DERIVED CHARACTER MAPS OF GROUP REPRESENTATIONS

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ABSTRACT. In this paper, we define and study derived character maps of finite-dimensional representations of homotopy simplicial groups, which are homotopy algebras over the algebraic theory of groups (in the sense of Badzioch [4]). We introduce cyclic, symmetric and representation homology for 'group algebras' $k[\Gamma]$ of such groups and construct canonical trace maps (natural transformations) relating these homology theories. We show that, in the case of one-dimensional representations, our trace maps are of topological origin: they are induced by some canonical maps of (iterated) loop spaces known in stable homotopy theory. Using this topological interpretation, we deduce some algebraic results on representation homology: in particular, we prove that the symmetric homology of group algebras and one-dimensional representation homology are naturally isomorphic, provided the base ring k is a field of characteristic zero. We also study the stable behavior of the derived character maps of n-dimensional representations as $n \to \infty$, in which case we show that these maps 'converge' to become isomorphisms.

1. Introduction

If Γ is a finite group and k is a field of characteristic zero, every finite-dimensional k-linear representation $\varrho : \Gamma \to \mathrm{GL}_n(k)$ is semi-simple and determined (up to equivalence) by its character: the trace function $\langle g \rangle \mapsto \mathrm{Tr}[\varrho(g)]$ defined on the set $\langle \Gamma \rangle$ of conjugacy classes of elements of Γ; moreover, for each $n \geq 0$, there are finitely many equivalence classes of such representations. These well-familiar facts from representation theory of finite groups generalize to arbitrary groups by means of algebraic geometry. For any discrete group Γ, the set of all n-dimensional representations of Γ can be naturally given the structure of an affine algebraic variety (more precisely, an affine k-scheme) $\mathrm{Rep}_n(\Gamma)$ called the *representation variety* of Γ. The equivalence classes of n-dimensional representations of Γ are classified by the orbits of the general linear group GL_n that acts algebraically on $\mathrm{Rep}_n(\Gamma)$ by conjugation. The classes of semi-simple representations correspond to the closed orbits[1] and are parametrized by the affine quotient scheme

$$\mathrm{Rep}_n(\Gamma)/\!/\mathrm{GL}_n(k) := \mathrm{Spec}\, \mathcal{O}[\mathrm{Rep}_n(\Gamma)]^{\mathrm{GL}_n}$$

called the *character variety* of Γ. Now, the characters of representations assemble into a linear map

$$\mathrm{Tr}_n(\Gamma) : \ k\langle \Gamma \rangle \to \mathcal{O}[\mathrm{Rep}_n(\Gamma)]^{\mathrm{GL}_n} \qquad (1.1)$$

defined on the k-vector space spanned by the conjugacy classes of elements of Γ. A well-known theorem of C. Procesi [59] asserts that the characters of Γ, i.e. the images of the map (1.1), generate $\mathcal{O}[\mathrm{Rep}_n(\Gamma)]^{\mathrm{GL}_n}$ as a commutative k-algebra, and thus, by Nullstellensatz, detect the semi-simple representations of Γ when k is algebraically closed. In general, the equivariant geometry of $\mathrm{Rep}_n(\Gamma)$ is closely related to representation theory of Γ, the geometric structure of GL_n-orbits in $\mathrm{Rep}_n(\Gamma)$ determining the algebraic structure of representations. Since the late 1980s this relation has been extensively studied and exploited in many areas of mathematics, most notably in geometric group theory and low-dimensional topology (see, e.g., [50, 66]).

Derived algebraic geometry allows one to extend — and in some sense to complete — this beautiful connection between representation theory and geometry. For any affine algebraic group G defined over a commutative ring k (e.g., $G = \mathrm{GL}_n(k)$), the classical representation scheme $\mathrm{Rep}_G(\Gamma)$ parametrizing the representations of Γ in G admits a natural derived extension $\mathrm{DRep}_G(\Gamma)$ called the *derived G-representation scheme*[2] of Γ. This derived scheme is represented by a simplicial commutative k-algebra $\mathcal{O}[\mathrm{DRep}_G(\Gamma)]$ whose

[1] At least when Γ is finitely generated.

[2] The first construction of this kind — the derived moduli space $\mathbf{R}\mathrm{Loc}_G(X)$ of G-local systems over a pointed connected space X — was introduced by Kapranov [43]. In recent years, several other constructions and generalizations of $\mathbf{R}\mathrm{Loc}_G(X)$

homotopy groups $\pi_i\mathcal{O}[\mathrm{DRep}_G(\Gamma)]$ are non-abelian homological invariants of Γ (or rather its classifying space $B\Gamma$). Following [14, 13], we set

$$\mathrm{HR}_*(\Gamma, G(k)) := \pi_*\mathcal{O}[\mathrm{DRep}_G(\Gamma)] \tag{1.2}$$

and call (1.2) the *representation homology of* Γ *with coefficients in* G. By definition, $\mathrm{HR}_*(\Gamma, G(k))$ is a graded commutative k-algebra, whose degree zero part is canonically isomorphic to the coordinate ring of $\mathrm{Rep}_G(\Gamma)$:

$$\mathrm{HR}_0(\Gamma, G(k)) \cong \mathcal{O}[\mathrm{Rep}_G(\Gamma)]\ . \tag{1.3}$$

Apart from groups, representation homology can be also defined for various types of algebras (e.g., associative and Lie algebras, see [9, 10, 8, 7]) as well as for topological spaces (see [14, 13, 12]). What is surprising perhaps is that, in the case of discrete groups, the representation homology admits a simple interpretation in terms of classical (abelian) homological algebra: namely, as shown in [14], there is a natural isomorphism

$$\mathrm{HR}_*(\Gamma, G(k)) \,\cong\, \mathrm{Tor}_*^{\mathfrak{G}}(k[\Gamma],\, \mathcal{O}(G))\,, \tag{1.4}$$

where $\mathfrak{G}$ is the (skeletal) category of f. g. free groups on which the group algebra $k[\Gamma]$ of the group Γ and the coordinate algebra $\mathcal{O}(G)$ of the algebraic group G are represented by monoidal functors: $\mathfrak{G} \to \mathrm{Mod}_k$ (contravariant and covariant, respectively). In the present paper, we will use formula (1.4) to *define* representation homology for *homotopy simplicial groups*, which are natural ('homotopy') generalizations of the usual ('strict') simplicial groups introduced in [4]. In addition to representation homology, we will also define the cyclic and symmetric homology for such groups, extending the classical approach of Connes [25] and Loday-Fiedorowicz [31] (see Section 3).

Now, returning to the character map (1.1) for a discrete group Γ, we observe that its domain is canonically isomorphic to the 0-th cyclic homology of the group algebra $k[\Gamma]$:

$$\mathrm{HC}_0(k[\Gamma]) \cong k\langle\Gamma\rangle \tag{1.5}$$

With isomorphisms (1.3) and (1.5), we can rewrite (1.1) in the form

$$\mathrm{Tr}_n(\Gamma):\ \mathrm{HC}_0(k[\Gamma]) \,\to\, \mathrm{HR}_0(\Gamma, \mathrm{GL}_n(k))^{\mathrm{GL}_n}\,, \tag{1.6}$$

which suggests that there might exist a natural extension of this map to higher cyclic homology with values in representation homology of Γ:

$$\mathrm{Tr}_n(\Gamma)_*:\ \mathrm{HC}_*(k[\Gamma]) \,\to\, \mathrm{HR}_*(\Gamma, \mathrm{GL}_n(k))^{\mathrm{GL}_n} \tag{1.7}$$

We call (1.7) the *derived character maps of* n*-dimensional representations of* Γ. Our goal is to define and study such maps for an arbitrary homotopy simplicial group Γ and an arbitrary affine algebraic group G (see Definition 3.15).

In the case of associative algebras, the derived character maps were originally constructed in [9], using non-abelian homological algebra. This construction was extended to Lie algebras in [8], where it was shown — among other things — that the derived character maps of Lie algebra representations are Koszul dual to the classical Loday-Quillen-Tsygan maps [49, 70]. The case of groups that we study in this paper is special for several reasons. First, as mentioned above, the representation homology of groups admits a natural construction in terms of functor homology that is similar to A. Connes' construction of cyclic homology. We will show that behind this 'similarity' there is actually a connection: a simple formula for the derived character maps (1.7) relating cyclic homology to representation homology via standard homological algebra (see Section 3.4).

Second, the cyclic homology of group algebras has a beautiful topological interpretation that goes back to the work of Goodwillie, Burghelea, Fiedorowicz and others (see [47, Chapter 7]). Specifically, there is a natural isomorphism

$$\mathrm{HC}_*(k[\Gamma]) \,\cong\, \mathrm{H}_*(ES^1 \times_{S^1} \mathcal{L}(B\Gamma);\, k)\,, \tag{1.8}$$

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have been studied in derived algebraic geometry (most notably, in the work of Toën and Vezzosi (see, e.g., [69, 58, 55, 54] and also [68]). A brief review and comparison of these constructions can be found in the Appendix of [12].

where the right-hand side is the S^1-equivariant homology of the free loop space $\mathcal{L}(B\Gamma) := \mathrm{Map}(S^1, B\Gamma)$ of the classifying space of Γ. In fact, (1.8) is just one on the list of several classical isomorphisms relating algebraic homology theories associated with so-called crossed simplicial groups [31] to (stable) homotopy theory:

$$
\begin{aligned}
\mathrm{HH}_*(k[\Gamma]) &\cong \mathrm{H}_*(\mathcal{L}(B\Gamma);\, k)\,,\\
\mathrm{HC}_*(k[\Gamma]) &\cong \mathrm{H}_*(ES^1 \times_{S^1} \mathcal{L}(B\Gamma);\, k)\,,\\
\mathrm{HS}_*(k[\Gamma]) &\cong \mathrm{H}_*(\Omega\,\Omega^\infty\Sigma^\infty(B\Gamma);\, k)\,,\\
\mathrm{HB}_*(k[\Gamma]) &\cong \mathrm{H}_*(\Omega^2\Sigma(B\Gamma);\, k)\,,\\
\mathrm{HO}_*(k[\Gamma]) &\cong \mathrm{H}_*(E(\mathbb{Z}/2)_+ \wedge_{\mathbb{Z}/2} \Omega\,\Omega^\infty\Sigma^\infty(B\Gamma);\, k)\,,
\end{aligned}
\tag{1.9}
$$

where Ω, Σ and $\Omega^\infty\Sigma^\infty$ denote the based loop, the (reduced) suspension, and the stable homotopy functors, respectively. The first two of the above isomorphisms (for Hochschild and cyclic homology) are well known: they were originally established in [33] and [20], and their proofs appear in Loday's textbook [47] (see also [48] for a nice self-contained exposition). The last three (for the symmetric HS_*, braided HB_* and hyperoctahedral HO_* homologies) are less known: they were discovered by Fiedorowicz [30] in the early 1990s, but detailed proofs were published only recently (see [2] and [37]).

