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CONSTRUCTIVE APPROXIMATION

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An Orthogonality Property of the Legendre Polynomials

L. Bos[1] · A. Narayan[2] · N. Levenberg[3] · F. Piazzon[4]

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Abstract We give a remarkable additional othogonality property of the classical Legendre polynomials on the real interval $[-1, 1]$: polynomials up to degree n from this family are mutually orthogonal under the arcsine measure weighted by the normalized degree-n Christoffel function.

Keywords Legendre polynomials · Christoffel function · Equilibrium measure

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✉ L. Bos
leonardpeter.bos@univr.it

A. Narayan
akil@sci.utah.edu

N. Levenberg
nlevenbe@indiana.edu

F. Piazzon
fpiazzon@math.unipd.it

1 Department of Computer Science, University of Verona, Verona, Italy

2 Mathematics Department, Scientific Computing and Imaging Institute (SCI), University of Utah, Salt Lake City, UT, USA

3 Department of Mathematics, Indiana University, Bloomington, IN, USA

4 Department of Mathematics, University of Padua, Padua, Italy

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Let $P_n(x)$ denote the classical Legendre polynomial of degree n and

$$P_n^*(x) := \frac{\sqrt{2n+1}}{\sqrt{2}} P_n(x)$$

its orthonormalized version. Thus, with $\delta_{i,j}$ the Kronecker delta, the family P_n^* satisfies

$$\int_{-1}^{1} P_i^*(x) P_j^*(x)\, \mathrm{d}x = \delta_{i,j}, \quad i, j \geq 0.$$

We consider the normalized (reciprocal of) the associated Christoffel function

$$K_n(x) := \frac{1}{n+1} \sum_{k=0}^{n} \left(P_k^*(x)\right)^2. \tag{1}$$

As is well known, $K_n(x)\mathrm{d}x$ tends weak-star to the equilibrium measure of complex potential theory for the interval $[-1, 1]$, and more precisely,

$$\lim_{n\to\infty} K_n(x) = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}, \quad x \in (-1, 1),$$

locally uniformly. In other words,

$$\lim_{n\to\infty} \frac{1}{K_n(x)} \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} = 1, \quad x \in (-1, 1),$$

locally uniformly, and it would not be unexpected that, at least asymptotically,

$$\int_{-1}^{1} P_i^*(x) P_j^*(x) \frac{1}{K_n(x)} \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} \mathrm{d}x \approx \delta_{ij}, \quad 0 \leq i, j \leq n.$$

The purpose of this paper is to show that the above is actually an identity.

Theorem 1 *With the above notation,*

$$\int_{-1}^{1} P_i^*(x) P_j^*(x) \frac{1}{K_n(x)} \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} \mathrm{d}x = \delta_{ij}, \quad 0 \leq i, j \leq n. \tag{2}$$

We expect this result to have use in applied approximation problems. For example, one application lies in polynomial approximation of functions from point-evaluations. Our result indicates that the functions $\left\{Q_j(x)\right\}_{j=0}^n := \left\{\frac{1}{\sqrt{K_n(x)}} P_j^*(x)\right\}_{j=0}^n$ are an orthonormal set on $(-1, 1)$ under the Lebesgue density $\frac{1}{\pi\sqrt{1-x^2}}$. If we generate Monte Carlo samples from this density, evaluate an unknown function at these samples, and perform least-squares regression using Q_j as a basis, then a stability factor for this problem is given by $\max_{x\in[-1,1]} \sum_{j=0}^{n} Q_j^2(x) = n+1$ [2]. In fact, this is the smallest attainable stability factor, and therefore this approximation strategy has optimal stability.

Note that (2) is equivalent to the following:

$$\int_{-1}^{1} P_i^*(x) P_j^*(x) \frac{1}{K_n(x)} \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} \mathrm{d}x = \delta_{ij}, \quad 0 \le i+j \le 2n,$$

and, for any polynomial p of degree at most $2n$,

$$\int_{-1}^{1} \frac{p(x)}{\pi K_n(x)\sqrt{1-x^2}} \mathrm{d}x = \int_{-1}^{1} p(x)\mathrm{d}x.$$

Written in this last form, our result (or rather, a transformed version on the unit circle shown later in (3)) bears a strong resemblance to Berstein–Szegö approximations (also referred to as a "Geronimus formula") for orthogonal polynomials on the unit circle, (e.g., [3], section V.2, p. 198). Given a measure $\mathrm{d}\mu$ on the unit circle with associated orthogonal polynomial family $\Phi_n(z)$, such results have the form

$$\int_{-\pi}^{\pi} \frac{p(z)}{|\Phi_n(z)|^2} \mathrm{d}\theta = \int_{-\pi}^{\pi} p(z)\mathrm{d}\mu(\theta)$$

for any polynomial p of degree $2n$ or less and $z = \exp(i\theta)$. There are also analogues of this type of formula on the real line, e.g., [4]. However, the precise formula we have derived has a factor of $K_n(z)$ in the denominator, instead of a squared orthogonal polynomial modulus. To our knowledge, the formula (2) cannot be concluded in a straightforward manner from existing Bernstein–Szegö results. Additionally, whereas the Bernstein–Szegö formula is valid for a general class of measures $\mathrm{d}\mu$, it is shown in [1] that formula (2) does not hold for any other orthogonal polynomial family.

