A global domination principle for $P-$pluripotential theory

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Abstract. We prove a global domination principle in the setting of $P-$pluripotential theory. This has many applications including a general product property for $P-$extremal functions. The key ingredient is the proof of the existence of a strictly plurisubharmonic $P-$potential.

In honor of 60 years of Tom Ransford

1. Introduction

Following [**1**], in [**2**] and [**4**] a pluripotential theory associated to plurisubharmonic (psh) functions on $\mathbb{C}^d$ having growth at infinity specified by $H_P(z) := \phi_P(\log|z_1|,...,\log|z_d|)$ where

$$\phi_P(x_1,...,x_d) := \sup_{(y_1,...,y_d)\in P}(x_1y_1+\cdots+x_dy_d)$$

is the indicator function of a convex body $P\subset(\mathbb{R}^+)^d$ was developed. Given P, the classes

$$L_P = L_P(\mathbb{C}^d) := \{u\in PSH(\mathbb{C}^d) : u(z)-H_P(z) = 0(1),\ |z|\to\infty\}$$

and

$$L_P^+ = L_P^+(\mathbb{C}^d) := \{u\in L_P : u(z)\geq H_P(z)+c_u\}$$

are of fundamental importance. These are generalizations of the standard Lelong classes $L(\mathbb{C}^d)$, the set of all plurisubharmonic (psh) functions u on $\mathbb{C}^d$ with $u(z)-\max[\log|z_1|,...,\log|z_d|] = 0(1),\ |z|\to\infty$, and

$$L^+(\mathbb{C}^d) = \{u\in L(\mathbb{C}^d) : u(z)\geq\max[0,\log|z_1|,...,\log|z_d|]+C_u\}$$

which correspond to $P=\Sigma$ where

$$\Sigma := \{(x_1,...,x_d)\in\mathbb{R}^d : x_1,...,x_d\geq 0,\ x_1+\cdots+x_d\leq 1\}.$$

For more on standard pluripotential theory, cf., [**7**].

Given $E\subset\mathbb{C}^d$, the $P-$*extremal function of* E is defined as

$$V_{P,E}^*(z) := \limsup_{\zeta\to z} V_{P,E}(\zeta)$$

where

$$V_{P,E}(z) := \sup\{u(z) : u\in L_P(\mathbb{C}^d),\ u\leq 0 \text{ on } E\}.$$

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For $P = \Sigma$, we write $V_E := V_{\Sigma,E}$. For E bounded and nonpluripolar, $V_E^* \in L^+(\mathbb{C}^d)$; $V_E^* = 0$ q.e. on E (i.e., on all of E except perhaps a pluripolar set); and $(dd^c V_E^*)^d = 0$ outside of $\bar{E}$ where $(dd^c V_E^*)^d$ is the complex Monge-Ampère measure of V_E^* (see section 2). A key ingredient in verifying a candidate function $v \in L^+(\mathbb{C}^d)$ is equal to V_E^* is the following global domination principle of Bedford and Taylor:

Proposition 1. [**3**] *Let $u \in L(\mathbb{C}^d)$ and $v \in L^+(\mathbb{C}^d)$ and suppose $u \leq v$ a.e.-$(dd^c v)^d$. Then $u \leq v$ on $\mathbb{C}^d$.*

Thus if one finds $v \in L^+(\mathbb{C}^d)$ with $v = 0$ a.e.-$(dd^c v)^d$ on $\bar{E}$ and $(dd^c v)^d = 0$ outside of $\bar{E}$ then $v = V_E^*$. For the proof of Proposition 1 in [**3**] the fact that in the definition of the Lelong classes $\max[\log|z_1|, ..., \log|z_d|]$ and $\max[0, \log|z_1|, ..., \log|z_d|]$ can be replaced by the Kähler potential

$$u_0(z) := \frac{1}{2}\log(1 + |z|^2) := \frac{1}{2}\log(1 + \sum_{j=1}^{d} |z_j|^2)$$

is crucial; this latter function is strictly psh and $(dd^c u_0)^d > 0$ on $\mathbb{C}^d$.