The second (and perhaps, the main) goal of this paper is to extend the above list of isomorphisms by adding to it representation homology. To be precise, for any commutative ring k, let $\mathrm{HR}_*(k[\Gamma]) := \mathrm{HR}_*(\Gamma, k^\times)$ denote the one-dimensional representation homology of Γ. We prove (see Lemma 4.1 and Theorem 4.2):

Theorem 1.1. *For any homotopy simplicial group* Γ*, there is a natural isomorphism*

$$\mathrm{HR}_*(k[\Gamma]) \cong \mathrm{H}_*(\Omega\,\mathrm{SP}^\infty(B\Gamma);\, k) \tag{1.10}$$

where $\mathrm{SP}^\infty(B\Gamma)$ *denotes the Dold-Thom space of the classifying space of* Γ *.*

Apart from the Hochschild and cyclic theories, most interesting on the list (1.9) is the *symmetric homology* theory HS_* introduced in [30] and studied in [2, 3]. Roughly speaking, HS_* is defined[3] in the same way as HC_*, with Connes' cyclic category ΔC replaced by the symmetric category ΔS, where the family of the symmetric groups $\{S_{n+1}^{\mathrm{op}}\}_{n\ge 0}$ is used instead of the cyclic groups $\{C_{n+1}\}_{n\ge 0}$. Now, the natural inclusions of groups $C_{n+1} \hookrightarrow S_{n+1}$ extend to a functor $\iota : \Delta C^{\mathrm{op}} \hookrightarrow \Delta S$, which, in turn, induces a natural map $\mathrm{HC}_*(k[\Gamma]) \to \mathrm{HS}_*(k[\Gamma])$. It turns out that, with identifications (1.9), this last map is induced (on homology) by a map of topological spaces

$$\mathtt{CS}_{B\Gamma} :\ ES^1 \times_{S^1} \mathcal{L}(B\Gamma) \to \Omega\,\Omega^\infty\Sigma^\infty(B\Gamma) \tag{1.11}$$

The map (1.11) is actually defined as a natural transformation $\mathtt{CS}_X$ on the (homotopy) category of all pointed spaces; it was originally constructed in the paper [21], and its relation to symmetric homology was noticed in [30]. We will refer to (1.11) as the *Carlsson-Cohen map* for $B\Gamma$.

We can now state our second observation that provides a topological interpretation of the derived character maps (1.7) for one-dimensional representations. To shorten notation we will write the maps (1.7) for $n = 1$ as

$$\mathrm{Tr}(\Gamma)_* :\ \mathrm{HC}_*(k[\Gamma]) \to \mathrm{HR}_*(k[\Gamma]) \tag{1.12}$$

The next theorem encapsulates the main results of Section 4.3 (see Proposition 4.8 and Corollary 4.10), Section 5.2 (see Proposition 5.2) and Section 5.3 (see Proposition 5.3).

Theorem 1.2. *With isomorphisms* (1.9) *and* (1.10)*, the derived character maps* (1.12) *are induced on homology by a natural map of topological spaces*

$$\mathtt{CR}_{B\Gamma} :\ ES^1 \times_{S^1} \mathcal{L}(B\Gamma) \to \Omega\,\mathrm{SP}^\infty(B\Gamma) \tag{1.13}$$

[3] See Sections 3.3 and 4.2 for precise definitions of $\mathrm{HC}_*(k[\Gamma])$ and $\mathrm{HS}_*(k[\Gamma])$ in the context of homotopy simplicial groups.

7 Nov 2023 14:19:46 PST
230102-Berest Version 2 - Submitted to Algebr. Geom. Topol.

The map (1.13) *factors* (*as a homotopy natural transformation*) *through the Carlsson-Cohen map* (1.11)*:*

$$ES^1 \times_{S^1} \mathcal{L}(B\Gamma) \xrightarrow{\mathtt{CS}_{B\Gamma}} \Omega\, \Omega^\infty\Sigma^\infty(B\Gamma) \xrightarrow{\mathtt{SR}_{B\Gamma}} \Omega\, \mathrm{SP}^\infty(B\Gamma) \tag{1.14}$$

where the induced map $\mathtt{SR}$ *is the* (*looped once*) *canonical natural transformation* $\Omega^\infty\Sigma^\infty \to \mathrm{SP}^\infty$ *relating stable homotopy to* (*reduced*) *singular homology of pointed spaces.*

Theorem 1.2 shows that, for any homotopy simplicial group Γ, the derived character map (1.12) factors through symmetric homology, and the induced map

$$\mathtt{SR}_{B\Gamma,*} :\ \mathrm{HS}_*(k[\Gamma]) \to \mathrm{HR}_*(k[\Gamma]) \tag{1.15}$$

is determined by a map of spaces that is well known in topology. Using topological results, we then conclude (see Corollary 5.5 and Remark 5.6):

Corollary 1.3. *If* k *is a field of characteristic* 0*, the map* (1.15) *is an isomorphism, at least when* $B\Gamma$ *is a simply connected space.*

The results stated above are all concerned with derived characters of one-dimensional representations. For higher dimensional representations ($n > 1$), the maps (1.7) are more complicated: in particular, they do not seem to factor through $\mathrm{HS}_*(k[\Gamma])$, and in general, the relation between symmetric homology and representation homology remains mysterious. However, when $n \to \infty$, things become more tractable. Assuming that k is a field of characteristic 0, we can naturally pass to the projective limit:

$$\mathrm{HR}_*(\Gamma,\, \mathrm{GL}_\infty(k))^{\mathrm{GL}_\infty} := \varprojlim_n \mathrm{HR}_*(\Gamma,\, \mathrm{GL}_n(k))^{\mathrm{GL}_n}$$

and construct the *stable* character maps

$$\mathrm{Tr}_\infty(\Gamma)_* :\ \overline{\mathrm{HC}}_*(k[\Gamma]) \to \mathrm{HR}_*(\Gamma,\, \mathrm{GL}_\infty(k))^{\mathrm{GL}_\infty}\,, \tag{1.16}$$

where $\overline{\mathrm{HC}}$ stands for the reduced cyclic homology. In this case, we have the following result, the proof of which is parallel to [10] and outlined in the last section of the paper (see Theorem 6.2).

Theorem 1.4. *Let* Γ *be a homotopy simplicial group such that* $B\Gamma$ *is a simply connected space of finite* (*rational*) *type. Then the stable character maps* (1.16) *induce an algebra isomorphism*

$$\Lambda\mathrm{Tr}_\infty(\Gamma)_* :\ \Lambda_k[\,\overline{\mathrm{HC}}_*(k[\Gamma])] \xrightarrow{\sim} \mathrm{HR}_*(\Gamma, \mathrm{GL}_\infty)^{\mathrm{GL}_\infty}\,, \tag{1.17}$$

where $\Lambda_k[\,\overline{\mathrm{HC}}_*(k[\Gamma])]$ *is the graded symmetric algebra generated by the reduced cyclic homology of* $k[\Gamma]$.

We close this Introduction by mentioning one application of stable character maps in derived Poisson geometry. If Γ is a simplicial group model of a simply-connected closed manifold X of dimension d (so that $X \simeq B\Gamma$), then, by (1.9), we can identify $\overline{\mathrm{HC}}_*(k[\Gamma])$ with the reduced S^1-equivariant homology $\bar{\mathrm{H}}_*^{S^1}(\mathcal{L}(X);k)$ of the free loop space of X. Thanks to the work of Chas and Sullivan, the latter is known to carry the so-called *string topology* Lie bracket, making the symmetric algebra $\Lambda_k[\,\overline{\mathrm{HC}}_*(k[\Gamma])] \cong \Lambda_k[\,\bar{\mathrm{H}}_*^{S^1}(\mathcal{L}(X);k)\,]$ a graded Poisson algebra. On the other hand, the representation homology ring $\mathrm{HR}_*(\Gamma, \mathrm{GL}_\infty)^{\mathrm{GL}_\infty}$ acquires a $(2-d)$-shifted graded Poisson structure from the Poincaré duality pairing on (the cohomology of) X. As an application of Theorem 1.4, we show that under the isomorphism (1.17), these two Poisson structures agree: i.e., the map (1.17) is an isomorphism of graded Poisson algebras (see Corollary 6.3).

The paper is organized as follows. In Section 2, we review basic facts from abstract homotopy theory concerning homotopy colimits. The new result proved in this section is Proposition 2.6, which we refer to as 'Shapiro Lemma for model categories'. This proposition provides a key step for proofs of main theorems in Section 4 and may be of independent interest. In Section 3, after reviewing basic theory of homotopy simplicial groups (Section 3.1), we define representation homology (Section 3.2), and cyclic homology (Section 3.3) for such groups and construct the derived character maps relating the two (Section 3.4). In Section 4, we prove Theorem 1.1 (Section 4.1) and then, after defining symmetric homology for homotopy simplicial groups (Section 4.2), we prove part of Theorem 1.2 (see Proposition 4.8 and Corollary 4.10 in Section 4.3). The proof of Theorem 1.2 is completed in Section 6, where we study the maps (1.13) and (1.14) in topological terms, using Goodwillie homotopy calculus and classical operads (see Proposition 5.2 and Proposition 5.3).

Finally, in Section 6, we describe the stabilization procedure for the derived character maps as $n \to \infty$ and sketch the proofs of Theorem 1.4 and Corollary 6.3. Each of the six sections begins with a short introduction that provides more details about its contents.

Acknowledgments. The work of the first author was partially supported by NSF grant DMS 1702372 and the Simons Collaboration Grant 712995. The second author was partially supported by NSF grant DMS 1702323.

2. Shapiro Lemma for Model Categories

In this section, we prove one general result in abstract homotopy theory concerning homotopy colimits that will provide a key step for our Theorem 1.1. We call this result (Proposition 2.6) 'Shapiro Lemma for model categories' as it appears to be a non-abelian generalization of the classical Shapiro Lemma in the context of model categories. We begin with a brief overview of the theory of homotopy colimits. The standard reference for this material is the last two chapters of Hirschhorn's book [39] but many results that we mention are classical and go back to Bousfield-Kan [19] and Quillen [61]. Our exposition is inspired by Cisinski's beautiful paper [24] that treats homotopy colimits axiomatically by analogy with derived direct image functors in algebraic geometry (unlike [24], however, we do not use the language of Grothendieck derivators). With exception of Proposition 2.6, which (to the best of our knowledge) is new, all results in this section are known.