The remainder of this document is devoted to the proof of (2).

Proof of Theorem 1 We change variables, letting $x = \cos(\theta)$, to arrive at the equivalent expression

$$\frac{1}{\pi} \int_{0}^{\pi} \frac{P_i^*(\cos(\theta)) P_j^*(\cos(\theta))}{K_n(\cos(\theta))} \mathrm{d}\theta = \delta_{ij}, \quad 0 \le i, j \le n,$$

which by symmetry is equivalent to

$$\frac{1}{2\pi} \int_{0}^{2\pi} \frac{P_i^*(\cos(\theta)) P_j^*(\cos(\theta))}{K_n(\cos(\theta))} \mathrm{d}\theta = \delta_{ij}, \quad 0 \le i, j \le n. \tag{3}$$

We will make use of the scaled Joukowski map J on $\mathbb{C}$ given by

$$J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

For $z = e^{i\theta}$ in the integral (3), we obtain $\mathrm{d}\theta = -iz^{-1}\mathrm{d}z$, $\cos(\theta) = J(z)$, and the equation becomes

$$\frac{1}{2\pi i}\int_C \frac{z^{-1}P_i^*(J(z))P_j^*(J(z))}{K_n(J(z))}\mathrm{d}z = \delta_{ij}, \quad 0 \le i, j \le n, \tag{4}$$

where C is the unit circle, oriented in the counter-clockwise direction.

The proof is a direct calculation of (4) based on the following lemmas.

First note that $K_n(\cos(\theta))$ is a *positive* trigonometric polynomial (of degree $2n$). By the Fejér–Riesz factorization theorem, there exists a trigonometric polynomial, $T_n(\theta)$ say, such that

$$K_n(\cos(\theta)) = |T_n(\theta)|^2.$$

In general, the coefficients of the factor polynomial, $T_n(\theta)$ in this case, are algebraic functions of the coefficients of the original polynomial. However, in our case we have the explicit (essentially) rational factorization.

Proposition 1 *(Féjer–Riesz Factorization of $K_n(J(z))$) Let*

$$F_n(z) := \frac{\mathrm{d}}{\mathrm{d}z}\left(z^{n+1}P_n(J(z))\right) = (n+1)z^nP_n(J(z)) + \frac{z^{n-1}(z^2-1)}{2}P_n'(J(z)). \tag{5}$$

Then

$$K_n(J(z)) = \frac{1}{2(n+1)}F_n(z)F_n(1/z). \tag{6}$$

Proof To begin, one may easily verify that

$$F_n(1/z) = z^{-2n}\left\{(n+1)z^nP_n(J(z)) - \frac{z^{n-1}\left(z^2-1\right)}{2}P_n'(J(z))\right\}. \tag{7}$$

Hence

$$\begin{aligned}F_n(z)F_n(1/z) &= z^{-2n}\left\{(n+1)^2z^{2n}(P_n(J(z)))^2 - z^{2(n-1)}\left(\frac{z^2-1}{2}\right)^2(P_n'(J(z))^2\right\}\\ &= (n+1)^2(P_n(J(z)))^2 - z^{-2}\left(\frac{z^2-1}{2}\right)^2\left(P_n'(J(z)\right)^2.\end{aligned}$$

Now notice that

$$z^{-2}\left(\frac{z^2-1}{2}\right)^2 = \frac{1}{4}\left(z-\frac{1}{z}\right)^2$$
$$= \frac{1}{4}\left(z^2+2+\frac{1}{z^2}-4\right)$$
$$= J^2(z)-1,$$

so that

$$F_n(z)F_n(1/z) = (n+1)^2(P_n(J(z)))^2 - (J(z)^2-1)\left(P_n'(J(z))\right)^2.$$

The result follows then from Lemma 1, below. □

Lemma 1 *For all $x \in \mathbb{C}$, we have*

$$K_n(x) = \frac{1}{2(n+1)}\left((n+1)^2(P_n(x))^2 - (x^2-1)\left(P_n'(x)\right)^2\right).$$

Proof First, we collect the following known identities concerning Legendre polynomials [5]:

(Christoffel–Darboux formula)

$$\sum_{k=0}^{n} P_k^2(x) = \frac{n+1}{2}\left[P_{n+1}'(x)P_n(x) - P_{n+1}(x)P_n'(x)\right], \tag{8a}$$

(Three-term recurrence)

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \tag{8b}$$

(Differentiated three-term recurrence)

$$(n+1)P_{n+1}'(x) = (2n+1)\left(P_n(x)+xP_n'(x)\right) - nP_{n-1}'(x), \tag{8c}$$
$$(x^2-1)P_n'(x) = n\left(xP_n(x) - P_{n-1}(x)\right), \tag{8d}$$
$$(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x). \tag{8e}$$

