S_{n+1})$ onto $C(0, 2^{-n-2})$. Therefore,

$$cap(K \cap (S_n \cup S_{n+1})) \geq cap(C(0, 2^{-n-2})) = 2^{-n-2},$$

and by [17], Theorem 5.1.4, we have then either

$$\frac{n}{\log 1/cap(K \cap S_n)} \geq \frac{n}{2(n+2)\log 2} \geq \frac{1}{6}$$

or

$$\frac{n+1}{\log 1/cap(K \cap S_{n+1})} \geq \frac{n+1}{2(n+2)\log 2} \geq \frac{1}{6}.$$

Thus the series in (5.7) contains infinitely many terms which are at least 1/6; hence (5.7) holds. ∎

We end this section with the proof of Lemma 5.3.

Proof. For simplicity we take $z = 0$ and $r \leq 1/2$. By Ancona's theorem [1] the set $K_n := K \cap (\overline{D(0, r/2 + 2^{-n})} \setminus D(0, r/2))$, if nonpolar, contains a regular compact set F_n such that

$$cap\,(K_n \setminus F_n) < e^{-n^3}.$$

Setting $F_n = \emptyset$ if K_n is polar, we define

$$L := \left(K \cap \overline{D(0,r/2)}\right) \bigcup (\cup_n F_n)\,.$$

We claim that L is regular. We need to prove that any outer boundary point z_0 of L is a regular point. From the Wiener criterion (5.7), we must show

$$\sum_n \frac{n}{\log\big(1/cap(L \cap S_n)\big)} = \infty \tag{5.8}$$

where $S_n = D(z_0, 2^{-n}) \setminus D(z_0, 2^{-n-1})$.

For $z_0 \in L$ outside the disk $\overline{D(0,r/2)}$ the union representing L is a locally finite union; thus (5.8) holds by regularity of the sets F_n. Also, by the strong regularity of K – note $L \subset K$ implies $cap(L \cap E) \le cap(K \cap E)$ for any set E – (5.8) is true for $z_0 \in L \cap D(0,r/2)$ (this statement is not necessarily true without the strong regularity hypothesis). It remains to prove (5.8) for $|z_0| = r/2$. By the strong regularity of K we have

$$\sum_n \frac{n}{\log\big(1/cap(K \cap S_n)\big)} = \infty.$$

Using Theorem 5.1.4 of [17], since $r \le 1/2$ it follows that either

$$\sum_n \frac{n}{\log\big(1/cap(K \cap \overline{D(0,r/2)} \cap S_n)\big)} = \infty, \tag{5.9}$$

or

$$\sum_n \frac{n}{\log\big(1/cap(K \cap (D(0,r) \setminus D(0,r/2)) \cap S_n)\big)} = \infty \tag{5.10}$$

(or both). If (5.9) holds then (5.8) is true since L contains $K \cap \overline{D(0,r/2)}$, so assume (5.10) is true. If $\mathcal{N}$ is the set of those n for which

$$\frac{n}{\log\big(1/cap(K \cap (D(0,r) \setminus D(0,r/2)) \cap S_n)\big)} > \frac{2}{n^2},$$

then we still have

$$\sum_{n \in \mathcal{N}} \frac{n}{\log\big(1/cap(K \cap (D(0,r) \setminus D(0,r/2)) \cap S_n)\big)} = \infty. \tag{5.11}$$

But again using Theorem 5.1.4 of [17],

$$\frac{1}{\log\big(1/cap(K \cap (D(0,r) \setminus D(0,r/2)) \cap S_n)\big)}$$

is no bigger than the sum of

$$\frac{1}{\log\big(1/cap(L \cap (D(0,r) \setminus D(0,r/2)) \cap S_n)\big)}$$

and

$$\frac{1}{\log\big(1/cap((K \setminus L) \cap (D(0,r) \setminus D(0,r/2)) \cap S_n)\big)}.$$