We prove a version of the global domination principle for very general L_P and L_P^+ classes. We consider convex bodies $P \subset (\mathbb{R}^+)^d$ satisfying

$$\Sigma \subset kP \text{ for some } k \in \mathbb{Z}^+. \tag{1.1}$$

Proposition 2. *For $P \subset (\mathbb{R}^+)^d$ satisfying (1.1), let $u \in L_P$ and $v \in L_P^+$ with $u \leq v$ a.e.-$(dd^c v)^d$. Then $u \leq v$ in $\mathbb{C}^d$.*

As a corollary, we obtain a generalization of Proposition 2.4 of [**4**] on $P-$extremal functions:

Proposition 3. *Given $P \subset (\mathbb{R}^+)^d$ satisfying (1.1), let $E_1, ..., E_d \subset \mathbb{C}$ be compact and nonpolar. Then*

$$V^*_{P,E_1\times\cdots\times E_d}(z_1, ..., z_d) = \phi_P(V^*_{E_1}(z_1), ..., V^*_{E_d}(z_d)). \tag{1.2}$$

The main issue in proving our version of the global domination principle (restated as Proposition 4 below) is the construction of a strictly psh $P-$potential u_P which can replace the logarithmic indicator function $H_P(z)$ used to define L_P and L_P^+. To do this, we utilize a classical result on subharmonic functions in the complex plane which we learned in Tom Ransford's beautiful book [**8**]; thus it is fitting that this article is written in his honor.

2. The global $P-$domination principle

Following [**2**] and [**4**], we fix a convex body $P \subset (\mathbb{R}^+)^d$; i.e., a compact, convex set in $(\mathbb{R}^+)^d$ with non-empty interior P^o. The most important example is the case where P is the convex hull of a finite subset of $(\mathbb{Z}^+)^d$ in $(\mathbb{R}^+)^d$ with $P^o \neq \emptyset$ (P is a non-degenerate convex polytope). Another interesting class consists of the $(\mathbb{R}^+)^d$ portion of an ℓ^q ball for $1 \leq q \leq \infty$; see (4.2). Recall that $H_P(z) := \phi_P(\log|z_1|, ..., \log|z_d|)$ where ϕ_P is the indicator function of P.

A C^2-function u on $D \subset \mathbb{C}^d$ is *strictly psh* on D if the complex Hessian $H(u) := [\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}]_{j,k=1,...,d}$ is positive definite on D. We define

$$dd^c u := 2i \sum_{j,k=1}^{d} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k$$

and

$$(dd^c u)^d = dd^c u \wedge \cdots \wedge dd^c u = c_d \det H(u) dV$$

where $dV = (\frac{i}{2})^d \sum_{j=1}^d dz_j \wedge d\bar{z}_j$ is the volume form on $\mathbb{C}^d$ and c_d is a dimensional constant. Thus u strictly psh on D implies that $(dd^c u)^d = f dV$ on D where $f > 0$. We remark that if u is a locally bounded psh function then $(dd^c u)^d$ is well-defined as a positive measure, the *complex Monge-Ampère measure* of u; this is the case, e.g., for functions $u \in L_P^+$.

Definition 2.1. We say that u_P is a *strictly psh $P-$potential* if

(1) $u_P \in L_P^+$ is strictly psh on $\mathbb{C}^d$ and
(2) there exists a constant C such that $|u_P(z) - H_P(z)| \leq C$ for all $z \in \mathbb{C}^d$.