We easily see from the Christoffel–Darboux formula that

$$K_n(x) = \frac{1}{2}\left(P_{n+1}'(x)P_n(x) - P_{n+1}(x)P_n'(x)\right).$$

Hence the result holds if and only if

$$\begin{gathered}
(n+1)P'_{n+1}(x)P_n(x) - (n+1)P_{n+1}(x)P'_n(x) \\
= (n+1)^2(P_n(x))^2 - (x^2-1)\left(P'_n(x)\right)^2 \\
\Updownarrow \text{ (8c), (8b), (8d)} \\
\left\{(2n+1)\left(P_n(x) + xP'_n(x)\right) - nP'_{n-1}(x)\right\} P_n(x) \\
- \{(2n+1)xP_n(x) - nP_{n-1}(x)\} P'_n(x) \\
= (n+1)^2(P_n(x))^2 - n\,(xP_n(x) - P_{n-1}(x))\,P'_n(x) \\
\Updownarrow \\
(2n+1)\,(P_n(x))^2 - nP'_{n-1}(x)P_n(x) + nP_{n-1}(x)P'_n(x) \\
= (n+1)^2(P_n(x))^2 - nxP_n(x)P'_n(x) + nP_{n-1}(x)P'_n(x) \\
\Updownarrow \\
-n^2\,(P_n(x))^2 - nP'_{n-1}(x)P_n(x) = -nxP_n(x)P'_n(x) \\
\Updownarrow \\
xP'_n(x) = nP_n(x) + P'_{n-1}(x) \\
\Updownarrow \text{ (8c)} \\
\tfrac{1}{2n+1}\left((n+1)P'_{n+1}(x) + nP'_{n-1}(x)\right) - P_n(x) = nP_n(x) + P'_{n-1}(x) \\
\Updownarrow \\
(n+1)P'_{n+1}(x) = (n+1)P'_{n-1}(x) + (2n+1)(n+1)P_n(x),
\end{gathered}$$

and this last relation is the same as the relation (8e). □

There is somewhat more that can be said about $F_n(z)$.

Lemma 2 *Let $F_n(z)$ be the polynomial of degree $2n$ defined in (5). Then all of its zeros are simple and lie in the* **interior** *of the unit disk.*

Proof The polynomial

$$Q_n(z) := z^{n+1}P_n(J(z)) = z\left\{z^nP_n(J(z))\right\}$$

has a zero at $z = 0$ and its other zeros are those of $P_n(J(z))$, namely, those $z \in \mathbb{C}$ for which $J(z) = r \in (-1, 1)$, a zero of $P_n(x)$. But

$$\begin{aligned}
& J(z) = r \in (-1,1) \\
\Longleftrightarrow\quad & (z + 1/z)/2 = r \\
\Longleftrightarrow\quad & z^2 - 2rz + 1 = 0 \\
\Longleftrightarrow\quad & z = r \pm i\sqrt{1-r^2}.
\end{aligned}$$

In particular $|z| = 1$ for the zeros of $z^nP_n(J(z))$. It follows then from the Gauss-Lucas Theorem that the zeros of $F_n(z)$ are in the convex hull of $z = 0$ and certain points on the unit circle, i.e., are all in the *closed* unit disk.

Suppose a zero of $F_n(z)$ satisfies $|z| = 1$. By Proposition 1,

$$K_n(J(z)) = \frac{1}{2(n+1)} F_n(z) F_n(1/z),$$

so that $K_n(J(z))$ also vanishes. But $|z| = 1$ implies that $J(z) \in [-1, 1]$, and $K_n(J(z))$ thus cannot vanish. Therefore, no zeros of F_n lie on the unit circle.

To see that the zeros are all simple, an elementary calculation and the ODE for Legendre polynomials gives us

$$F_n'(z) = 2n(n+1)z^{n-1} P_n(J(z)) + \left\{ nz^n - (n+1)z^{n-2} \right\} P_n'(J(z)).$$

Hence $F_n(z) = F_n'(z) = 0$ if and only if

$$\begin{pmatrix} n+1 & \frac{z^2-1}{2} \\ \frac{2n(n+1)}{z} & nz - \frac{n+1}{z} \end{pmatrix} \begin{pmatrix} z^n P_n(J(z)) \\ z^{n-1} P_n'(J(z)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

But the determinant of this matrix is

$$(n+1)(nz - (n+1)/z) - n(n+1)(z^2-1)/z = -(n+1)/z \neq 0.$$

Hence $F_n(z) = F_n'(z) = 0$ if and only if $z^n P_n(J(z)) = z^{n-1} P_n'(J(z)) = 0$ if and only if $P_n(J(z)) = P_n'(J(z)) = 0$, which is not possible as $P_n(x)$ has only simple zeros. □

The integral (4) can therefore be expressed as

$$\begin{aligned} \frac{1}{2\pi i} \int_C \frac{z^{-1} P_i^*(J(z)) P_j^*(J(z))}{K_n(J(z))} \mathrm{d}z &= \frac{1}{2\pi i} \int_C \frac{2(n+1)z^{-1} P_i^*(J(z)) P_j^*(J(z))}{F_n(z) F_n(1/z)} \mathrm{d}z \\ &= \frac{1}{2\pi i} \int_C \frac{2(n+1)z^{2n-1} P_i^*(J(z)) P_j^*(J(z))}{F_n(z) z^{2n} F_n(1/z)} \mathrm{d}z \\ &= \frac{1}{2\pi i} \int_C \frac{2(n+1)z^{2n-1} P_i^*(J(z)) P_j^*(J(z))}{F_n(z) G_n(z)} \mathrm{d}z, \end{aligned}$$

where we define the *polynomial* of degree $2n$,

$$G_n(z) := z^{2n} F_n(1/z). \tag{9}$$

As all the zeros of $F_n(z)$ are in the interior of the unit disk, the zeros of $G_n(z)$ are all exterior to the (closed) unit disk.