This latter quantity is no bigger than

$$\frac{1}{\log\Big(1/cap\left(\left[K \cap (\overline{D(0,r/2+2^{-n})} \setminus D(0,r/2))\right] \setminus F_n\right)\Big)},$$

which is smaller than $1/n^3$ by our choice of F_n. Thus for $n \in \mathcal{N}$, we necessarily have

$$\frac{n}{\log\big(1/cap(L \cap (D(0,r) \setminus D(0,r/2)) \cap S_n)\big)} > \frac{n/2}{\log\big(1/cap(K \cap (D(0,r) \setminus D(0,r/2)) \cap S_n)\big)}.$$

Hence, in this case, (5.8) is a consequence of (5.11). ■

6 Probabilistic Results in Compact Case

In this section, we work with K a compact, nonpolar subset of $\mathbb{C}$ satisfying (4.11) for a fixed f holomorphic on U; that is, we are in the case $g(z) = z$ (for the rest of this article). In this setting, we let ν be a measure on K with $\nu(K) < \infty$. Fix $Q \in \mathcal{A}(K)$. Define

$$\begin{aligned} Z_k &:= \int_{K^{k+1}} |VDM_k^Q(z_0,\ldots,z_k)| \mathrm{d}\nu(z_0)\cdots\mathrm{d}\nu(z_k) \\ &= \int_{K^{k+1}} |VDM(z_0,\ldots,z_k)| \mathrm{e}^{-k[Q(z_0)+\cdots+Q(z_k)]} |VDM(f(z_0),\ldots,f(z_k))| \mathrm{d}\nu(z_0)\cdots\mathrm{d}\nu(z_k) \end{aligned} \quad (6.1)$$

(recall (3.1)). A Bernstein–Markov property (5.1) for ν gives asymptotics of $\{Z_k\}$.

Proposition 6.1. Let $K \subset \mathbb{C}$ be a compact, nonpolar set satisfying (4.11). Suppose $Q \in \mathcal{A}(K)$ and ν is a measure on K with $\nu(K) < \infty$ satisfying (5.1). Then

$$\lim_{k\to\infty} Z_k^{2/k(k+1)} = \delta^Q(K) = \exp\left(-E^Q(\mu_{K,Q})\right). \tag{6.2}$$

□

Proof. Since

$$Z_k \leq \left(\max_{z_0,\ldots,z_k\in K} |VDM_k^Q(z_0,\ldots,z_k)|\right) \cdot \nu(K)^{k+1},$$

the upper bound

$$\limsup_{k\to\infty} Z_k^{2/k(k+1)} \leq \delta^Q(K) = \exp\left(-E^Q(\mu_{K,Q})\right)$$

follows from part 2. of Theorem 3.1. To prove the lower bound

$$\liminf_{k\to\infty} Z_k^{2/k(k+1)} \geq \delta^Q(K) = \exp\left(-E^Q(\mu_{K,Q})\right),$$

fix $\epsilon > 0$ and a set of weighted Fekete points $(a_0,\ldots,a_k)$ of order k for K, Q. Writing

$$|VDM_k^Q(a_0,\ldots,a_k)| = \prod_{i<j}|a_i - a_j| \cdot \prod_{i<j}|f(a_i) - f(a_j)| \cdot \exp\Big(-k[Q(a_0) + \cdots + Q(a_k)]\Big),$$

we recall from the beginning of Section 4 that

$$\begin{aligned} h_k(t) &:= VDM_k(t,a_1,\ldots,a_k) \cdot VDM_k(f(t),f(a_1),\ldots,f(a_k)) \cdot \exp\Big(-k[Q(a_1) + \cdots + Q(a_k)]\Big) \\ &= VDM_k^Q(t,a_1,\ldots,a_k)\mathrm{e}^{kQ(t)} = p_k(t)q_k(f(t)) \in \mathcal{F}_k \end{aligned}$$

as in (4.4) with $g(z) = z$ since p_k, q_k are polynomials of degree at most k. Then

$$h_k(t)\exp\left(-kQ(t)\right)$$

attains its maximum modulus on K at $t = a_0$. Applying the weighted Bernstein–Markov type inequality (5.1) gives

$$|VDM_k^Q(a_0,\ldots,a_k)| \leq C\mathrm{e}^{\epsilon k}\int_K |VDM_k^Q(t,a_1,\ldots,a_k)|\mathrm{d}\nu(t). \tag{6.3}$$