This property implies that u_P can replace H_P in defining the L_P and L_P^+ classes:

$$L_P = \{u \in PSH(\mathbb{C}^d) : u(z) - u_P(z) = 0(1), \ |z| \to \infty\}$$

and

$$L_P^+ = \{u \in L_P : u(z) \geq u_P(z) + c_u\}.$$

Given the existence of a strictly psh $P-$potential, we can follow the proof of Proposition 1 in [**3**] to prove:

Proposition 4. *For $P \subset (\mathbb{R}^+)^d$ satisfying (1.1), let $u \in L_P$ and $v \in L_P^+$ with $u \leq v$ a.e. $(dd^c v)^d$. Then $u \leq v$ in $\mathbb{C}^d$.*

Proof. Suppose the result is false; i.e., there exists $z_0 \in \mathbb{C}^d$ with $u(z_0) > v(z_0)$. Since $v \in L_P^+$, by adding a constant to v we may assume $v(z) \geq u_P(z)$ in $\mathbb{C}^d$. Note that $(dd^c u_P)^d > 0$ on $\mathbb{C}^d$. Fix $\delta, \ \epsilon > 0$ with $\delta < \epsilon/2$ in such a way that the set

$$S := \{z \in \mathbb{C} : u(z) + \delta u_P(z) > (1+\epsilon) v(z)\}$$

contains z_0. Then S has positive Lebesgue measure. Moreover, since $\delta < \epsilon$ and $v \geq u_P$, S is bounded. By the comparison principle (cf., Theorem 3.7.1 [**7**]), we conclude that

$$\int_S (dd^c[u + \delta u_P])^d \leq \int_S (dd^c (1+\epsilon) v)^d.$$

But $\int_S (dd^c \delta u_P)^d > 0$ since S has positive Lebesgue measure, so

$$(1+\epsilon)^d \int_S (dd^c v)^d > 0.$$

By hypothesis, for a.e.-$(dd^c v)^d$ points in $\text{supp}(dd^c v)^d \cap S$ (which is not empty since $\int_S (dd^c v)^d > 0$), we have

$$(1+\epsilon) v(z) < u(z) + \delta u_P(z) \leq v(z) + \delta u_P(z),$$

i.e., $v(z) < \frac{1}{2} u_P(z)$ since $\delta < \epsilon/2$. This contradicts the normalization $v(z) \geq u_P(z)$ in $\mathbb{C}^d$. □

In the next section, we show how to construct u_P in Definition 2.1 for a convex body in $(\mathbb{R}^+)^d$ satisfying (1.1).

3. Existence of strictly psh $P-$potential

For the P we consider, $\phi_P \geq 0$ on $(\mathbb{R}^+)^d$ with $\phi_P(0) = 0$. We write $z^J = z_1^{j_1} \cdots z_d^{j_d}$ where $J = (j_1, ..., j_d) \in P$ (the components j_k need not be integers) so that

$$H_P(z) := \sup_{J \in P} \log |z^J| := \phi_P(\log^+ |z_1|, ..., \log^+ |z_d|)$$

with $|z^J| := |z_1|^{j_1} \cdots |z_d|^{j_d}$. To construct a strictly psh $P-$potential u_P, we first assume P is a convex polytope in $(\mathbb{R}^+)^d$ satisfying (1.1). Thus

$$(a_1, 0, ..., 0), \ldots, (0, ..., 0, a_d) \in \partial P$$

for some $a_1, ..., a_d > 0$. A calculation shows that

$$\log(1 + |z_1|^{2a_1} + \cdots + |z_d|^{2a_d})$$

is strictly psh in $\mathbb{C}^d$.

We claim then that

$$u_P(z) := \frac{1}{2} \log(1 + \sum_{J \in Extr(P)} |z^J|^2) \tag{3.1}$$

is strictly psh in $\mathbb{C}^d$ and the L_P, L_P^+ classes can be defined using u_P instead of H_P; i.e., u_P satisfies (1) and (2) of Definition 2.1. Here, $Extr(P)$ denotes the extreme points of P but we omit the origin $\mathbf{0}$. Note that $(a_1, 0, ..., 0), \ldots, (0, ..., 0, a_d) \in Extr(P)$.