2.1. **Notation and conventions.** Throughout this section, $\mathcal{M}$ will denote a fixed model category which we assume to be cofibrantly generated and having all small limits and colimits. Unless stated otherwise, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ... will denote small categories that we will use to index diagrams in $\mathcal{M}$. For a small category $\mathcal{A}$, the category of $\mathcal{A}$-diagrams in $\mathcal{M}$ (i.e. all functors $\mathcal{A} \to \mathcal{M}$) will be denoted $\mathcal{M}^{\mathcal{A}}$. As usual, Cat will stand for the category of all small categories with morphisms being arbitrary functors.

2.2. **Homotopy colimits.** For any small category $\mathcal{A}$, the category $\mathcal{M}^{\mathcal{A}}$ has a projective (aka Bousfield-Kan) model structure inherited from $\mathcal{M}$: the weak equivalences and fibrations are defined in this model structure objectwise, while the cofibrations are determined by the Lifting Axiom of model categories (specifically, as morphisms having the left lifting property with respect to fibrations which are also weak equivalences in $\mathcal{M}^{\mathcal{A}}$). Since $\mathcal{M}$ is cofibrantly generated, such a model structure on $\mathcal{M}^{\mathcal{A}}$ always exists and is cofibrantly generated (see [39, Theorem 11.6.1]).

Any functor $f : \mathcal{A} \to \mathcal{B}$ (a morphism in Cat) defines the *pullback functor* on the diagram categories $f^* : \mathcal{M}^{\mathcal{B}} \to \mathcal{M}^{\mathcal{A}}$, which is obtained by restricting diagrams $\mathcal{B} \to \mathcal{M}$ along f. This pullback functor preserves objectwise weak equivalences and fibrations and — since $\mathcal{M}$ has small colimits – admits a left adjoint:

$$f_! : \mathcal{M}^{\mathcal{A}} \rightleftarrows \mathcal{M}^{\mathcal{B}} : f^* \tag{2.1}$$

defined on a diagram $X : \mathcal{A} \to \mathcal{M}$ as the left Kan extension $f_!(X) := \mathrm{Lan}_f(X)$ of X along f. Thus, the functors (2.1) form a Quillen pair between the model categories $\mathcal{M}^{\mathcal{A}}$ and $\mathcal{M}^{\mathcal{B}}$. Then, by Quillen's Adjunction Theorem (see [39, 8.5.8]), they admit total (left and right) derived functors

$$\boldsymbol{L}f_! : \mathtt{Ho}(\mathcal{M}^{\mathcal{A}}) \rightleftarrows \mathtt{Ho}(\mathcal{M}^{\mathcal{B}}) : f^* \tag{2.2}$$

that form an adjunction between the homotopy categories of diagrams induced by (2.1).

The derived pushforward functor $\boldsymbol{L}f_!$ is called the *homotopy left Kan extension* along f. It is a generalization of the classical *homotopy colimit functor* $\mathrm{hocolim}_{\mathcal{A}} : \mathtt{Ho}(\mathcal{M}^{\mathcal{A}}) \to \mathtt{Ho}(\mathcal{M})$ that corresponds to the trivial map $\mathcal{A} \to *$, where "$*$" denotes the one-point category (the terminal object in Cat). In this last case, we will use the classical notation writing $\mathrm{hocolim}_{\mathcal{A}}(X)$ instead of $\boldsymbol{L}(\mathcal{A} \to *)_!(X)$ for $X : \mathcal{A} \to \mathcal{M}$. We summarize the main properties of this construction in the following theorem.

Theorem 2.1 (see [24]). *Let $\mathcal{M}$ be a model category with all small limits and colimits.*

(1) 2-Functoriality: *The pullback functors f^* fit together to give a strict, weakly product-preserving 2-functor*[4] $\mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$ *that takes a small category $\mathcal{A} \in \mathrm{Cat}$ to the homotopy category $\mathtt{Ho}(\mathcal{M}^{\mathcal{A}})$. By adjunction, this implies, in particular, the existence of natural weak equivalences*

$$\boldsymbol{L}(fg)_! \simeq \boldsymbol{L}f_!\, \boldsymbol{L}g_! \tag{2.3}$$

for any composable morphisms f and g in Cat.

(2) Reflexivity: *For any $\mathcal{A} \in \mathrm{Cat}$, the functor $i^* : \mathtt{Ho}(\mathcal{M}^{\mathcal{A}}) \to \mathtt{Ho}(\mathcal{M}^{\mathcal{A}^\delta})$ corresponding to the inclusion of the underlying discrete subcategory $\mathcal{A}^\delta \subset \mathcal{A}$ is conservative, i.e. reflects the weak equivalences in $\mathcal{M}^{\mathcal{A}^\delta}$.*

(3) Base change: *For any $f : \mathcal{A} \to \mathcal{B}$ and any object $b \in \mathcal{B}$, the 2-commutativity of the fibre square*

$$\begin{array}{ccc} f \downarrow b & \xrightarrow{\pi} & \mathcal{A} \\ {\scriptstyle p}\downarrow & \swarrow & \downarrow{\scriptstyle f} \\ * & \xrightarrow{b} & \mathcal{B} \end{array}$$

induces a change-of-base natural transformation that is a natural weak equivalence:

$$\boldsymbol{L}p_!\, \pi^* \stackrel{\sim}{\to} b^*\, \boldsymbol{L}f_!$$

For a diagram $X : \mathcal{A} \to \mathcal{M}$, this simply says that

$$\boldsymbol{L}f_! X(b) \simeq \mathrm{hocolim}_{f\downarrow b}(\pi^* X)\,, \tag{2.4}$$

where $f \downarrow b$ is the comma category of the functor $f : \mathcal{A} \to \mathcal{B}$ over the object $b \in \mathcal{B}$.

Remark 2.2. In terminology of [24] (*cf.* Definition 1.6, pp. 205-206), the properties (1)-(3) of Theorem 2.1 can be summarized by saying that the 2-functor $\mathtt{Ho}(\mathcal{M}^-) : \mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$ is a weak left derivator (*un dérivateur faible à gauche*) associated to the model category $\mathcal{M}$.

The properties of homotopy colimits listed in Theorem 2.1 are essentially formal. The next result — called the Cofinality Theorem — gives a deeper property of homotopy-theoretic nature that is very useful in computations. To state this result we recall that a functor $f : \mathcal{A} \to \mathcal{B}$ is *right homotopy cofinal* if its comma-category $b \downarrow f$ under each object $b \in \mathcal{B}$ is (weakly) contractible, i.e. $B(b \downarrow f) \simeq \mathrm{pt}$. As an example, we point out that every right adjoint functor is right homotopy cofinal: indeed, if $f : \mathcal{A} \to \mathcal{B}$ admits a left adjoint, say $g : \mathcal{B} \to \mathcal{A}$, then each comma-category $b \downarrow f$ has an initial object (namely, (b, η_b), where $\eta_b : b \to fg(b)$ is the unit of the adjunction evaluated at $b \in \mathcal{B}$), hence $b \downarrow f$ is contractible for any $b \in \mathcal{B}$.

Theorem 2.3 (Cofinality Theorem)**.** *If $f : \mathcal{A} \to \mathcal{B}$ is right homotopy cofinal, then the natural map*

$$\mathrm{hocolim}_{\mathcal{A}}(f^* X) \stackrel{\sim}{\to} \mathrm{hocolim}_{\mathcal{B}}(X)$$

is a weak equivalence for any diagram $X : \mathcal{B} \to \mathcal{M}$.

For the proof of Theorem 2.3 we refer to [39, Theorem 19.6.7]. As an application, we prove one simple lemma that we will need for our computations. Given a functor $f : \mathcal{A} \to \mathcal{B}$, we recall that its *fibre category* $f^{-1}(b)$ over an object $b \in \mathcal{B}$ is the subcategory of $\mathcal{A}$ consisting of all objects $a \in \mathcal{A}$ such that $f(a) = b$ and all morphisms $\varphi \in \mathrm{Hom}_{\mathcal{A}}(a, a')$ such that $f(\varphi) = \mathrm{Id}_b$. Note that the fibre inclusion functor $i : f^{-1}(b) \hookrightarrow \mathcal{A}$ factors through the comma-category $f \downarrow b$ over b:

$$\begin{array}{ccc} f^{-1}(b) & \xhookrightarrow{i} & \mathcal{A} \\ {\scriptstyle j}\downarrow & \nearrow{\scriptstyle \pi} & \\ f \downarrow b & & \end{array} \tag{2.5}$$

defining the 'comparison' functor

$$j :\ f^{-1}(b) \to f \downarrow b \qquad a \mapsto (a, f(a) = b \stackrel{\mathrm{Id}}{\to} b) \tag{2.6}$$

[4] Here, $\mathrm{Cat}^{\mathrm{op}}$ stands for the opposite 2-category of small categories, while CAT denotes the '2-category' of all (not necessarily small) categories.

Recall that a functor $f:\mathcal{A}\to\mathcal{B}$ is *precofibred* if (2.6) has a left adjoint for every object $b\in\mathcal{B}$ (see [60, §1]).