The following formulas for $F_n(z)$ and $G_n(z)$ will be useful.

Lemma 3 *We have*

$$F_n(z) = \frac{z^n}{z^2-1} \left\{ \left((2n+1)z^2 - 1 \right) P_n(J(z)) - 2nz P_{n-1}(J(z)) \right\},$$

and

$$G_n(z) = \frac{z^n}{z^2-1}\left\{\left(z^2-(2n+1)\right)P_n(J(z)) + 2nzP_{n-1}(J(z))\right\}.$$

Proof From the formula (5), we have

$$F_n(z) = (n+1)z^n P_n(J(z)) + z^{n-1}\frac{z^2-1}{2}P_n'(J(z)),$$

and from (7),

$$G_n(z) = (n+1)z^n P_n(J(z)) - z^{n-1}\frac{z^2-1}{2}P_n'(J(z)).$$

From the Legendre polynomial identity (8d) with $x = J(z)$, we obtain

$$z^{n-1}\frac{z^2-1}{2}P_n'(J(z)) = 2n\frac{z^{n+1}}{z^2-1}J(z)P_n(J(z)) - 2n\frac{z^{n+1}}{z^2-1}P_{n-1}(J(z)).$$

Combining these gives the result. □

It is also interesting to note that $F_n(z)$ is a certain hypergeometric function.

Lemma 4 *We have*

1. *The polynomial* $y = F_n(z)$ *is a solution of the ODE*

$$(1-z^2)y'' + 2\frac{(n-2)z^2-n}{z}y' + 6ny = 0.$$

2. *If* $F_n(z) =: f_n(z^2)$*, then* $y = f_n(z)$ *is a solution of the hypergeometric ODE*

$$z(1-z)y'' + (c-(a+b+1)z)y' - aby = 0,$$

   with $a = -n$*,* $b = 3/2$*, and* $c = 1/2 - n$*.*
3. $f_n(z) = 2^{-2n}\binom{2n}{n}{}_2F_1(a,b;c;z)$.
4. $F_n(z) = 2^{-2n}\binom{2n}{n}{}_2F_1(a,b;c;z^2)$.
5.

$$F_n(z) = 2^{-2n}\binom{2n}{n}\sum_{k=0}^{n}\frac{(2k+1)\binom{n}{k}^2}{\binom{2n}{2k}}z^{2k}$$
$$= 2^{-2n}\sum_{k=0}^{n}(2k+1)\binom{2k}{k}\binom{2n-2k}{n-k}z^{2k}.$$

6. *If we write* $F_n(z) = \sum_{k=0}^{n} c_k z^{2k}$, *then*

$$G_n(z) = \sum_{k=0}^{n} c_{n-k} z^{2k} = \sum_{k=0}^{n} \frac{2(n-k)+1}{2k+1} c_k z^{2k}.$$

Proof Equation (1) is easily verified using the ODE for $P_n(x)$ and the definition of $F_n(z)$, (5).

(2) follows by changing variables $z' = z^2$.

(3) follows as ${}_2F_1(a, b; c; z)$ is the only polynomial solution of the hypergeometric ODE. The constant of proportionality is calculated by noting that the leading coefficient of ${}_2F_1(a, b; c; z)$ is $2n+1$, whereas that of $f_n(z)$ is $(2n+1)/2^n$ times the leading coefficient of $P_n(x)$, i.e., $(2n+1)/2^n \times \binom{2n}{n}/2^n$.

(4) is trivial from the definition $f_n(z^2) := F_n(z)$.

(5) follows from the fact that

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{n} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

with $(a)_k$ the rising factorial, and calculating

$$(a)_k = (-1)^k \frac{n!}{(n-k)!}, \quad (b)_k = 2^{-2k} \frac{(2k+1)!}{k!}, \quad (c)_k = (-1)^k 2^{-2k} \frac{(2n)!(n-k)!}{(2n-2k)!n!},$$

so that

$$\frac{1}{k!} \frac{(a)_k (b)_k}{(c)_k} = \frac{(2k+1)\binom{n}{k}^2}{\binom{2n}{2k}}.$$

(6) follows easily from the fact that $G_n(z) := z^n F_n(1/z) = \sum_{k=0}^{n} c_{n-k} z^k$ and that c_{n-k} is easily computed from the formula for c_k given in (5). □

Returning to the proof of the theorem, we will actually show that:

(a) $$\frac{1}{2\pi i} \int_C \frac{2(n+1) z^{2n-1} P_k(J(z))}{F_n(z) G_n(z)} \mathrm{d}z = 0, \quad 1 \le k \le 2n,$$