Now for each fixed $t \in K$, we consider $\widetilde{h}_k(s) := VDM_k^Q(t, s, a_2, \ldots, a_k) \in \mathcal{F}_k$. Then

$$|VDM_k^Q(t, a_1, \ldots, a_k)| = |\widetilde{h}_k(a_1)| \le \max_{s \in K} |\widetilde{h}_k(s)|$$

and we apply (5.1) in the right-hand-side integral in (6.3). Continuing this process in each variable, and using (3.4) for weighted Fekete points, we obtain the lower bound. ■

Given $K \subset \mathbb{C}$ compact, $Q \in \mathcal{A}(K)$, and a measure ν on K, we define a probability measure $Prob_k$ on K^{k+1}: for a Borel set $A \subset K^{k+1}$,

$$Prob_k(A) := \frac{1}{Z_k} \cdot \int_A |VDM_k^Q(\mathbf{X_k})| \mathrm{d}\nu(\mathbf{X_k}) \tag{6.4}$$

where $\mathbf{X_k} = (x_0, \ldots, x_k)$ and $\mathrm{d}\nu(\mathbf{X_k}) = \mathrm{d}\nu(x_0) \cdots \mathrm{d}\nu(x_k)$. Directly from (6.2) and (6.4) we obtain the following estimate.

Corollary 6.2. Let $K \subset \mathbb{C}$ be a compact, nonpolar set satisfying (4.11). For $Q \in \mathcal{A}(K)$ and ν a finite measure on K satisfying (6.2), given $\eta > 0$, define

$$A_{k,\eta} := \{\mathbf{X_k} \in K^{k+1} : |VDM_k^Q(\mathbf{X_k})| \ge (\delta^Q(K) - \eta)^{k(k+1)/2}\}. \tag{6.5}$$

Then there exists $k^* = k^*(\eta)$ such that for all $k > k^*$,

$$Prob_k(K^{k+1} \setminus A_{k,\eta}) \le \left(1 - \eta/(2\delta^Q(K))\right)^{k(k+1)/2} \nu(K^{k+1}).$$ □

We get the induced product probability measure $\mathbf{P}$ on the space of arrays on K,

$$\chi := \{X = \{\mathbf{X_k} \in K^{k+1}\}_{k \ge 1}\},$$

namely,

$$(\chi, \mathbf{P}) := \prod_{k=1}^{\infty} (K^{k+1}, Prob_k).$$

As an immediate consequence of Corollary 6.2, the Borel–Cantelli lemma, and 3. of Theorem 3.1, we obtain:

Corollary 6.3. Let $Q \in \mathcal{A}(K)$ and ν a finite measure on K satisfying (6.2). For $\mathbf{P}$-a.e. array $X = \{x_j^{(k)}\}_{j=0,\ldots,k;\ k=2,3,\ldots} \in \chi$,

$$\frac{1}{k+1} \sum_{j=0}^{k} \delta_{x_j^{(k)}} \to \mu_{K,Q} \text{ weakly as } k \to \infty.$$ □

We remark that $\mathcal{M}(K)$, with the weak topology, is a Polish space; that is, a separable, complete metrizable space. A neighborhood basis of $\mu \in \mathcal{M}(K)$ is given by sets of the form

$$G(\mu, k, \epsilon) := \{\sigma \in \mathcal{M}(K) : |\int_K x^a y^b (d\mu(z) - d\sigma(z))| < \epsilon$$
$$\text{for } 0 \leq a + b \leq k\}, \text{ where } z = x + iy.$$

We have all of the ingredients needed to follow the arguments of Section 6 of [4] to prove the analog of Theorem 6.6 there and hence an LDP (Definition 6.6 and Theorem 6.8 below) which quantifies the statement of $\mathbf{P}$-a.e. convergence for arrays $X = \{x_j^{(k)}\}$ of $\frac{1}{k+1}\sum_{j=0}^{k} \delta_{x_j^{(k)}}$ to $\mu_{K,Q}$. Given $G \subset \mathcal{M}(K)$, for each $k = 1, 2, \ldots$ we let

$$\widetilde{G}_k := \{\mathbf{a} = (a_0, \ldots, a_k) \in K^{k+1}, \frac{1}{k+1}\sum_{j=0}^{k} \delta_{a_j} \in G\}, \tag{6.6}$$

and set

$$J_k^Q(G) := \left[\int_{\widetilde{G}_k} |VDM_k^Q(\mathbf{a})| d\nu(\mathbf{a})\right]^{2/k(k+1)}. \tag{6.7}$$