Lemma 2.4. *If $f:\mathcal{A}\to\mathcal{B}$ is precofibred, then, for any diagram $X:\mathcal{A}\to\mathcal{M}$,*

$$(\boldsymbol{L}f_!X)(b)\simeq \mathrm{hocolim}_{f^{-1}(b)}(i^*X)$$

Proof. By assumption, the inclusion functor $j:f^{-1}(b)\to f\downarrow b$ is right adjoint, hence right homotopy cofinal. By the base change formula (2.4) and Cofinality Theorem 2.3, we conclude

$$\begin{aligned}(\boldsymbol{L}f_!X)(b) &\simeq \mathrm{hocolim}_{f\downarrow b}(\pi^*X)\\ &\simeq \mathrm{hocolim}_{f^{-1}(b)}(j^*\pi^*X)\\ &= \mathrm{hocolim}_{f^{-1}(b)}((\pi j)^*X)\\ &= \mathrm{hocolim}_{f^{-1}(b)}(i^*X)\end{aligned}$$

where the last identification follows from (2.5). □

In practice, precofibred functors arise from the so-called Grothendieck construction (see [67]). Given a functor $F:\mathcal{C}\to\mathrm{Cat}$ (i.e., a strict diagram of small catgories), its *Grothendieck construction* is defined to be the small category $\mathcal{C}\smallint F$ with $\mathrm{Ob}(\mathcal{C}\smallint F):=\{(c,x):c\in\mathcal{C},\,x\in F(c)\}$ and morphism sets

$$\mathrm{Hom}_{\mathcal{C}\smallint F}((c,x),\,(c',x')):=\{(\varphi,f):\varphi\in\mathrm{Hom}_{\mathcal{C}}(c,c'),\,f\in\mathrm{Hom}_{F(c')}(F(\varphi)x,x')\}\ . \tag{2.7}$$

The composition in $\mathcal{C}\smallint F$ is given by $(\varphi,f)\circ(\varphi',f')=(\varphi\varphi',\ f\,F(\varphi)f')$. The category $\mathcal{C}\smallint F$ comes equipped with a natural (forgetful) functor

$$p:\mathcal{C}\smallint F\to\mathcal{C}\ ,\qquad (c,x)\mapsto c$$

which is precofibred (in fact, cofibred) over $\mathcal{C}$. Notice that $p^{-1}(c)=F(c)$ for any object $c\in\mathcal{C}$. Hence, by Lemma 2.4, for any functor $X:\mathcal{C}\smallint F\to\mathcal{M}$,

$$(\boldsymbol{L}p_!X)(c)\simeq \mathrm{hocolim}_{F(c)}[X(c)]\ , \tag{2.8}$$

where $X(c):=i_c^*X$ is the restriction of X to $F(c)$ via the inclusion functor

$$i_c:\ F(c)\to\mathcal{C}\smallint F,\qquad x\mapsto(c,x)\ ,\quad (x\xrightarrow{f}x')\mapsto(\mathrm{Id}_c,f)\ .$$

Note that, by 2-functoriality of homotopy Kan extensions (see (2.3)), (2.8) implies the weak equivalence

$$\mathrm{hocolim}_{\mathcal{C}\smallint F}(X)\simeq \mathrm{hocolim}_{c\in\mathcal{C}}(\mathrm{hocolim}_{F(c)}X(c)) \tag{2.9}$$

which is known as *Thomason's formula* for homotopy colimits over $\mathcal{C}\smallint F$ (see [23, Theorem 26.8]).

An important special case arises when we apply the Grothendieck construction to a set-valued functor $F:\mathcal{C}\to\mathrm{Set}$, regarding sets as discrete categories (i.e. by embedding $\mathrm{Set}\hookrightarrow\mathrm{Cat}$). In this case, the category $\mathcal{C}\smallint F$ is usually denoted $\mathcal{C}_F$ and called the *category of elements of* F as its object set $\mathrm{Ob}(\mathcal{C}_F)$ can be identified with $\coprod_{c\in\mathcal{C}}F(c)$ (we will still write the objects of $\mathcal{C}_F$ as pairs (c,x), where $c\in\mathcal{C}$ and $x\in F(c)$). The Hom-sets in $\mathcal{C}_F$ are given by $\mathrm{Hom}_{\mathcal{C}_F}((c,x),\,(c',x'))=\{\varphi\in\mathrm{Hom}_{\mathcal{C}}(c,c')\,:\,F(\varphi)x=x'\}$ (*cf.* (2.7)). If we take $\mathcal{M}=s\mathrm{Set}$ to be the category of simplicial sets (equipped with standard Quillen model structure) and apply Thomason's formula (2.9) to the trivial diagram $X:\mathcal{C}_F\to *$ in $\mathcal{M}$, then for any functor $F:\mathcal{C}\to\mathrm{Set}$, we get

$$\mathrm{hocolim}_{\mathcal{C}}(F)\cong N_*(\mathcal{C}_F) \tag{2.10}$$

where $N_*(\mathcal{C}_F)$ denotes the simplicial nerve of the category $\mathcal{C}_F$. Formula (2.10) is known as the *Bousfield-Kan construction* for homotopy colimits in $s\mathrm{Set}$ (see [19]).

2.3. **Homotopy coends.** Homotopy coends are special kinds of homotopy colimits defined for *bifunctors*, i.e. the diagrams of the form $\mathcal{C}^{\rm op} \times \mathcal{C} \to \mathcal{M}$. There is a broader range of techniques for manipulating with such homotopy colimits, which makes them more accessible for computations. The homotopy coends are defined in terms of the so-called *factorization category* $\mathcal{F}(\mathcal{C})$ introduced by Quillen [61]. It can be described as the category of elements of the bifunctor $\mathrm{Hom} : \mathcal{C}^{\rm op} \times \mathcal{C} \to \mathrm{Set}$ of the given category $\mathcal{C}$:

$$\mathcal{F}(\mathcal{C}) := (\mathcal{C}^{\rm op} \times \mathcal{C}) \smallint \mathrm{Hom}\ . \tag{2.11}$$

We will be actually dealing with the opposite category $\mathcal{F}(\mathcal{C})^{\rm op}$ which can be explicitly described as follows: the objects of $\mathcal{F}(\mathcal{C})^{\rm op}$ are the morphisms $\{\varphi : c \to d\}$ in $\mathcal{C}$, and the Hom-sets are commutative squares

$$\begin{array}{ccc} d & \xleftarrow{\ \beta\ } & d' \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi'} \\ c & \xrightarrow{\ \alpha\ } & c' \end{array} \tag{2.12}$$

i.e., $\mathrm{Hom}_{\mathcal{F}(\mathcal{C})^{\rm op}}(\varphi, \varphi')$ consists of the pairs of morphisms (α, β) in $\mathcal{C}$ such that $\varphi = \beta\varphi'\alpha$, with compositions defined in the obvious way. Note that $\mathcal{F}(\mathcal{C})^{\rm op} \not\cong \mathcal{F}(\mathcal{C}^{\rm op})$ in general. Now, there are two natural functors

$$s^{\rm op} :\ \mathcal{F}(\mathcal{C})^{\rm op} \to \mathcal{C}\,, \qquad (c \xrightarrow{\varphi} d) \mapsto c \tag{2.13}$$

$$t^{\rm op} :\ \mathcal{F}(\mathcal{C})^{\rm op} \to \mathcal{C}^{\rm op}\,, \qquad (c \xrightarrow{\varphi} d) \mapsto d \tag{2.14}$$

called the (opposite) source and target functors, respectively. We have

Lemma 2.5 (Quillen)**.** *The functors* (2.13) *and* (2.14) *are both right homotopy cofinal.*

Proof. Since $\mathcal{F}(\mathcal{C})$ is defined by Grothendieck construction (2.11), the canonical (forgetful) functor

$$s \times t : \mathcal{F}(\mathcal{C}) \to \mathcal{C}^{\rm op} \times \mathcal{C}$$

is precofibred. It follows (*cf.* [61, Example, p. 94]) that both $s : \mathcal{F}(\mathcal{C}) \to \mathcal{C}^{\rm op}$ and $t : \mathcal{F}(\mathcal{C}) \to \mathcal{C}$ are precofibred. Hence the inclusions $s^{-1}(c) \hookrightarrow s \downarrow c$ and $t^{-1}(d) \hookrightarrow t \downarrow d$ induce weak equivalences of classifying spaces

$$B(s^{-1}(c)) \simeq B(s \downarrow c) \quad , \quad B(t^{-1}(d)) \simeq B(t \downarrow d)\ . \tag{2.15}$$

On the other hand, by inspection, $s^{-1}(c) = c \downarrow \mathcal{C}$ and $t^{-1}(d) = (\mathcal{C} \downarrow d)^{\rm op}$ are the slice and coslice categories respectively. Since both $c \downarrow \mathcal{C}$ and $(\mathcal{C} \downarrow d)^{\rm op}$ have initial objects, they are contractible for all $c, d \in \mathcal{C}$. To complete the proof it remains to note that $(c \downarrow s^{\rm op}) = (s \downarrow c)^{\rm op}$ and $(d \downarrow t^{\rm op}) = (t \downarrow d)^{\rm op}$, where $s^{\rm op}$ and $t^{\rm op}$ are the functors (2.13) and (2.14). Hence

$$B(c \downarrow s^{\rm op})\ =\ B(s \downarrow c)^{\rm op} \simeq B(s \downarrow c) \simeq B(s^{-1}(c)) \simeq \mathrm{pt}$$

and similarly $B(d \downarrow t^{\rm op}) \simeq \mathrm{pt}$. This shows that $s^{\rm op}$ and $t^{\rm op}$ are right homotopy cofinal. □

In view of Lemma 2.5, the Cofinality Theorem gives two natural weak equivalences

$$s^* : \mathrm{hocolim}_{\mathcal{F}(\mathcal{C})^{\rm op}}(s^*Y) \xrightarrow{\sim} \mathrm{hocolim}_{\mathcal{C}}(Y) \tag{2.16}$$

$$t^* : \mathrm{hocolim}_{\mathcal{F}(\mathcal{C})^{\rm op}}(t^*X) \xrightarrow{\sim} \mathrm{hocolim}_{\mathcal{C}^{\rm op}}(X) \tag{2.17}$$

for any diagrams $X : \mathcal{C}^{\rm op} \to \mathcal{M}$ and $Y : \mathcal{C} \to \mathcal{M}$. These equivalences can be used to express arbitrary homotopy colimits over $\mathcal{C}$ and $\mathcal{C}^{\rm op}$ as homotopy coends which we introduce next. Set

$$\pi^{\rm op} := t^{\rm op} \times s^{\rm op} : \mathcal{F}(\mathcal{C})^{\rm op} \to \mathcal{C}^{\rm op} \times \mathcal{C}$$

and for a bifunctor $D : \mathcal{C}^{\rm op} \times \mathcal{C} \to \mathcal{M}$, define its *homotopy coend* by

$$\int_{\boldsymbol{L}}^{c \in \mathcal{C}} D(c, c)\ :=\ \mathrm{hocolim}_{\mathcal{F}(\mathcal{C})^{\rm op}}(\pi^* D)\,, \tag{2.18}$$

where $\pi^*D := D \circ \pi^{\mathrm{op}} : \mathcal{F}(\mathcal{C})^{\mathrm{op}} \to \mathcal{M}$. This is indeed the (left) derived functor of the classical coend functor which is usually denoted

$$\int^{c \in \mathcal{C}} D(c,c) \;:=\; \mathrm{colim}_{\mathcal{F}(\mathcal{C})^{\mathrm{op}}}(\pi^* D)\ . \tag{2.19}$$

The notation (2.18) is very convenient as it suggests the analogy with (definite) integrals in Calculus. For example, for a bifunctor $D : (\mathcal{A} \times \mathcal{B})^{\mathrm{e}} \to M$ defined on a product of two small categories $(\mathcal{A} \times \mathcal{B})^{\mathrm{e}} := \mathcal{A}^{\mathrm{op}} \times \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \times \mathcal{B}$ there is a natural weak equivalence

$$\int_{\boldsymbol{L}}^{(a,b)\in \mathcal{A}\times\mathcal{B}} D(a,b;a,b) \simeq \int_{\boldsymbol{L}}^{a\in\mathcal{A}} \int_{\boldsymbol{L}}^{b \in \mathcal{B}} D(a,b;a,b)$$

which is analogous to the classical Fubini Theorem in Calculus (and thus called the Fubini Theorem for homotopy coends). Another useful formula that we will need is

$$\int_{\boldsymbol{L}}^{c\in\mathcal{C}} \boldsymbol{L}F[D(c,c)] \;\simeq\; \boldsymbol{L}F\left[\int_{\boldsymbol{L}}^{c\in\mathcal{C}} D(c,c)\right] \tag{2.20}$$

where F is a left Quillen functor between model categories. This formula is a consequence of a more general (well-known) result that the derived functors of left Quillen functors preserve homotopy colimits (for a short proof, see e.g. [74, Proposition 3.15]).