(b) $$\frac{1}{2\pi i} \int_C \frac{2(n+1) z^{2n-1} P_0(J(z))}{F_n(z) G_n(z)} \mathrm{d}z = 2.$$

The theorem follows directly, as, for $i, j \le n$, we may expand

$$P_i^*(x) P_j^*(x) = \sum_{k=0}^{2n} a_k P_k(x)$$

for certain coefficients a_k. From (a) and (b), we then conclude that

$$\frac{1}{2\pi i}\int_C \frac{2(n+1)z^{2n-1}P_i^*(J(z))P_j^*(J(z))}{F_n(z)G_n(z)}\mathrm{d}z = 2a_0.$$

But for $i \neq j$,

$$a_0 = \frac{1}{2}\int_{-1}^{1} P_i^*(x)P_j^*(x)P_0(x)\mathrm{d}x = 0,$$

as $P_0(x) = 1$ and $P_i^*(x)$ and $P_j^*(x)$ are orthogonal. While for $i = j$, we have

$$a_0 = \frac{1}{2}\int_{-1}^{1} (P_i^*(x))^2 P_0(x)\mathrm{d}x = \frac{1}{2}.$$

We will now compute the partial fraction decomposition of the integrands in (a) and (b),

$$\frac{2(n+1)z^{2n-1}P_k(J(z))}{F_n(z)G_n(z)},$$

which will involve the following two pairs of families of functions.

Definition 1 We define

$$A_0^{(n)}(z) := z^{n-1}, \quad A_1^{(n)}(z) := z^{n-2},$$
$$B_0^{(n)}(z) := z^{n-1}, \quad B_1^{(n)}(z) := z^{n},$$

with

$$(n+k+1)A_{k+1}^{(n)} := (2(n+k)+1)J(z)A_k^{(n)}(z) - (n+k)A_{k-1}^{(n)}(z), \quad k = 1, 2, \dots,$$

and

$$(n+k+1)B_{k+1}^{(n)} := (2(n+k)+1)J(z)B_k^{(n)}(z) - (n+k)B_{k-1}^{(n)}(z), \quad k = 1, 2, \dots.$$

Further, we let

$$C_0^{(n)}(z) := z^{n-1}, \quad C_1^{(n)}(z) := z^{n-2}\frac{(2n+1)z^2-1}{2n},$$
$$D_0^{(n)}(z) := z^{n-1}, \quad D_1^{(n)}(z) := z^{n-2}\frac{(2n+1)-z^2}{2n},$$

with

$$(n-k)C_{k+1}^{(n)}(z) := (2(n-k)+1)J(z)C_k^{(n)}(z) - (n-k+1)C_{k-1}^{(n)}(z), \quad 1 \le k \le n-1,$$

and

$$(n-k)D_{k+1}^{(n)}(z) := (2(n-k)+1)J(z)D_k^{(n)}(z) - (n-k+1)D_{k-1}^{(n)}(z), \\ 1 \le k \le n-1.$$

Proposition 2 *We have*

$$2(n+1)z^{2n-1}P_{n+k}(J(z)) = A_k^{(n)}(z)G_n(z) + B_k^{(n)}(z)F_n(z), \quad k = 0, 1, 2, \dots, \quad (10)$$

so that

$$\frac{2(n+1)z^{2n-1}P_{n+k}(J(z))}{F_n(z)G_n(z)} = \frac{A_k^{(n)}(z)}{F_n(z)} + \frac{B_k^{(n)}(z)}{G_n(z)}, \quad k = 0, 1, 2, \dots,$$

and

$$2(n+1)z^{2n-1}P_{n-k}(J(z)) = C_k^{(n)}(z)G_n(z) + D_k^{(n)}(z)F_n(z), \quad 0 \le k \le n, \quad (11)$$

so that

$$\frac{2(n+1)z^{2n-1}P_{n-k}(J(z))}{F_n(z)G_n(z)} = \frac{C_k^{(n)}(z)}{F_n(z)} + \frac{D_k^{(n)}(z)}{G_n(z)}, \quad 0 \le k \le n.$$

Proof (by induction). As $A_0^{(n)} = C_0^{(n)} = z^{n-1}$ and $B_0^{(n)} = D_0^{(n)} = z^{n-1}$, the $k = 0$ case is in common, and we calculate

$$\begin{aligned} &= z^{n-1}(F_n(z) + G_n(z)) \\ &\overset{\text{(Lemma 3)}}{=} z^{n-1}\frac{z^n}{z^2-1}\left\{\left((2n+1)z^2-1\right) + \left(z^2-(2n+1)\right)\right\} P_n(J(z)) \\ &= z^{2n-1}\frac{1}{z^2-1}\left\{2(n+1)z^2 - 2(n+1)\right\} P_n(J(z)) \\ &= 2(n+1)z^{2n-1}P_n(J(z)), \end{aligned}$$

as desired.