Indeed, in this case,

$$H_P(z) = \sup_{J \in P} \log |z^J| = \max[0, \max_{J \in Extr(P)} \log |z^J|]$$

so clearly for $|z|$ large, $|u_P(z) - H_P(z)| = 0(1)$ while on any compact set K,

$$\sup_{z \in K} |u_P(z) - H_P(z)| \leq C = C(K)$$

which gives (2) (and therefore that $u_P \in L_P^+$).

It remains to verify the strict psh of u_P in (3.1). We use reasoning based on a classical univariate result which is exercise 4 in section 2.6 of [**8**]: if u, v are nonnegative with $\log u$ and $\log v$ subharmonic (shm) – hence u, v are shm – then $\log(u + v)$ is shm. The usual proof is to show $(u + v)^a$ is shm for any $a > 0$ – which is exercise 3 in section 2.6 of [**8**] – which trivially follows since u, v are shm and $a > 0$. However, assume u, v are smooth and compute the Laplacian $\Delta \log(u + v)$ on $\{u, v > 0\}$:

$$\big(\log(u + v)\big)_{z\bar{z}} = \frac{(u + v)(u_{z\bar{z}} + v_{z\bar{z}}) - (u_z + v_z)(u_{\bar{z}} + v_{\bar{z}})}{(u + v)^2}$$

$$= \frac{[uu_{z\bar{z}} - |u_z|^2 + vv_{z\bar{z}} - |v_z|^2] + [uv_{z\bar{z}} + vu_{z\bar{z}} - 2\Re(u_z v_{\bar{z}})]}{(u + v)^2}.$$

Now $\log u$, $\log v$ shm implies $uu_{z\bar{z}} - |u_z|^2 \geq 0$ and $vv_{z\bar{z}} - |v_z|^2 \geq 0$ with strict inequality in case of strict shm. Since $\log(u + v)$ is shm, the entire numerator is nonnegative:

$$[uu_{z\bar{z}} - |u_z|^2 + vv_{z\bar{z}} - |v_z|^2] + [uv_{z\bar{z}} + vu_{z\bar{z}} - 2\Re(u_z v_{\bar{z}})] \geq 0$$

so that the "extra term"

$$uv_{z\bar{z}} + vu_{z\bar{z}} - 2\Re(u_z v_{\bar{z}})$$

is nonnegative whenever

$$(\log u)_{z\bar{z}} + (\log v)_{z\bar{z}} = uu_{z\bar{z}} - |u_z|^2 + vv_{z\bar{z}} - |v_z|^2 = 0.$$

We show $\Delta \log(u+v)$ is strictly positive on $\{u, v > 0\}$ if one of $\log u$ or $\log v$ is strictly shm.

Proposition 5. *Let $u, v \geq 0$ with $\log u$ and $\log v$ shm. If one of $\log u$ or $\log v$ is strictly shm, e.g., $\Delta \log u > 0$, then $\Delta \log(u+v) > 0$ on $\{u, v > 0\}$.*

Proof. We have $u, v \geq 0$, $u_{z\bar{z}}, v_{z\bar{z}} \geq 0$, $vv_{z\bar{z}} - |v_z|^2 \geq 0$ and $uu_{z\bar{z}} - |u_z|^2 > 0$ if $u > 0$. We want to show that

$$uv_{z\bar{z}} + vu_{z\bar{z}} - 2\Re(u_z v_{\bar{z}}) = uv_{z\bar{z}} + vu_{z\bar{z}} - (u_z v_{\bar{z}} + v_z u_{\bar{z}}) > 0$$

on $\{u, v > 0\}$. We start with the identity

$$(uv_z - vu_z)(uv_{\bar{z}} - vu_{\bar{z}}) = u^2 v_z v_{\bar{z}} + v^2 u_z u_{\bar{z}} - uv(u_z v_{\bar{z}} + v_z u_{\bar{z}}) \geq 0. \tag{3.2}$$