We are now in a position to state the main result of this section.

Proposition 2.6 (Shapiro Lemma for model categories)**.** *Let $\mathcal{M}$ be a model category, $\mathcal{C}$ a small category, and $F : \mathcal{C} \to \mathrm{Set}$ a set-valued functor on $\mathcal{C}$. For any contravariant diagram $X : \mathcal{C}^{\mathrm{op}} \to \mathcal{M}$, such that $X(c)$ is cofibrant in $\mathcal{M}$ for all $c \in \mathcal{C}$, there is a natural weak equivalence*

$$\mathrm{hocolim}_{\mathcal{C}_F^{\mathrm{op}}}(p^*X) \;\simeq\; \int_{\boldsymbol{L}}^{c\in\mathcal{C}} X(c) \otimes F(c) \tag{2.21}$$

where $\mathcal{C}_F$ is the category of elements of F, and "$\otimes$" denotes the natural (tensor) action[5] *of* Set *on $\mathcal{M}$.*

For the proof of Proposition 2.6, we need the following observation.

Lemma 2.7. *For any set-valued functor $F : \mathcal{C} \to \mathrm{Set}$, the functor $\mathcal{F}(p)^{\mathrm{op}} : \mathcal{F}(\mathcal{C}_F)^{\mathrm{op}} \to \mathcal{F}(\mathcal{C})^{\mathrm{op}}$ induced by the canonical projection $p : \mathcal{C}_F \to \mathcal{C}$ is precofibred.*

Proof. The proof is by direct verification: we give some details in order to introduce notation and make a few observations that we will use later. We set $f := \mathcal{F}(p)^{\mathrm{op}}$ and describe first the fibre category $f^{-1}(\varphi)$ for $(\varphi : c \to d) \in \mathcal{F}(\mathcal{C})^{\mathrm{op}}$. The objects of $f^{-1}(\varphi)$ are the morphisms in $\mathcal{C}_F$ of the form $(c,x) \xrightarrow{\varphi} (d,y)$ such that $y = F(\varphi)(x)$. We will write the object $(c,x) \xrightarrow{\varphi} (d, F(\varphi)(x))$ of $\mathcal{F}(\mathcal{C}_F)^{\mathrm{op}}$ as (φ, x). Thus,

$$\mathrm{Ob}(f^{-1}(\varphi)) \;=\; \{(\varphi,x) : x \in F(c)\}\ .$$

Further, the morphisms $(\varphi,x) \to (\varphi,y)$ in $f^{-1}(\varphi)$ are precisely the morphisms in $\mathcal{F}(\mathcal{C}_F)^{\mathrm{op}}$, i.e., commutative diagrams of the form

$$\begin{array}{ccc} (d, F(\varphi)(x)) & \xleftarrow{\ \beta\ } & (d, F(\varphi)(y)) \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi} \\ (c,x) & \xrightarrow{\ \alpha\ } & (c,y) \end{array}$$

mapped to the identity by f. This last condition implies that $\alpha = \mathrm{Id}_c$ and $\beta = \mathrm{Id}_d$. Hence,

$$\mathrm{Hom}_{f^{-1}(\varphi)}\big((\varphi,x),(\varphi,y)\big) = \begin{cases} \{\mathrm{Id}\} & \text{if } x = y \\ \varnothing & \text{otherwise} \end{cases}\ .$$

Hence, $f^{-1}(\varphi) \cong F(c)$, where the set $F(c)$ is viewed as a discrete category.

[5] That is, $\otimes$ is the bifunctor $\mathcal{M} \times \mathrm{Set} \to \mathcal{M}$ defined by $A \otimes S = \coprod_S A$, where $\coprod_S A$ is the coproduct of copies of A indexed by the elements of S.

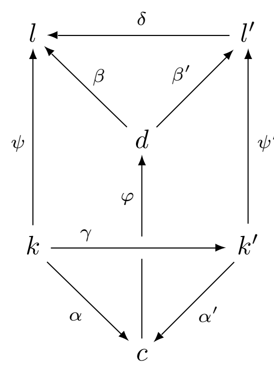
Next, for $(\varphi : c \to d) \in \mathcal{F}(\mathcal{C})^{\mathrm{op}}$, the objects of $f \downarrow \varphi$ are given by

$$\mathrm{Ob}(f \downarrow \varphi) = \{[\psi : k \to l, z, \alpha, \beta] : (\psi, z) \in \mathcal{F}(\mathcal{C}_F)^{\mathrm{op}}, (\alpha, \beta) \in \mathrm{Hom}_{\mathcal{F}(\mathcal{C})^{\mathrm{op}}}(\psi, \varphi)\}\,,$$

while the morphisms $[\psi, z, \alpha, \beta] \to [\psi', z', \alpha', \beta']$ in $f \downarrow \varphi$ are the commutative diagrams in $\mathcal{C}_F$ of the form

$$\begin{array}{ccc} (l, F(\psi)(z)) & \xleftarrow{\delta} & (l', F(\psi')(z')) \\ {\scriptstyle\psi}\uparrow & & \uparrow{\scriptstyle\psi'} \\ (k, z) & \xrightarrow{\gamma} & (k', z') \end{array}$$

such that



commutes in $\mathcal{C}$. In particular, a morphism $[\psi, z, \alpha, \beta] \to [\varphi, x, \mathrm{Id}_c, \mathrm{Id}_d]$ in $f \downarrow \varphi$ is represented by

$$\begin{array}{ccc} (l, F(\psi)(z)) & \xleftarrow{\beta} & (d, F(\varphi)(x)) \\ {\scriptstyle\psi}\uparrow & & \uparrow{\scriptstyle\varphi} \\ (k, z) & \xrightarrow{\alpha} & (c, x) \end{array}$$

Such a diagram exists if and only if $x = F(\alpha)(z)$, in which case it is unique. Hence,

$$\mathrm{Hom}_{f\downarrow\varphi}([\psi, z, \alpha, \beta], [\varphi, x, \mathrm{Id}_c, \mathrm{Id}_d]) \;=\; \begin{cases} \{(\alpha, \beta)\} & \text{if } x = F(\alpha)(z) \\ \varnothing & \text{otherwise}\;. \end{cases}$$

where (α, β) is viewed as a morphism $(\psi, z) \to (\varphi, x)$ in $\mathcal{F}(\mathcal{C}_F)^{\mathrm{op}}$ (rather than $\mathcal{F}(\mathcal{C})^{\mathrm{op}}$).

Now, consider the assignment

$$\Phi : f \downarrow \varphi \to f^{-1}(\varphi)\,, \qquad [\psi, z, \alpha, \beta] \mapsto (\varphi, F(\alpha)(z))\;.$$

If $(\gamma, \delta) : (\psi, z) \to (\psi', z')$ is a morphism in $f \downarrow \varphi$, then $z' = F(\gamma)(z)$ and $\alpha' \circ \gamma = \alpha$. Hence, letting Φ map (γ, δ) to the identity on $(\varphi, F(\alpha)(z))$ makes Φ a *functor*. We then note that

$$\mathrm{Hom}_{f^{-1}(\varphi)}(\Phi([\psi, z, \alpha, \beta]), (\varphi, x)) \;=\; \mathrm{Hom}_{f^{-1}(\varphi)}((\varphi, F(\alpha)(z)), (\varphi, x)) = \begin{cases} \{\mathrm{Id}\} & \text{if } x = F(\alpha)(z) \\ \varnothing & \text{otherwise} \end{cases}\,.$$

Hence, there is a natural bijection

$$\mathrm{Hom}_{f^{-1}(\varphi)}(\Phi([\psi, z, \alpha, \beta]), (\varphi, x)) \cong \mathrm{Hom}_{f\downarrow\varphi}([\psi, z, \alpha, \beta], [\varphi, x, \mathrm{Id}_c, \mathrm{Id}_d])\,,$$

showing that Φ is left adjoint to the canonical inclusion

$$f^{-1}(\varphi) \hookrightarrow f \downarrow \varphi\,, \qquad (\varphi, x) \mapsto [\varphi, x, \mathrm{Id}_c, \mathrm{Id}_d]\;.$$

This shows that f is precofibred, as desired. $\square$

Proof of Proposition 2.6. By formula (2.17) (applied to the category $\mathcal{C}_F$), there is a natural weak equivalence

$$t^* : \mathrm{hocolim}_{\mathcal{F}(\mathcal{C}_F)^{\mathrm{op}}}(t^* p^* X) \xrightarrow{\sim} \mathrm{hocolim}_{\mathcal{C}_F^{\mathrm{op}}}(p^* X)$$

where

$$t^*p^*X:\ \mathcal{F}(\mathcal{C}_F)^{\mathrm{op}} \xrightarrow{t^{\mathrm{op}}} \mathcal{C}_F^{\mathrm{op}} \xrightarrow{p^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{X} \mathcal{M}\,.$$

On the other hand, by definition (2.18),

$$\int_{\boldsymbol{L}}^{c\in\mathcal{C}} X(c)\otimes F(c) \;=\; \mathrm{hocolim}_{\mathcal{F}(\mathcal{C})^{\mathrm{op}}}[\pi^*(X\otimes F)]$$

where

$$\pi^*(X\otimes F):\ \mathcal{F}(\mathcal{C})^{\mathrm{op}} \xrightarrow{\pi^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}}\times\mathcal{C} \xrightarrow{X\times F} \mathcal{M}\times\mathrm{Set} \xrightarrow{\otimes} \mathcal{M}\,.$$