Definition 6.4. For $\mu \in \mathcal{M}(K)$ we define

$$\overline{J}^Q(\mu) := \inf_{G \ni \mu} \overline{J}^Q(G), \text{ where } \overline{J}^Q(G) := \limsup_{k\to\infty} J_k^Q(G);$$
$$\underline{J}^Q(\mu) := \inf_{G \ni \mu} \underline{J}^Q(G), \text{ where } \underline{J}^Q(G) := \liminf_{k\to\infty} J_k^Q(G),$$

where the infimum is taken over all open neighborhoods $G \subset \mathcal{M}(K)$ of μ. If $Q = 0$ we simply write $\overline{J}(\mu), \underline{J}(\mu)$. □

Following the steps in Section 6 of [4] with Corollary 5.3 there replaced by our approximation result, Lemma 3.3, we obtain equality of the $\overline{J}^Q$ and $\underline{J}^Q$ functionals for any admissible weight Q provided ν is a strong Bernstein–Markov measure for $\mathcal{F}_k$ on K (see Theorem 6.6 in [4]).

Theorem 6.5. Let $K \subset \mathbb{C}$ be a compact, nonpolar set satisfying (4.11). Let $\nu \in \mathcal{M}(K)$ be a strong Bernstein–Markov measure for $\mathcal{F}_k$ on K (e.g., if ν satisfies a mass density

condition (5.2) and K is strongly regular).
(i) For any $\mu \in \mathcal{M}(K)$,

$$\log \overline{J}(\mu) = \log \underline{J}(\mu) = -I(\mu) - I(f_*\mu).$$

(ii) Let $Q \in \mathcal{A}(K)$. Then

$$\overline{J}^Q(\mu) = \overline{J}(\mu) \cdot \mathrm{e}^{-2\int_K Q \mathrm{d}\mu}$$

(and with the $\underline{J}, \underline{J}^Q$ functionals as well) so that,

$$\log \overline{J}^Q(\mu) = \log \underline{J}^Q(\mu) = -E^Q(\mu). \tag{6.8}$$

□

Thus we simply write J, J^Q without an underline or overline.

Define $j_k : K^{k+1} \to \mathcal{M}(K)$ via

$$j_k(x_0, \dots, x_k) = \frac{1}{k+1} \sum_{j=0}^{k} \delta_{x_j}. \tag{6.9}$$

The push-forward $\sigma_k := (j_k)_*(Prob_k)$ is a probability measure on $\mathcal{M}(K)$: for a Borel set $G \subset \mathcal{M}(K)$,

$$\sigma_k(G) = \frac{1}{Z_k} \int_{\widetilde{G}_k} |VDM_k^Q(x_0, \dots, x_k)| \mathrm{d}\nu(x_0) \cdots \mathrm{d}\nu(x_k). \tag{6.10}$$

Definition 6.6. The sequence $\{\sigma_k\}$ of probability measures on $\mathcal{M}(K)$ satisfies an LDP with good rate function $\mathcal{I}$ and speed $\{s_k\}$ with $s_k \to \infty$ if for all measurable sets $\Gamma \subset \mathcal{M}(K)$,

$$-\inf_{\mu \in \Gamma^0} \mathcal{I}(\mu) \leq \liminf_{k \to \infty} \frac{1}{s_k} \log \sigma_k(\Gamma) \text{ and} \tag{6.11}$$

$$\limsup_{k \to \infty} \frac{1}{s_k} \log \sigma_k(\Gamma) \leq -\inf_{\mu \in \bar{\Gamma}} \mathcal{I}(\mu). \tag{6.12}$$

□

We will give an interpretation of our specific LDP (Theorem 6.8) after its statement. On $\mathcal{M}(K)$, to prove a LDP it suffices to work with a base for the weak topology. The following is a special case of a basic general existence result, Theorem 4.1.11 in [9].