Since $uu_{z\bar{z}} - |u_z|^2 > 0$ and $vv_{z\bar{z}} - |v_z|^2 \geq 0$,

$$uu_{z\bar{z}} > u_z u_{\bar{z}}, \ vv_{z\bar{z}} \geq v_z v_{\bar{z}}.$$

Thus

$$uv_{z\bar{z}} + vu_{z\bar{z}} = \frac{u}{v} vv_{z\bar{z}} + \frac{v}{u} uu_{z\bar{z}} > \frac{u}{v} v_z v_{\bar{z}} + \frac{v}{u} u_z u_{\bar{z}}.$$

Thus it suffices to show

$$\frac{u}{v} v_z v_{\bar{z}} + \frac{v}{u} u_z u_{\bar{z}} \geq u_z v_{\bar{z}} + u_{\bar{z}} v_z.$$

Multiplying both sides by uv, this becomes

$$u^2 v_z v_{\bar{z}} + v^2 u_z u_{\bar{z}} \geq uv(u_z v_{\bar{z}} + u_{\bar{z}} v_z).$$

This is (3.2).

□

This proof actually shows that

$$uv_{z\bar{z}} + vu_{z\bar{z}} - 2\Re(u_z v_{\bar{z}}) > 0$$

under the hypotheses of the proposition.

Remark 3.1. To be precise, this shows strict shm only on $\{u, v > 0\}$. In the multivariate case, this shows the restriction of $\log(u+v)$ to the intersection of a complex line and $\{u, v > 0\}$ is strictly shm if one of $\log u, \log v$ is strictly psh so that $\log(u+v)$ is strictly psh on $\{u, v > 0\}$.

Now with u_P in (3.1) we may write

$$u_P(z) = \log(u+v)$$

where

$$u(z) = 1 + |z_1|^{2a_1} + \cdots + |z_d|^{2a_d} \tag{3.3}$$

– so that $\log u$ is strictly psh in $\mathbb{C}^d$ – and

$$v(z) = \sum_{J \in Extr(P)} |z^J|^2 - |z_1|^{2a_1} - \cdots - |z_d|^{2a_d}.$$

If $v \equiv 0$ (e.g., if $P = \Sigma$) we are done. Otherwise $v \geq 0$ (being a sum of non-negative terms) and $\log v$ is psh (being the logarithm of a sum of moduli squared of holomorphic functions) showing that $u_P(z) := \frac{1}{2} \log(1 + \sum_{J \in Extr(P)} |z^J|^2)$ is strictly psh where $v > 0$. There remains an issue at points where $v = 0$ (coordinate axes). However, if we simply replace the decomposition $u_P(z) = \log(u + v)$ by $u_P(z) = \log(u_\epsilon + v_\epsilon)$ where

$$u_\epsilon := 1 + (1-\epsilon)(|z_1^a|^2 + ... + |z_d^a|^2) \text{ and}$$
$$v_\epsilon := \sum_{J \in ExtrP} |z^J|^2 - (1-\epsilon)(|z_1^a|^2 + ... + |z_d^a|^2)$$

for $\epsilon > 0$ sufficiently small, then the result holds everywhere. We thank F. Piazzon for this last observation.

If $P \subset (\mathbb{R}^+)^d$ is a convex body satisfying (1.1), we can approximate P by a monotone decreasing sequence of convex polytopes P_n satisfying the same property. Since $P_{n+1} \subset P_n$ and $\cap_n P_n = P$, the sequence $\{u_{P_n}\}$ decreases to the function $u_P \in L_P^+$. Since each u_{P_n} is of the form

$$u_{P_n}(z) = \log(u_n + v_n)$$

where $u_n(z) = 1 + |z_1|^{2a_{n1}} + \cdots + |z_d|^{2a_{nd}}$ and $a_{nj} \geq a_j$ for all n and each $j = 1, ..., d$ in (3.3), it follows that u_P is strictly psh and hence satisfies Definition 2.1. This concludes the proof of Proposition 4.

Remark 3.2. Another construction of a strictly psh $P-$potential as in Definition 2.1 which is based on solving a real Monge-Ampère equation and which works in more general situations was recently given by C. H. Lu [**5**]. Indeed, his construction, combined with Corollary 3.10 of [**6**], yields a new proof of the global domination principle, Proposition 4.