We now prove (10) for the case $k = 1$. We calculate, using Lemma 3,

$$\begin{aligned} &A_1^{(n)}(z)G_n(z) + B_1^{(n)}(z)F_n(z) \\ &\quad = z^{n-2}G_n(z) + z^nF_n(z) \\ &\quad = z^{n-2}(G_n(z) + z^2F_n(z)) \\ &\quad = z^{n-2}\frac{z^n}{z^2-1}\left\{\left[\left(z^2-(2n+1)\right)P_n(J(z)) + 2nzP_{n-1}(J(z))\right]\right. \\ &\qquad \left. + z^2\left[\left((2n+1)z^2-1\right)P_n(J(z)) - 2nzP_{n-1}(J(z))\right]\right\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{z^{2n-2}}{z^2-1}\left\{(2n+1)(z^4-1)P_n(J(z)) - 2nz(z^2-1)P_{n-1}(J(z))\right\}\\
&= z^{2n-2}\left\{(2n+1)(z^2+1)P_n(J(z)) - 2nzP_{n-1}(J(z))\right\}\\
&= z^{2n-1}\{(2n+1)(z+1/z)P_n(J(z)) - 2nP_{n-1}(J(z))\}\\
&= z^{2n-1}\{(2n+1)2J(z)P_n(J(z)) - 2nP_{n-1}(J(z))\}\\
&\overset{(8b)}{=} 2(n+1)z^{2n-1}P_{n+1}(J(z)).
\end{aligned}
$$

Now for (11) for $k=1$, calculate, again using Lemma 3,

$$
\begin{aligned}
&C_1^{(n)}(z)G_n(z) + D_1^{(n)}(z)F_n(z)\\
&\quad= z^{n-2}\left\{\frac{(2n+1)z^2-1}{2n}G_n(z) + \frac{(2n+1)-z^2}{2n}F_n(z)\right\}\\
&\quad= z^{n-2}\frac{z^n}{z^2-1}\left\{\frac{(2n+1)z^2-1}{2n}\left[\left(z^2-(2n+1)\right)P_n(J(z)) + 2nzP_{n-1}(J(z))\right]\right.\\
&\qquad\left. + \frac{(2n+1)-z^2}{2n}\left[\left((2n+1)z^2-1\right)P_n(J(z)) - 2nzP_{n-1}(J(z))\right]\right\}\\
&\quad= \frac{z^{2n-2}}{z^2-1}2nz\left\{\frac{(2n+1)z^2-1}{2n} - \frac{(2n+1)-z^2}{2n}\right\}P_{n-1}(J(z))\\
&\quad= \frac{z^{2n-1}}{z^2-1}\left(\left(2(n+1)z^2-2(n+1)\right)P_{n-1}(J(z))\right)\\
&\quad= 2(n+1)z^{2n-1}P_{n-1}(J(z)).
\end{aligned}
$$

The rest of the proof proceeds by induction. Assuming that (10) and (11) hold from 0 up to a certain k, we prove that they also hold for $k+1$. To this end, we calculate

$$
\begin{aligned}
&A_{k+1}^{(n)}(z)G_n(z) + B_{k+1}^{(n)}(z)F_n(z)\\
&\quad= \frac{(2(n+k)+1)J(z)A_k^{(n)}(z) - (n+k)A_{k-1}^{(n)}(z)}{n+k+1}G_n(z)\\
&\qquad+ \frac{(2(n+k)+1)J(z)B_k^{(n)}(z) - (n+k)B_{k-1}^{(n)}(z)}{n+k+1}F_n(z)\\
&\quad= \frac{1}{n+k+1}\left\{(2(n+k)+1)J(z)\left[A_k^{(n)}(z)G_n(z) + B_k^{(n)}(z)F_n(J(z))\right]\right.\\
&\qquad\left. -(n+k)\left[A_{k-1}^{(n)}(z)G_n(z) + B_{k-1}^{(n)}F_n(J(z))\right]\right\}\\
&\quad= 2(n+1)z^{2n-1}\frac{1}{n+k+1}\{(2(n+k)+1)J(z)P_{n+k}(J(z))\\
&\qquad-(n+k)P_{n+k-1}(J(z))\}\quad \text{(by the induction hypothesis)}\\
&\quad= 2(n+1)z^{2n-1}P_{n+k+1}(J(z)),
\end{aligned}
$$

by the three-term recursion formula for Legendre polynomials (8b) with degree $m = n + k$.

Similarly,

$$\begin{aligned}
&C_{k+1}^{(n)}(z)G_n(z) + D_{k+1}^{(n)}(z)F_n(z) \\
&\quad = \frac{(2(n-k)+1)J(z)C_k^{(n)}(z) - (n-k+1)C_{k-1}^{(n)}(z)}{n-k} G_n(z) \\
&\qquad + \frac{(2(n-k)+1)J(z)D_k^{(n)}(z) - (n-k+1)D_{k-1}^{(n)}(z)}{n-k} F_n(z) \\
&\quad = \frac{1}{n-k}\Big\{(2(n-k)+1)J(z)\Big[C_k^{(n)}(z)G_n(z) + D_k^{(n)}(z)F_n(J(z))\Big] \\
&\qquad - (n-k+1)\Big[C_{k-1}^{(n)}(z)G_n(z) + D_{k-1}^{(n)}F_n(J(z))\Big]\Big\} \\
&\quad = 2(n+1)z^{2n-1}\frac{1}{n-k}\Big\{(2(n-k)+1)J(z)P_{n-k}(J(z)) \\
&\qquad - (n-k+1)P_{n-(k-1)}(J(z))\Big\} \quad \text{(by the induction hypothesis)} \\
&= 2(n+1)z^{2n-1}P_{n-(k+1)}(J(z)),
\end{aligned}$$