To prove the desired proposition we thus need to show that

$$\mathrm{hocolim}_{\mathcal{F}(\mathcal{C}_F)^{\mathrm{op}}}(t^*p^*X) \;\simeq\; \mathrm{hocolim}_{\mathcal{F}(\mathcal{C})^{\mathrm{op}}}[\pi^*(X\otimes F)] \tag{2.22}$$

By Theorem 2.1(1) (see (2.3)), it suffices to show that there is an weak equivalence of $\mathcal{F}(\mathcal{C})^{\mathrm{op}}$-diagrams

$$\boldsymbol{L}f_!(t^*p^*X) \;\simeq\; \pi^*(X\otimes F)\,, \tag{2.23}$$

where $f:\mathcal{F}(\mathcal{C}_F)^{\mathrm{op}}\to\mathcal{F}(\mathcal{C})^{\mathrm{op}}$ is the functor induced by the canonical projection $p:\mathcal{C}_F\to C$. Thanks to Lemma 2.7, we can use Lemma 2.4 to evaluate the homotopy Kan extension in (2.23) in terms of homotopy colimits over fibre categeories. Specifically, for any $\varphi\in\mathcal{F}(\mathcal{C})^{\mathrm{op}}$, we have

$$\boldsymbol{L}f_!(t^*p^*X)(\varphi) \;\simeq\; \mathrm{hocolim}_{f^{-1}(\varphi)}(i^*t^*p^*X)$$

where $i: f^{-1}(\varphi)\hookrightarrow\mathcal{F}(\mathcal{C}_F)^{\mathrm{op}}$. In the proof of Lemma 2.7, we have described the fibre category $f^{-1}(\varphi)$: namely, $f^{-1}(\varphi)$ is isomorphic to the discrete category $F(c)$ for any $(\varphi: c\to d)\in\mathcal{F}(\mathcal{C})^{\mathrm{op}}$. Now, since $i^*t^*p^*X = i^*f^*t^*X = (fi)^*t^*X = t^*X(\varphi) = X(d)$ and since X is objectwise cofibrant in $\mathcal{M}$, we have

$$\mathrm{hocolim}_{f^{-1}(\varphi)}(i^*t^*p^*X) \simeq \coprod_{F(c)}^{\boldsymbol{L}} X(d) \simeq \coprod_{F(c)} X(d)\,,$$

which is precisely the value of $\pi^*(X\otimes F)$ at φ. Thus, $\boldsymbol{L}f_!(t^*p^*X)\varphi\simeq\pi^*(X\otimes F))\varphi$ in $\mathcal{M}$ for all $\varphi\in\mathcal{F}(\mathcal{C})^{\mathrm{op}}$. By Theorem 2.1(2), this implies (2.23). Summing up, we have constructed the pullback-pushforward diagram

$$\begin{array}{ccc} & \mathrm{hocolim}_{\mathcal{F}(\mathcal{C}_F)^{\mathrm{op}}}(t^*p^*X) & \\ {\scriptstyle t^*}\swarrow & & \searrow{\scriptstyle \boldsymbol{L}f_!} \\ \mathrm{hocolim}_{\mathcal{C}_F^{\mathrm{op}}}(p^*X) & & \mathrm{hocolim}_{\mathcal{F}(\mathcal{C})^{\mathrm{op}}}[\pi^*(X\otimes F)] \end{array}$$

each arrow in which is a weak equivalence. This shows that the objects in both sides of (2.21) are weakly equivalent in $\mathcal{M}$ as claimed by the proposition. □

Remark 2.8. The proof of Proposition 2.6 shows that the additional assumption on the diagram X to be objectwise cofibrant in $\mathcal{M}$ is not needed if the coproducts in $\mathcal{M}$ preserve weak equivalences (e.g., if $\mathcal{M}$ is a cofibrant model category such as the category sSet of simplicial sets with Quillen model structure).

In the special case, if we take $\mathcal{M} = \mathrm{Ch}(\mathrm{Mod}_k)$ to be the category of chain complexes of k-modules equipped with standard projective model structure (see [40, 2.3.11]), Proposition 2.6 implies the following classical result in homological algebra.

Corollary 2.9 (Shapiro Lemma)**.** *Let k be a commutative ring, $\mathcal{C}$ a small category, and $\mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}})$ the (abelian) category of $\mathcal{C}^{\mathrm{op}}$-diagrams of k-modules. Then, for any functor $F:\mathcal{C}\to\mathrm{Set}$, and for any module $X\in\mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}})$, such that $X(c)$ is k-projective for all $c\in\mathcal{C}$, there is a natural isomorphism*

$$\mathrm{Tor}_*^{\mathcal{C}_F}(p^*X,k) \;\cong\; \mathrm{Tor}_*^{\mathcal{C}}(X,k[F])$$

where $k[F]:\ \mathcal{C}\xrightarrow{F}\mathrm{Set}\xrightarrow{k[-]}\mathrm{Mod}_k$ is the k-linear functor generated by F.

Shapiro Lemma appears in [47, Appendix C.12], where it is proven in the special case $X = k$ (the constant $\mathcal{C}^{\rm op}$-diagram valued at k); in the general form, the result of Corollary 2.9 is stated, for example, in [26].

As another immediate consequence of Proposition 2.6, we get a derived version of the classical 'coend formula' for left Kan extensions (see [51, Theorem X.4.1]).

Corollary 2.10. *Let $f : \mathcal{A} \to \mathcal{B}$ be a functor between small categories. Let $X : \mathcal{A} \to \mathcal{M}$ be an $\mathcal{A}$-diagram in a model category $\mathcal{M}$ such that $X(a)$ is cofibrant for all $a \in \mathcal{A}$. Then, for all objects $b \in \mathcal{B}$,*

$$\boldsymbol{L}f_!(X)(b) \simeq \int_{\boldsymbol{L}}^{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(f(a), b) \otimes X(a) \tag{2.24}$$

Proof. To apply Proposition 2.6 take $\mathcal{C} = \mathcal{A}^{\rm op}$ and $F = \mathrm{Hom}_{\mathcal{B}}(f(-), b) : \mathcal{C} \to \mathrm{Set}$. Then $\mathcal{C}_F^{\rm op} \cong f \downarrow b$ and the equivalence (2.24) is obtained as a combination of (2.4) and (2.21). □

The result of Corollary 2.10 must be well known to experts, although we could not find an exact reference in the literature.

3. Representation and Cyclic Homology of Homotopy Simplicial Groups

In this section, we define representation homology of groups with coefficients in a commutative Hopf algebra $\mathcal{H}$, following the approach of [13, 14]. Taking $\mathcal{H} = \mathcal{O}(G)$, where G is an affine algebraic group, we then construct the derived character maps for G-representations of Γ. In the case when $G = \mathrm{GL}_n$, these maps specialize to the character maps (1.7) announced in the Introduction. Unlike in [13, 14], we will work here with *homotopy* simplicial groups (in the sense of Badzioch [4]), which are more general and flexible objects than the usual (strict) simplicial groups. In Section 3.1, we define the classifying spaces for such groups, and in Section 3.3, the cyclic bar construction and cyclic homology, both of which may be of independent interest. We begin by reviewing the main results of [4] specializing to the algebraic theory of groups.

3.1. **Homotopy simplicial groups.** Let $\mathfrak{G}$ denote the small category whose objects $\langle n\rangle$ are the finitely generated free groups $\mathbb{F}_n = \mathbb{F}\langle x_1, x_2, \ldots, x_n\rangle$, one for each $n \ge 0$ (with convention that $\langle 0\rangle$ is the trivial group), and the morphisms are arbitrary group homomorphisms. Every discrete group Γ defines a contravariant functor $\underline{\Gamma} : \mathfrak{G}^{\rm op} \to \mathrm{Set}$, $\langle n\rangle \mapsto \Gamma^n$, which is simply the restriction of the Yoneda functor $\mathrm{Hom}(-, \Gamma) : \mathrm{Gr}^{\rm op} \to \mathrm{Set}$ to $\mathfrak{G} \subset \mathrm{Gr}$. More generally, every simplicial group $\Gamma \in \mathrm{sGr}$ (i.e. a simplicial object in Gr) defines a functor

$$\underline{\Gamma} : \mathfrak{G}^{\rm op} \to \mathrm{sSet}\,, \qquad \langle n\rangle \mapsto \Gamma^n\,, \tag{3.1}$$

where Γ^n denotes the product of n copies of the underlying simplicial set of Γ. The functors (3.1) can be characterized by the property of being product-preserving. To make it precise, observe that the category $\mathfrak{G}$ carries a (strict) monoidal structure $\amalg : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ given by the coproduct (free product) of free groups: $\langle n\rangle \amalg \langle m\rangle = \langle n+m\rangle$. The opposite category $\mathfrak{G}^{\rm op}$ is thus equipped with the dual monoidal structure which we simply denote by $\amalg : \mathfrak{G}^{\rm op} \times \mathfrak{G}^{\rm op} \to \mathfrak{G}^{\rm op}$. Every object $\langle n\rangle^{\rm o} \in \mathfrak{G}^{\rm op}$ comes equipped with n natural projections:

$$p_{n,k} : \langle n\rangle^{\rm o} \to \langle 1\rangle^{\rm o}\,, \qquad 1 \leqslant k \leqslant n\,, \tag{3.2}$$

that correspond to the canonical inclusions $i_{n,k} : \langle 1\rangle \hookrightarrow \langle n\rangle$, $x_1 \mapsto x_k$, in $\mathfrak{G}$. We say that a functor $\mathcal{F} : \mathfrak{G}^{\rm op} \to \mathrm{sSet}$ is *product-preserving* if the maps induced by (3.2)

$$\mathcal{F}(p_n) := \prod_{k=1}^{n} \mathcal{F}(p_{n,k}) : \mathcal{F}\langle n\rangle \to (\mathcal{F}\langle 1\rangle)^n \tag{3.3}$$

are isomorphisms in sSet for all $n \geqslant 0$. It is easy to show that assigning to a simplicial group $\Gamma \in \mathrm{sGr}$ the functor (3.1) defines an equivalence of categories

$$\mathrm{sGr} \xrightarrow{\sim} \mathrm{sSet}_{\otimes}^{\mathfrak{G}^{\rm op}} \tag{3.4}$$

where $\mathrm{sSet}_{\otimes}^{\mathfrak{G}^{\rm op}}$ denotes the full subcategory of product-preserving functors in the diagram category $\mathrm{sSet}^{\mathfrak{G}^{\rm op}}$. We will use (3.4) to identify $\mathrm{sGr} = \mathrm{sSet}_{\otimes}^{\mathfrak{G}^{\rm op}}$, thus regarding the simplicial groups as functors of the form (3.1). Now, the homotopy simplicial groups are obtained by replacing the assumption that the maps (3.3) are isomorphisms in sSet with that of being weak equivalences, which is a more natural condition from the point of view of homotopy theory. Precisely,

Definition 3.1 (Badzioch [4])**.** A *homotopy simplicial group* is a functor $\mathcal{F} : \mathfrak{G}^{\rm op} \to \mathrm{sSet}$ that is weakly product-preserving in the sense that the maps (3.3) are weak equivalences in sSet for all $n \geqslant 0$ (with convention that $\mathcal{F}\langle 0\rangle \simeq \mathrm{pt}$).