4. The product property

In this section, we prove the product property stated in the introduction:

Proposition 6. *For $P \subset (\mathbb{R}^+)^d$ satisfying (1.1), let $E_1, ..., E_d \subset \mathbb{C}$ be compact and nonpolar. Then*

$$V^*_{P,E_1\times\cdots\times E_d}(z_1, ..., z_d) = \phi_P(V^*_{E_1}(z_1), ..., V^*_{E_d}(z_d)). \tag{4.1}$$

Remark 4.1. One can verify the formula

$$V_{P,T^d}(z) = H_P(z) = \sup_{J\in P} \log |z^J|$$

for the $P-$extremal function of the torus

$$T^d := \{(z_1, ..., z_d) : |z_j| = 1,\ j = 1, ..., d\}$$

for a general convex body by modifying the argument in [**7**] for the standard extremal function of a ball in a complex norm. Indeed, let $u \in L_P$ with $u \leq 0$ on T^d. For $w = (w_1, ..., w_d) \notin T^d$ and $w_j \neq 0$, we consider

$$v(\zeta_1, ..., \zeta_d) := u(w_1/\zeta_1, ..., w_d/\zeta_d) - H_P(w_1/\zeta_1, ..., w_d/\zeta_d).$$

This is psh on $0 < |\zeta_j| < |w_j|,\ j = 1, ..., d$. Since $u \in L_P$, v is bounded above near the pluripolar set given by the union of the coordinate planes in this polydisk and hence extends to the full polydisk. On the boundary $|\zeta_j| = |w_j|$, $v \leq 0$ so at $(1, 1, ..., 1)$ we get $u(w_1, ..., w_d) \leq H_P(w_1, ..., w_d)$. Note

$$H_P(z) = \sup_{J\in P} \log |z^J| = \phi_P(\log^+ |z_1|, ..., \log^+ |z_d|)$$

and $V_{T^1}(\zeta) = \log^+ |\zeta|$ so this is a special case of Proposition 6.

PROOF. For simplicity we consider the case $d = 2$ with variables (z, w) on $\mathbb{C}^2$. As in [**4**], we may assume V_E and V_F are continuous. Also, by approximation we may assume ϕ_P is smooth. We write

$$v(z,w) := \phi_P(V_E(z), V_F(w)).$$

An important remark is that, since $P \subset (\mathbb{R}^+)^2$, P is convex, and P contains $k\Sigma$ for some $k > 0$, the function ϕ_P on $(\mathbb{R}^+)^2$ satisfies

(1) $\phi_P \geq 0$ and $\phi_P(x, y) = 0$ only for $x = y = 0$;
(2) ϕ_P is nondecreasing in each variable; i.e., $(\phi_P)_x, (\phi_P)_y \geq 0$;
(3) ϕ_P is convex; i.e., the real Hessian $H_{\mathbb{R}}(\phi_P)$ of ϕ_P is positive semidefinite; and, more precisely, by the homogenity of ϕ_P; i.e., $\phi_P(tx, ty) = t\phi_P(x, y)$,

$$\det H_{\mathbb{R}}(\phi_P) = 0 \text{ away from the origin.}$$

As in [**4**], to see that

$$v(z,w) \leq V_{P,E\times F}(z,w),$$

since $\phi_P(0, 0) = 0$, it suffices to show that $\phi_P(V_E(z), V_F(w)) \in L_P(\mathbb{C}^2)$. From the definition of ϕ_P,

$$\phi_P(V_E(z), V_F(w)) = \sup_{(x,y)\in P} [xV_E(z) + yV_F(w)]$$

which is a locally bounded above upper envelope of plurisubharmonic functions. As ϕ_P is convex and V_E, V_F are continuous, $\phi_P(V_E(z), V_F(w))$ is continuous. Since $V_E(z) = \log|z| + 0(1)$ as $|z| \to \infty$ and $V_F(w) = \log|w| + 0(1)$ as $|w| \to \infty$, it follows that $\phi_P(V_E(z), V_F(w)) \in L_P(\mathbb{C}^2)$.