using the reverse three-term recursion

$$mP_{m-1}(x) = (2m+1)xP_m(x) - (m+1)P_{m+1}(x),$$

with $m = n - k$. □

Due to the $J(z)$ factor in the recursive definitions of Definition 1, the functions $A_k^{(n)}(z)$, $B_k^{(n)}(z)$, $C_k^{(n)}(z)$, and $D_k^{(n)}(z)$ are all Laurent polynomials. It is easy to verify that they have the forms:

- $A_k^{(n)}(z) = \sum_{j=n-(k+1)}^{n+(k-3)} a_j z^j, \; k \geq 1,$
- $B_k^{(n)}(z) = \sum_{j=n-(k-1)}^{n+(k-1)} b_j z^j, \; k \geq 1,$
- $C_k^{(n)}(z) = \sum_{j=n-(k+1)}^{n+(k-1)} c_j z^j, \; 1 \leq k \leq n,$
- $D_k^{(n)}(z) = \sum_{j=n-(k+1)}^{n+(k-1)} d_j z^j, \; 1 \leq k \leq n.$

In particular, for $0 \leq k \leq n-1$, the functions $A_k^{(n)}(z)$, $B_k^{(n)}(z)$, $C_k^{(n)}(z)$ are all *polynomials* of degree at most $2n-2$.

Hence, for $1 \le k \le 2n-1$,

$$\frac{1}{2\pi i}\int_C \frac{2(n+1)z^{2n-1}P_k(J(z))}{F_n(z)G_n(z)}\mathrm{d}z = \frac{1}{2\pi i}\left\{\int_C \frac{p(z)}{F_n(z)}\mathrm{d}z + \int_C \frac{q(z)}{G_n(z)}\mathrm{d}z\right\}$$

for certain polynomials $p(z)$ and $q(z)$ of degree at most $2n-2$.

Now, $\int_C \frac{q(z)}{G_n(z)}\mathrm{d}z = 0$ as all the zeros of $G_n(z)$ lie outside the (closed) unit disk. Further, if we let z_j $1 \le j \le 2n$ be the (simple) zeros of $F_n(z)$, we may write

$$\frac{p(z)}{F_n(z)} = \frac{p(z)/c_n}{F_n(z)/c_n} = \sum_{j=1}^{2n} \frac{R_j}{z-z_j},$$

where c_n is the leading coefficient of $F_n(z)$. Hence

$$\frac{1}{2\pi i}\int_C \frac{p(z)}{F_n(z)}\mathrm{d}z = \frac{1}{2\pi i}2\pi i\sum_{j=1}^{2n} R_j.$$

But $\sum_{j=1}^{2n} R_j$ is the leading coefficient (of z^{2n-1}) of $p(z)/c_n$, i.e., 0, as $p(z)$ is of degree at most $2n-2$. It follows that

$$\frac{1}{2\pi i}\int_C \frac{2(n+1)z^{2n-1}P_k(J(z))}{F_n(z)G_n(z)}\mathrm{d}z = 0, \quad 1 \le k \le 2n-1.$$

The cases $P_0(J(z))$ and $P_{2n}(J(z))$ are special.

First consider the case $P_{2n}(J(z))$. We have from Proposition 2, with $k=n$,

$$2(n+1)z^{2n-1}P_{2n}(J(z)) = A_n^{(n)}(z)G_n(z) + B_n^{(n)}(z)F_n(z).$$

However, $A_n^{(n)}(z) = \sum_{j=-1}^{2n-3} a_j z^j$ has a z^{-1}, while $B_n^{(n)}(z) = \sum_{j=+1}^{2n-1} b_j z^j$ is still a polynomial, of degree at most $2n-1$. Therefore it is still the case that $\int_C \frac{B_n^{(n)}(z)}{G_n(z)}\mathrm{d}z = 0$ (the zeros of $G_n(z)$ being all outside the unit disk). We need to show that $\int_C \frac{A_n^{(n)}(z)}{F_n(z)}\mathrm{d}z = 0$. We write $A_n^{(n)}(z) = q(z) + c/z$, where $q(z)$ is a polynomial of degree $2n-3$. Then

$$\int_C \frac{A_n^{(n)}(z)}{F_n(z)}\mathrm{d}z = \int_C \frac{q(z)}{F_n(z)}\mathrm{d}z + c\int_C \frac{1}{zF_n(z)}\mathrm{d}z.$$