Proposition 6.7. Let $\{\sigma_\epsilon\}$ be a family of probability measures on $\mathcal{M}(K)$. Let $\mathcal{B}$ be a base for the topology of $\mathcal{M}(K)$. For $\mu \in \mathcal{M}(K)$ let

$$\mathcal{I}(\mu) := -\inf_{\{G\in\mathcal{B}:\mu\in G\}} \left(\liminf_{\epsilon\to 0} \epsilon \log \sigma_\epsilon(G)\right).$$

Suppose for all $\mu \in \mathcal{M}(K)$,

$$\mathcal{I}(\mu) := -\inf_{\{G\in\mathcal{B}:\mu\in G\}} \left(\limsup_{\epsilon\to 0} \epsilon \log \sigma_\epsilon(G)\right).$$

Then $\{\sigma_\epsilon\}$ satisfies an LDP with rate function $\mathcal{I}(\mu)$ and speed $1/\epsilon$. □

Following Section 7 of [4], Theorem 6.5 immediately yields an LDP:

Theorem 6.8. Assume ν is a strong Bernstein–Markov measure for $\mathcal{F}_k$ on K, $Q \in \mathcal{A}(K)$ and ν satisfies (6.2). The sequence $\{\sigma_k = (j_k)_*(Prob_k)\}$ of probability measures on $\mathcal{M}(K)$ satisfies an LDP with speed $k^2/2$ and good rate function $\mathcal{I} := \mathcal{I}_{K,Q}$ where, for $\mu \in \mathcal{M}(K)$,

$$\mathcal{I}(\mu) := \log J^Q(\mu_{K,Q}) - \log J^Q(\mu) = E^Q(\mu) - E^Q(\mu_{K,Q}). \qquad \square$$

Intuitively, this says the following. Given any $\mu \in \mathcal{M}(K)$ with $\mu \neq \mu_{K,Q}$, we know the probability that a random array $X = \{x_j^{(k)}\}_{j=0,\ldots,k;\ k=2,3,\ldots} \in \chi$ has the property that $\frac{1}{k+1}\sum_{j=0}^k \delta_{x_j^{(k)}} \to \mu$ is zero; the "rate" at which the probability that this sequence lies in small neighborhoods of μ tends to zero as $k \to \infty$ like $\exp\left[-k^2/2 \cdot \mathcal{I}(\mu)\right]$.

7 Some Results for K Unbounded

In this section, we let K be closed and unbounded; more specifically, recalling Remark 4.9, we take

$$K \subset \mathbb{C} \text{ for } f \text{ entire or } K \subset (0,\infty) \text{ for } f \text{ holomorphic in } H \text{ with } f(x) > 0 \text{ if } x \in (0,\infty), \tag{7.1}$$

where H is the right half plane. We only consider these two situations. We let Q be an f-admissible weight on K as in Definition 2.1: the function

$$\psi(x) := Q(x) - \frac{1}{2}\log\left[(1+|x|^2)(1+|f(x)|^2)\right]$$

satisfies $\liminf_{|x|\to\infty,\ x\in K} \psi(x) = \infty$.

We show that Theorem 3.1 remains valid in this setting. The first part of Theorem 3.1,

(1) if $\{z_j^{(k)}\}_{j=0,\ldots,k;k=2,3,\ldots} \subset K$ and $\{\mu_k = \frac{1}{k+1}\sum_{j=0}^{k}\delta_{z_j^{(k)}}\} \subset \mathcal{M}(K)$ converge weakly to $\mu \in \mathcal{M}(K)$, then (3.2), that is,

$$\limsup_{k\to\infty} |VDM_k^Q(z_0^{(k)},\ldots,z_k^{(k)})|^{2/k(k+1)} \le \exp\left(-E^Q(\mu)\right)$$