By Proposition 4, it remains to show $(dd^c v)^2 = 0$ outside of $E \times F$. Since we can approximate v from above uniformly by a decreasing sequence of smooth psh functions by convolving v with a smooth bump function, we assume v is smooth and compute the following derivatives:

$$v_z = (\phi_P)_x(V_E)_z,\ v_w = (\phi_P)_y(V_F)_w;$$
$$v_{z\bar{z}} = (\phi_P)_{xx}|(V_E)_z|^2 + (\phi_P)_x(V_E)_{z\bar{z}};$$
$$v_{z\bar{w}} = (\phi_P)_{xy}(V_E)_z(V_F)_{\bar{w}};$$
$$v_{w\bar{w}} = (\phi_P)_{yy}|(V_F)_w|^2 + (\phi_P)_y(V_F)_{w\bar{w}}.$$

It follows from (2) that $v_{z\bar{z}}, v_{w\bar{w}} \geq 0$. Next, we compute the determinant of the complex Hessian of v on $(\mathbb{C} \setminus E) \times (\mathbb{C} \setminus F)$ (so $(V_E)_{z\bar{z}} = (V_F)_{w\bar{w}} = 0$):

$$\begin{gathered} v_{z\bar{z}}v_{w\bar{w}} - |v_{z\bar{w}}|^2 \\ = (\phi_P)_{xx}|(V_E)_z|^2(\phi_P)_{yy}|(V_F)_w|^2 - [(\phi_P)_{xy}]^2|(V_E)_z|^2|(V_F)_w|^2 = \\ = |(V_E)_z|^2|(V_F)_w|^2[(\phi_P)_{xx}(\phi_P)_{yy} - (\phi_P)_{xy}]^2]. \end{gathered}$$

This is nonnegative by the convexity of ϕ_P and, indeed, it vanishes on $(\mathbb{C} \setminus E) \times (\mathbb{C} \setminus F)$ by (3). The general formula for the determinant of the complex Hessian of v is

$$\begin{gathered} v_{z\bar{z}}v_{w\bar{w}} - |v_{z\bar{w}}|^2 \\ = |(V_E)_z|^2|(V_F)_w|^2[(\phi_P)_{xx}(\phi_P)_{yy} - (\phi_P)_{xy}]^2] + (\phi_P)_{xx}|(V_E)_z|^2(\phi_P)_y(V_F)_{w\bar{w}} \\ +(\phi_P)_{yy}|(V_F)_w|^2(\phi_P)_x(V_E)_{z\bar{z}} + (\phi_P)_x(V_E)_{z\bar{z}}(\phi_P)_y(V_F)_{w\bar{w}}. \end{gathered}$$

If, e.g., $z \in E$ and $w \in (\mathbb{C} \setminus F)$,

$$|(V_E)_z|^2|(V_F)_w|^2[(\phi_P)_{xx}(\phi_P)_{yy} - (\phi_P)_{xy}]^2] = 0$$

by (3) (since $(V_E(z), V_F(w)) = (0, a) \neq (0, 0)$) and $(V_F)_{w\bar{w}} = 0$ so

$$v_{z\bar{z}}v_{w\bar{w}} - |v_{z\bar{w}}|^2 = (\phi_P)_{yy}|(V_F)_w|^2(\phi_P)_x(V_E)_{z\bar{z}}.$$

However, we claim that

$$(\phi_P)_{yy}(0, a) = 0 \text{ if } a > 0$$

since we have $\phi_P(0, ty) = t\phi_P(0, y)$. Hence

$$v_{z\bar{z}}v_{w\bar{w}} - |v_{z\bar{w}}|^2 = 0$$

if $z \in E$ and $w \in (\mathbb{C} \setminus F)$. Similarly,

$$(\phi_P)_{xx}(a, 0) = 0 \text{ if } a > 0$$

so that

$$v_{z\bar{z}}v_{w\bar{w}} - |v_{z\bar{w}}|^2 = 0$$

if $z \in (\mathbb{C} \setminus E)$ and $w \in F$.