The first integral on the right is zero, as the coefficient in $q(z)$ of z^{2n-1} is 0. For the second integral, decompose

$$
\begin{aligned}
c\int_C \frac{1}{zF_n(z)}\mathrm{d}z &= c\int_C \frac{1}{F_n(0)}\left\{\frac{1}{z} - \frac{(F_n(z)-F_n(0))/z}{F_n(z)}\right\}\mathrm{d}z \\
&= \frac{1}{F_n(0)}\left\{\int_C \frac{1}{z}\mathrm{d}z - \int_C \frac{(F_n(z)-F_n(0))/z}{F_n(z)}\right\}\mathrm{d}z.
\end{aligned}
$$

The first integral is trivially $2\pi i$, while the second is $2\pi i$ times the coefficient of z^{2n-1} in $(F_n(z)-F_n(0))/z$ divided by the leading coefficient (of z^{2n}) in $F_n(z)$, i.e., $2\pi i \times 1$. Hence, indeed,

$$
\int_C \frac{A_n^{(n)}(z)}{F_n(z)}\mathrm{d}z = 0.
$$

Lastly we calculate

$$
\frac{1}{2\pi i}\int_C \frac{2(n+1)z^{2n-1}P_0(J(z))}{F_n(z)G_n(z)}\mathrm{d}z = \frac{1}{2\pi i}\int_C \frac{2(n+1)z^{2n-1}}{F_n(z)G_n(z)}\mathrm{d}z.
$$

We still have (11) from Proposition 2,

$$
2(n+1)z^{2n-1}P_{n-n}(z) = C_n^{(n)}(z)G_n(z) + D_n^{(n)}(z)F_n(z).
$$

However, $C_n^{(n)}(z)$ and $D_n^{(n)}(z)$ have the form

$$
C_n^{(n)}(z) = \sum_{j=-1}^{2n-1} c_k z^k, \quad D_n^{(n)}(z) = \sum_{j=-1}^{2n-1} d_k z^k,
$$

i.e., are both of the form $p(z) + c/z$, where $p(z)$ is a polynomial of degree $2n-1$. Specifically, we write $C_n^{(n)}(z) = p(z) + c/z$ and $D_n^{(n)}(z) = q(z) + d/z$. We thus have the following expression:

$$
\begin{aligned}
\frac{1}{2\pi i}\int_C \frac{2(n+1)z^{2n-1}P_0(z)}{F_n(z)G_n(z)}\mathrm{d}z &= \frac{1}{2\pi i}\int_C \frac{C_n^{(n)}(z)}{F_n(z)}\mathrm{d}z + \frac{1}{2\pi i}\int_C \frac{D_n^{(n)}(z)}{G_n(z)}\mathrm{d}z \\
&= \frac{1}{2\pi i}\left[\int_C \frac{p(z)}{F_n(z)}\mathrm{d}z + \int_C \frac{c}{zF_n(z)}\mathrm{d}z \right.\\
&\quad \left. + \int_C \frac{q(z)}{G_n(z)}\mathrm{d}z + \int_C \frac{d}{zG_n(z)}\mathrm{d}z\right].
\end{aligned}
$$

We need to show that this expression has value 2. Now we have already shown that $\int_C \frac{1}{zF_n(z)}\mathrm{d}z = 0$ and also remarked that $\int_C \frac{q(z)}{G_n(z)}\mathrm{d}z = 0$. Hence we need to calculate $\frac{1}{2\pi i}\int_C \frac{p(z)}{F_n(z)}\mathrm{d}z$ and $\frac{1}{2\pi i}\int_C \frac{d}{zG_n(z)}\mathrm{d}z$. But the first of these is just the (leading) coefficient of z^{2n-1} in $p(z)$, i.e., in $C_n^{(n)}(z)$ divided by the leading coefficient of $F_n(z)$. From Lemma 4, we have that

$$F_n(z) = 2^{-2n}(2n+1)\binom{2n}{n}z^{2n} + \cdots ,$$

and it is easy to verify by induction that also $C_n^{(n)}(z) = 2^{-2n}(2n+1)\binom{2n}{n}z^{2n} + \cdots$. Hence

$$\frac{1}{2\pi i}\int_C \frac{p(z)}{F_n(z)}\mathrm{d}z = 1.$$

For the second integral, decompose as before,

$$\frac{1}{2\pi i}\int_C \frac{d}{zG_n(z)}\mathrm{d}z = \frac{d}{2\pi i}\frac{1}{G_n(0)}\int_C \left\{\frac{1}{z} - \frac{(G_n(z)-G_n(0))/z}{G_n(z)}\right\}\mathrm{d}z.$$

The second integral above is zero, as all the zeros of $G_n(z)$ are outside the closed unit disk. The first integral is trivially $d/G_n(0)$. From Lemma 4 parts 4 and 6, we have that

$$G_n(0) = (2n+1)F_n(0) = 2^{-2n}(2n+1)\binom{2n}{n},$$

whereas the coefficient of z^{-1} in $B_n^{(n)}(z)$ is easily verified by induction to have the same value. Hence

$$\frac{1}{2\pi i}\int_C \frac{d}{zG_n(z)}\mathrm{d}z = 1,$$

and we have shown that

$$\frac{1}{2\pi i}\int_C \frac{2(n+1)z^{2n-1}P_0(J(z))}{F_n(z)G_n(z)}\mathrm{d}z = 1+1 = 2,$$

as claimed. □

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