The category of homotopy simplicial groups (i.e. the full subcategory of all weakly product-preserving functors in $\mathrm{sSet}^{\mathfrak{G}^{\rm op}}$) does not carry any model structure as it is not closed under colimits. Instead, as suggested in [4], one can put a new model structure on the diagram category $\mathrm{sSet}^{\mathfrak{G}^{\rm op}}$ in which the homotopy simplicial groups are exhibited as fibrant objects (*cf.* [4, Proposition 5.5]). We call this model structure the *Badzioch model structure* and denote it sGr^h. To be precise, sGr^h is defined by localizing (i.e. taking the left Bousfield localization of) the standard projective model structure on $\mathrm{sSet}^{\mathfrak{G}^{\rm op}}$ with respect to the set of maps

$$S = \{i_n :\ \amalg_{k=1}^{n} \mathrm{Hom}_{\mathfrak{G}}(-,\langle 1\rangle) \to \mathrm{Hom}_{\mathfrak{G}}(-,\langle n\rangle)\}_{n\geq 0}$$

induced by the natural inclusions $i_{n,k} : \langle 1\rangle \to \langle n\rangle$ in $\mathfrak{G}$. By definition, the underlying category of sGr^h is that of $\mathrm{sSet}^{\mathfrak{G}^{\rm op}}$ but its class of weak equivalences is larger: in addition to all weak equivalences of $\mathrm{sSet}^{\mathfrak{G}^{\rm op}}$(which are objectwise equivalences of diagrams of simplicial sets), the weak equivalences of sGr^h include the set S and thus called the S-*local* weak equivalences. There is a canonical localization functor $\mathcal{L}_S : \mathrm{sSet}^{\mathfrak{G}^{\rm op}} \to \mathrm{sGr}^h$ that takes a diagram $\Gamma \in \mathrm{sSet}^{\mathfrak{G}^{\rm op}}$ to its functorial fibrant replacement in the model structure sGr^h. In this way, one can make any diagram in $\mathrm{sSet}^{\mathfrak{G}^{\rm op}}$ a homotopy simplicial group. On the other hand, the model category of (strict) simplicial groups sGr is related to sGr^h by a Quillen adjunction:

$$K : \mathrm{sGr}^h \rightleftarrows \mathrm{sGr} : J \tag{3.5}$$

which is obtained by localizing (at S) the Quillen adjunction $K : \mathrm{sSet}^{\mathfrak{G}^{\rm op}} \rightleftarrows \mathrm{sGr} : J$ between sGr and the model category of all diagrams $\mathrm{sSet}^{\mathfrak{G}^{\rm op}}$. In particular, the right adjoint functor in (3.5) is given by the inclusion $J(\Gamma) = \underline{\Gamma}$ (see (3.1)), while the left adjoint — called the Badzioch rigidification functor — is described explicitly in Lemma 3.5 below. Now, the main result of [4] reads:

Theorem 3.2 (Badzioch)**.** *The adjunction* (3.5) *is a Quillen equivalence.*

Remark 3.3. Theorem 3.2 was proved in [4] (see *loc. cit.*, Theorem 6.4) for an arbitrary one-sorted algebraic theory. It was extended to all multi-sorted theories in [17], and further to limit theories and to diagrams in model categories other than sSet in [63].

Next, recall that there is a classical adjunction — called the *Kan loop group construction* [42] — that relates the model category sGr of (strict) simplicial groups to that of (reduced) simplicial sets:

$$\mathbb{G} : \mathrm{sSet}_0 \rightleftarrows \mathrm{sGr} : \bar{W} \tag{3.6}$$

The left adjoint $\mathbb{G}$ is called the *Kan loop group functor*, and the right adjoint $\bar{W}$ is the *classifying complex functor* on simplicial groups. The properties of these functors are well known and discussed in detail, for example, in [32, Chapter V] (see also [14, Section 2.2]). Here, we mention only two important facts: first, the pair (3.6) is a Quillen equivalence, both $\mathbb{G}$ and $\bar{W}$ being homotopy invariant functors (see [32, V.6.4]). Second, for any reduced simplicial set X, there is a weak homotopy equivalence (see [32, V.5.11])

$$|\mathbb{G}(X)| \simeq \Omega|X| \tag{3.7}$$

where $\Omega|X|$ is the (Moore) based loop space of the geometric realization of X. The equivalence (3.7) clarifies the topological meaning of the Kan loop group functor $\mathbb{G}$ (and justifies its name). Combining now Badzioch's Theorem 3.2 with Kan's construction, we get natural equivalences of homotopy categories

$$\mathtt{Ho}(\mathrm{sGr}^h) \xrightarrow{\boldsymbol{L}K} \mathtt{Ho}(\mathrm{sGr}) \xrightarrow{\bar{W}} \mathtt{Ho}(\mathrm{sSet}_0) \xrightarrow{|-|} \mathtt{Ho}(\mathrm{Top}_{0,*}) \tag{3.8}$$

induced by the above indicated functors. This leads us to the following definition.

Definition 3.4. For a homotopy simplicial group $\Gamma \in \mathrm{sGr}^h$, we define its *classifying space* $B\Gamma$ by

$$B\Gamma := |\bar{W}\boldsymbol{L}K(\Gamma)| \tag{3.9}$$

where $\boldsymbol{L}K : \mathtt{Ho}(\mathrm{sGr}^h) \to \mathtt{Ho}(\mathrm{sGr})$ is the derived rigidification functor (see (3.11)).

Note that if Γ is a (strict) simplicial group, i.e. $\Gamma = J(\Gamma)$, then $B\Gamma \cong |\bar{W}\Gamma|$, since $\boldsymbol{L}K \circ J \cong \mathrm{Id}$. Thus the above definition is a natural extension of Kan's definition of classifying spaces for simplicial groups (which is, in turn, an extension of the classical definition of $B\Gamma$ for ordinary discrete groups).

We conclude this section by giving a simple formula for the Badzioch rigidification functor that did not seem to appear explicitly in [4].

Lemma 3.5. *The functor* $K : \mathrm{sGr}^h \to \mathrm{sGr}$ *in* (3.5) *is given by the coend*

$$K(\Gamma) = \int^{\langle n\rangle \in \mathfrak{G}} \Gamma\langle n\rangle \otimes \mathbb{F}\langle n\rangle\,, \tag{3.10}$$

where $\mathbb{F} : \mathfrak{G} \hookrightarrow \mathrm{sGr}$, $\langle n\rangle \mapsto \mathbb{F}_n$, *is the natural inclusion functor, and* $\otimes : \mathrm{sSet} \times \mathrm{sGr} \to \mathrm{sGr}$ *is the standard simplicial tensor action on the category of simplicial groups.*

It follows from Lemma 3.5 that the derived functor $\boldsymbol{L}K$ can be written as the homotopy coend

$$\boldsymbol{L}K(\Gamma) = \int_{\boldsymbol{L}}^{\langle n\rangle \in \mathfrak{G}} \Gamma\langle n\rangle \otimes \mathbb{F}\langle n\rangle\ . \tag{3.11}$$

For the proof of Lemma 3.5 and formula (3.11) (in the general setting of [4]) we refer to our forthcoming paper [11].

3.2. **Representation homology.** Let k be a commutative ring. Recall that, for a small category $\mathcal{C}$, we denote by $\mathrm{Mod}_k(\mathcal{C})$ and $\mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}})$ the categories of all covariant and contravariant functors from $\mathcal{C}$ to Mod_k, respectively. It is well known that these are abelian categories with sufficiently many projective and injective objects. Recall also (see, e.g., [47, Appendix C.10]) that there is a natural bi-additive functor

$$-\otimes_{\mathcal{C}} - : \mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}}) \times \mathrm{Mod}_k(\mathcal{C}) \to \mathrm{Mod}_k$$

called the *functor tensor product*. Explicitly, for $\mathcal{M} : \mathcal{C} \to \mathrm{Mod}_k$ and $\mathcal{N} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_k$, it is defined by

$$\mathcal{N} \otimes_{\mathcal{C}} \mathcal{M} := \big[\bigoplus_{c\in\mathcal{C}} \mathcal{N}(c) \otimes_k \mathcal{M}(c)\big]/R \tag{3.12}$$

where R is the k-submodule spanned by elements of the form $\mathcal{N}(\varphi)x \otimes y - x \otimes \mathcal{M}(\varphi)y$ for all $x \in \mathcal{N}(c')$, $y \in \mathcal{M}(c)$ and $\varphi \in \mathrm{Hom}_{\mathcal{C}}(c, c')$. The functor (3.12) is right exact (with respect to each argument), preserves sums, and is left balanced. Its classical (left) derived functors with respect to each argument are canonically isomorphic and their common value is denoted by $\mathrm{Tor}_*^{\mathcal{C}}(\mathcal{N}, \mathcal{M})$. More generally, we can extend the bifunctor (3.12) to chain complexes of $\mathcal{C}$-modules, i.e. the categories $\mathrm{Ch}(\mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}}))$ and $\mathrm{Ch}(\mathrm{Mod}_k(\mathcal{C}))$, and define

$$\mathrm{Tor}_*^{\mathcal{C}}(\mathcal{N}, \mathcal{M}) := \mathrm{H}_*(\mathcal{N} \otimes^{\boldsymbol{L}}_{\mathcal{C}} \mathcal{M}) \tag{3.13}$$

for any $\mathcal{N} \in \mathrm{Ch}(\mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}}))$ and $\mathcal{M} \in \mathrm{Ch}(\mathrm{Mod}_k(\mathcal{C}))$. Note that $\mathcal{N} \otimes^{\boldsymbol{L}}_{\mathcal{C},k} \mathcal{M}$ is an object in the (unbounded) derived category $\mathcal{D}(k) = \mathcal{D}(\mathrm{Mod}_k)$ of k-modules, and (3.13) is just the usual hyper-Tor functor on chain complexes. Next, observe that there is a natural functor

$$\mathrm{sSet}^{\mathcal{C}^{\mathrm{op}}} \xrightarrow{k[-]} \mathrm{sMod}_k(\mathcal{C}^{\mathrm{op}}) \overset{N}{\hookrightarrow} \mathrm{Ch}(\mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}})) \tag{3.14}$$

transforming the $\mathcal{C}^{\mathrm{op}}$-diagrams in sSet (simplicial presheaves on $\mathcal{C}$) to chain complexes over $\mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}})$. Here N stands for the classical Dold-Kan normalization functor that identifies simplicial objects in $\mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}})$ with non-negatively graded chain complexes in $\mathrm{Ch}(\mathrm{Mod}_k(\mathcal{C}^{\mathrm{op}}))$. Abusing notation, we will write the functor (3.14) simply as $k[-]$.