□

Remark 4.2. In [**4**], a (much different) proof of Proposition 6 was given under the additional hypothesis that $P \subset (\mathbb{R}^+)^d$ be a *lower set*: for each $n = 1, 2, ...,$ whenever $(j_1, ..., j_d) \in nP \cap (\mathbb{Z}^+)^d$ we have $(k_1, ..., k_d) \in nP \cap (\mathbb{Z}^+)^d$ for all $k_l \leq j_l,\ l = 1, ..., d$.

Finally, although computation of the $P-$extremal function of a product set is now rather straightforward, even qualitative properties of the corresponding Monge-Ampère measure are less clear. To be concrete, for $q \geq 1$, let

$$P_q := \{(x_1, ..., x_d) : x_1, ..., x_d \geq 0,\ x_1^q + \cdots + x_d^q \leq 1\} \tag{4.2}$$

be the $(\mathbb{R}^+)^d$ portion of an ℓ^q ball. Then for $1/q' + 1/q = 1$ we have $\phi_{P_q}(x) = ||x||_{\ell^{q'}}$ (for $q = \infty$ we take $q' = 1$ and vice-versa). Hence if $E_1, ..., E_d \subset \mathbb{C}$,

$$\begin{aligned} V^*_{P_q, E_1\times\cdots\times E_d}(z_1, ..., z_d) &= \|[V^*_{E_1}(z_1), V^*_{E_2}(z_2), \ldots, V^*_{E_d}(z_d)]\|_{\ell^{q'}} \\ &= [V^*_{E_1}(z_1)^{q'} + \cdots + V^*_{E_d}(z_d)^{q'}]^{1/q'}. \end{aligned}$$

In the standard case $q = 1$, $P_1 = \Sigma$ and we have the well-known result that

$$V^*_{E_1\times\cdots\times E_d}(z_1, ..., z_d) = \max[V^*_{E_1}(z_1), V^*_{E_2}(z_2), \ldots, V^*_{E_d}(z_d)].$$

Then if none of the sets E_j are polar,

$$(dd^cV^*_{E_1\times\cdots\times E_d})^d = \mu_{E_1} \times \cdots \times \mu_{E_d}$$

where $\mu_{E_j} = \Delta V^*_{E_j}$ is the classical equilibrium measure of E_j.

Question 7. What can one say about $\text{supp}(dd^cV^*_{P_q, E_1\times\cdots\times E_d})^d$ in the case when $q > 1$?

As examples, for $T^d = \{(z_1, ..., z_d) : |z_j| = 1,\ j = 1, ..., d\}$ we have $V_T(z_j) = \log^+ |z_j|$ and hence for $q \geq 1$

$$V_{P_q, T^d}(z) = \phi_{P_q}(\log^+ |z_1|, ..., \log^+ |z_d|) = [\sum_{j=1}^{d}(\log^+ |z_j|)^{q'}]^{1/q'}.$$

The measure $(dd^cV_{P_q, T^d})^d$ is easily seen to be invariant under the torus action and hence is a positive constant times Haar measure on T^d. Thus in this case $\text{supp}(dd^cV_{P_q, T^d})^d = T^d$ for $q \geq 1$.

For the set $[-1,1]^d$ we have $V_{[-1,1]}(z_j) = \log|z_j + \sqrt{z_j^2 - 1}|$ and hence for $q \geq 1$

$$V_{P_q,[-1,1]^d}(z_1, ..., z_d) = \left\{ \sum_{j=1}^{d} \left(\log \left| z_j + \sqrt{z_j^2 - 1} \right| \right)^{q'} \right\}^{1/q'}.$$

In this case, it is not clear for $q > 1$ whether supp $(dd^c V_{P_q,[-1,1]^d})^d = [-1,1]^d$.

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