follows as in the case where K is compact. In order to verify the validity of the rest of Theorem 3.1 in this situation, recall that Proposition 4.8 gives the weighted Bernstein–Walsh estimate (4.14). Under (7.1), Remark 4.9 shows that the set S in (4.13) is compact. We will use (4.14) to show that the sequence of probability measures $\mu_k := \frac{1}{k+1}\sum_{j=0}^{k}\delta_{z_j^{(k)}}$ associated to an array $\{z_j^{(k)}\}_{j=0,\ldots,k;\ k=1,2,\ldots}$ of asymptotically weighted Fekete points for K, Q (see (3.4) and Remark 3.2) is *uniformly tight*, that is, given $\epsilon > 0$, there exists a compact set C_ϵ such that $\mu_k(K \setminus C_\epsilon) < \epsilon$ for all k. Indeed, we prove a stronger statement.

Proposition 7.1. For K and f as in (7.1), let $\{z_j^{(k)}\}_{j=0,\ldots,k;\ k=1,2,\ldots}$ be an array of asymptotically weighted Fekete points for K, Q, where Q is f-admissible. Let S be as in (4.13). For any $M > 0$ with $S \subset D(0,M)$ and any $\delta > 0$, there exists k_0 such that for all $k > k_0$,

$$\frac{\#\{j : z_j^{(k)} \in D(0,M)\}}{k} > 1 - \delta. \qquad \square$$

Proof. Fix $M > 0$ with $S \subset D(0,M)$ and k. Suppose $|z_0^{(k)}| > M$. Now

$$z_0^{(k)} \to VDM_k^Q(z_0^{(k)},\ldots,z_k^{(k)}) =: p_k(z_0^{(k)})q_k(f(z_0^{(k)}))\mathrm{e}^{-kQ(z_0^{(k)})}$$

for polynomials p_k, q_k of degree k. Let

$$H_{K,Q}(z) := W^*_{K,Q}(z) - Q(z).$$

By (4.14),

$$\begin{aligned}|p_k(z_0^{(k)})q_k(f(z_0^{(k)}))|\mathrm{e}^{-kQ(z_0^{(k)})} &\le \left(\max_{w\in S}|p_k(w)q_k(f(w))|\mathrm{e}^{-kQ(w)}\right)\cdot \mathrm{e}^{k[H_{K,Q}(z_0^{(k)})]}\\ &\le \left(\max_{w\in S}|p_k(w)q_k(f(w))|\mathrm{e}^{-kQ(w)}\right)\cdot \rho^k,\end{aligned}$$

where, by definition of S and M,

$$\rho = \exp[\sup\{[H_{K,Q}(z)] : z \in K,\ |z| > M\}] < 1.$$

Thus we can find $\widetilde{z}_0^{(k)} \in K \cap D(0,M)$ with

$$|VDM_k^Q(\widetilde{z}_0^{(k)},\ldots,z_k^{(k)})| = |p_k(\widetilde{z}_0^{(k)})q_k(f(\widetilde{z}_0^{(k)}))|\mathrm{e}^{-kQ(\widetilde{z}_0^{(k)})} = \max_{w\in K\cap D(0,M)}|p_k(w)q_k(f(w))|\mathrm{e}^{-kQ(w)}$$

so that

$$|VDM_k^Q(\widetilde{z}_0^{(k)},\ldots,z_k^{(k)})| \geq |VDM_k^Q(z_0^{(k)},\ldots,z_k^{(k)})|/\rho^k.$$

If $\#\{j : z_j^{(k)} > M\}/k > \delta$, by applying the same reasoning for each such point $z_j^{(k)}$, we obtain a set of k points $\widetilde{z}_0^{(k)},\ldots,\widetilde{z}_k^{(k)} \in K$ where $\lfloor \delta k \rfloor$ of the "tilde" points are new and lie in $K \cap D(0,M)$ with

$$|VDM_k^Q(\widetilde{z}_0^{(k)},\ldots,\widetilde{z}_k^{(k)})| \geq |VDM_k^Q(z_0^{(k)},\ldots,z_k^{(k)})|/\rho^{\lfloor \delta k \rfloor \cdot k}.$$

Taking $k(k+1)/2$ roots, we get that

$$\liminf_{k\to\infty} |VDM_k^Q(\widetilde{z}_0^{(k)},\ldots,\widetilde{z}_k^{(k)})|^{2/k(k+1)} \geq \frac{\lim_{k\to\infty} |VDM_k^Q(z_0^{(k)},\ldots,z_k^{(k)})|^{2/k(k+1)}}{\rho^{2\delta}}$$
$$= \delta^Q(K)/\rho^{2\delta} > \delta^Q(K),$$

a contradiction. ■

The importance of Proposition 7.1 is that the rest of Theorem 3.1; that is, parts 2. and 3., follows for this setting of $K \subset \mathbb{C}$ unbounded with Q an f-admissible weight on K. The uniform tightness allows one to extract a subsequence converging in the weak topology on $\mathcal{M}(K)$; we omit the details.

Corollary 7.2. For K and f as in (7.1) and Q an f-admissible weight on K,

(1) we have

$$\delta^Q(K) := \lim_{k\to\infty} \delta_k^Q(K) = \exp\left(-E^Q(\mu_{K,Q})\right);$$

(2) if $\{z_j^{(k)}\}_{j=0,\ldots,k;\ k=2,3,\ldots} \subset K$ and

$$\lim_{k\to\infty} |VDM_k^Q(z_0^{(k)},\ldots,z_k^{(k)})|^{2/k(k+1)} = \exp\left(-E^Q(\mu_{K,Q})\right)$$

then

$$\mu_k = \frac{1}{k+1}\sum_{j=0}^{k} \delta_{z_j^{(k)}} \to \mu_{K,Q} \text{ weakly.}$$ □

In the setting of an unbounded set K and an f-admissible weight Q, in order to have an analog of the $\{Z_k = Z_k(\nu)\}$ asymptotics in Proposition 6.1 we need some restriction on allowable measures ν related to Q ensuring finiteness of these quantities. Given a σ-finite measure ν on K one can impose the condition that

$$\exists\alpha > 0, \quad \int_K \epsilon(z)^\alpha \mathrm{d}\nu(z) < \infty \tag{7.2}$$

where $\epsilon(z)$ is some nonnegative continuous function that tends to 0 as $|z|$ tends to ∞ through points in K. For simplicity, we take

$$-\log \epsilon(z) \leq Q(z) - \log|zf(z)| \tag{7.3}$$

where the right-hand side goes to infinity as $|z|$ tends to ∞ through points in K by the f-admissibility of Q. For such triples (K, Q, ν), we use the same definition of a weighted Bernstein–Markov type inequality as in Definition 5.1.

We note that if ν satisfies a weighted Bernstein–Markov type inequality on any compact neighborhood of S in (4.13), then it satisfies a weighted Bernstein–Markov type inequality on K. Combining this observation with the examples given in Example 5.7, we see that, for appropriate unbounded $K \subset \mathbb{R}$ or $\mathbb{C}$, Lebesgue measure is a strong Bernstein–Markov measure in the setting of (7.1).

Using (4.14) in Proposition 4.8 one can prove the analog of Lemma 8.2 from [4] in our setting.

Lemma 7.3. For K and f as in (7.1), let Q be f-admissible and let ν be a σ-finite measure such that (K, Q, ν) satisfies (7.2) and a weighted Bernstein–Markov type inequality. We can find a closed neighborhood N of S (see (4.13)) and a constant $c > 0$ independent of k such that, for all $h_k \in \mathcal{F}_k$,

$$\int_K |h_k(z)|\mathrm{e}^{-kQ(z)}\mathrm{d}\nu(z) \leq (1 + \mathcal{O}(\mathrm{e}^{-ck})) \int_N |h_k(z)|\mathrm{e}^{-kQ(z)}\mathrm{d}\nu(z). \tag{7.4}$$

□

From the lemma, as in Section 6, one immediately obtains analogs of free energy asymptotics (Proposition 6.1) and hence Corollary 6.2 and the **P**-a.e. convergence result for arrays as in Corollary 6.3. An LDP can also be obtained when Q is strongly f-admissible; here, Lemma 2.6 can be utilized.

Finally, we remark that using the methods of this article, many results can be extended to cases where the discrete weighted energy minimization problem (see 3.1) involves products of three or more Vandermonde factors.

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