We are now in a position to define representation homology of homotopy simplicial groups with coefficients in commutative Hopf algebras. We recall the well-known fact (see, e.g., [62, Proposition 14.1.6]) that every such algebra $\mathcal{H}$ defines a covariant functor (a left $\mathfrak{G}$-module) by the rule

$$\underline{\mathcal{H}} : \mathfrak{G} \to \mathrm{Mod}_k\,, \quad \langle n\rangle \mapsto \mathcal{H}^{\otimes n}\ . \tag{3.15}$$

In particular if G is an affine algebraic group (e.g., $G = \mathrm{GL}_n(k)$) with coordinate ring $\mathcal{H} = \mathcal{O}(G)$, then (3.15) can be written in the form $\langle n\rangle \mapsto \mathcal{O}[\mathrm{Rep}_G(\langle n\rangle)]$ which makes the functoriality clear.

Definition 3.6. The *representation homology* of a homotopy simplicial group $\Gamma \in \mathrm{sGr}^h$ with coefficients in $\mathcal{H}$ is defined by

$$\mathrm{HR}_*(\Gamma, \mathcal{H}) \;:=\; \mathrm{Tor}^{\mathfrak{G}}_*(k[\Gamma], \underline{\mathcal{H}})\,,$$

where $k[\Gamma]$ and $\underline{\mathcal{H}}$ are viewed as chain complexes of $\mathfrak{G}$-modules and 'Tor' stands for the hyper-Tor functor over $\mathfrak{G}$ defined by (3.13).

In the special case when G is an affine algebraic group over k and $\mathcal{H} = \mathcal{O}(G)$, we simply write $\mathrm{HR}_*(\Gamma, G)$ instead of $\mathrm{HR}_*(\Gamma, \mathcal{O}(G))$.

The next lemma shows that the above definition agrees with the Badzioch model structure on sGr^h.

Lemma 3.7. *If two homotopy simplicial groups Γ and Γ' are weakly equivalent in sGr^h, then*

$$\mathrm{HR}_*(\Gamma, \mathcal{H}) \;\cong\; \mathrm{HR}_*(\Gamma', \mathcal{H}) \tag{3.16}$$

for any commutative Hopf algebra $\mathcal{H}$.

Proof. By [4, Proposition 5.6], if two homotopy simplicial groups Γ and Γ' are S-locally weakly equivalent, then their underlying diagrams are, in fact, weakly equivalent in $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$. It therefore suffices to show that (3.16) holds for any objectwise weak equivalent diagrams $\Gamma, \Gamma' : \mathfrak{G}^{\mathrm{op}} \to \mathrm{sSet}$. To this end, observe that the linearization functor

$$k[-]\,:\,\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}} \to \mathrm{sMod}_k(\mathfrak{G}^{\mathrm{op}}) \tag{3.17}$$

is left Quillen with respect to the projective model structures (its right adjoint is the forgetful functor). Since the weak equivalences in $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$ are defined objectwise and the model structure on sSet is cofibrant, being left Quillen, the functor (3.17) is actually homotopy invariant: i.e., it maps weakly equivalent objects in $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$ to weakly equivalent objects in $\mathrm{sMod}_k(\mathfrak{G}^{\mathrm{op}})$, which, in turn, are transformed by the normalization functor N to quasi-isomorphic complexes in $\mathrm{Ch}(\mathrm{Mod}_k(\mathfrak{G}^{\mathrm{op}}))$. Thus if $\Gamma \simeq \Gamma'$ in $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$, then $k[\Gamma] \otimes^{\boldsymbol{L}}_{\mathfrak{G},k} \underline{\mathcal{H}} \simeq k[\Gamma'] \otimes^{\boldsymbol{L}}_{\mathfrak{G},k} \underline{\mathcal{H}}$ in $\mathcal{D}(k)$, which implies (3.16). $\square$

Remark 3.8. Recall that the category sGr^h is obtained from $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$ via a left Bousfield localization: its objects are arbitrary diagrams of simplicial sets $\Gamma : \mathfrak{G}^{\mathrm{op}} \to \mathrm{sSet}$ (not just homotopy simplicial groups). The result of Lemma 3.7 does *not* hold for arbitrary diagrams in sGr^h, since the functor (3.17) does not map all S-local weak equivalences to objectwise weak equivalences in $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}}$. This last fact can be easily seen by evaluating (3.17) on representable simplicial presheaves on $\mathfrak{G}$.

An important consequence of Lemma 3.7 is that the representation homology of a homotopy simplicial group Γ depends only on the homotopy type of its classifying space $B\Gamma$ (Definition 3.4). In fact, we have

$$\mathrm{HR}_*(\Gamma, \mathcal{H}) \;\cong\; \mathrm{HR}_*(B\Gamma, \mathcal{H}) \tag{3.18}$$

where the 'HR' in the right-hand side stands for representation homology of topological spaces as defined in [14], using a (non-abelian) derived representation functor (see *loc. cit.*, Definition 3.1). Indeed, by Badzioch's results (*cf.* [4, Theorem 3.1]), every homotopy simplicial group Γ is weakly equivalent to a strict one, say Γ'; hence

$$B\Gamma \;\simeq\; B\Gamma' \;\simeq\; \bar{W}\Gamma'\,. \tag{3.19}$$

On the other hand, by [14, Theorem 4.2], $\mathrm{HR}_*(\Gamma', \mathcal{H}) \cong \mathrm{HR}_*(\bar{W}\Gamma', \mathcal{H})$, which together with (3.19) and the isomorphism (3.16) of Lemma 3.7 implies (3.18).

We conclude this section by briefly explaining how our approach (Definition 3.6) relates to derived algebraic geometry (DAG). For a model of DAG, we will take the simplicial presheaf model developed in [69]. Given a homotopy simplicial group $\Gamma \in \mathrm{sGr}^h$ and an affine algebraic group (scheme) G over k with coordinate algebra $\mathcal{H} = \mathcal{O}(G)$, we introduce the *derived representation scheme of* Γ *in* G:

$$\mathrm{DRep}_G(\Gamma) := \mathbf{R}\mathrm{Spec}\,(k[\Gamma] \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \mathcal{O}(G))\,. \tag{3.20}$$

Here '$\mathbf{R}$Spec' denotes the Toën-Vezzosi derived Yoneda functor that assigns to a (homotopy) simplicial commutative algebra A — a derived ring in terminology of [69] — the simplicial presheaf (prestack)

$$\mathbf{R}\mathrm{Spec}(A) : \mathrm{dAff}_k^{\mathrm{op}} := \mathrm{sComm}_k \to \mathrm{sSet} , \quad B \mapsto \mathrm{Map}(QA, B) ,$$

where QA is a cofibrant replacement of A and 'Map' is the simplicial mapping space (function complex) in sComm_k. The prestack $\mathbf{R}\mathrm{Spec}(A)$ satisfies the descent condition for étale hypercoverings and hence defines a derived stack (which is a derived affine scheme in the sense of [69]). On the other hand, for any pointed space (simplicial set) X, we can define the *pointed* mapping stack $\mathbf{Map}_*(X, BG)$ to be the homotopy fibre of the canonical map in the (homotopy) category of derived stacks:

$$\mathbf{Map}_*(X, BG) := \mathrm{hofib}\,[\mathbf{Map}(X, BG) \to BG] , \tag{3.21}$$

where $\mathbf{Map}(X, BG)$ stands for the (unpointed) derived mapping stack defined in [69, 2.2.6.2]. This last mapping stack is a basic object of derived algebraic geometry that plays an important role in applications (see, e.g., [55]). Now, its relation to representation homology is clarified by the following

Proposition 3.9 (see [12]). *There is a (weak) equivalence of derived stacks*

$$\mathrm{DRep}_G(\Gamma) \simeq \mathbf{Map}_*(B\Gamma, BG)$$

For a detailed proof of Proposition 3.9 and more explanations we refer to [12, Appendix A.1].

3.3. **Cyclic homology.** We now define cyclic homology for homotopy simplicial groups. To this end, we will associate to each $\Gamma \in \mathrm{sGr}^h$ a cyclic module $k[B^{\mathrm{cyc}}\Gamma]$ that generalizes the classical cyclic bar construction $C_*(k[\Gamma])$ when Γ is an ordinary discrete group. We begin by recalling basic definitions.

Let Δ denote the (co)simplicial category whose objects are finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ and morphisms are (nonstrictly) order preserving maps. The category Δ is generated by two families of maps $d_n^i : [n-1] \to [n]$ $(0 \leqslant i \leqslant n,\ n \geqslant 1)$ and $s_n^j : [n+1] \to [n]$ $(0 \leqslant j \leqslant n,\ n \geqslant 0)$, called the (co)face and (co)degeneracy maps respectively. These maps satisfy the standard (co)simplicial relations listed, for example, in [47, Appendix B.3]. Connes' *cyclic category* ΔC is a natural extension of Δ that has the same objects and is generated by the morphisms of Δ and the cyclic maps $\tau_n : [n] \to [n]$, $n \geqslant 0$, satisfying $\tau_n^{n+1} = \mathrm{Id}$ (see [47, 6.11]). Formally, the category ΔC can be characterized by the two properties:

(Cyc1) For each $n \geqslant 0$, $\mathrm{Aut}_{\Delta C}([n]) \cong C_{n+1}$, where $C_{n+1} = \mathbb{Z}/(n+1)\mathbb{Z}$,

(Cyc2) Any morphism $f : [n] \to [m]$ in ΔC can be factored uniquely as $f = g \circ \varphi$, where $g \in \mathrm{Hom}_\Delta([n], [m])$ and $\varphi \in \mathrm{Aut}_{\Delta C}([n])$,

which show that it is a crossed simplicial category associated to the family of cyclic groups $\{C_{n+1}\}_{n\geqslant 0}$ (see [47, 6.3.0]). A *cyclic set* (resp., a *cyclic module*) is defined to be a functor $\Delta C^{\mathrm{op}} \to \mathrm{Set}$ (resp, $\Delta C^{\mathrm{op}} \to \mathrm{Mod}_k$), while a *cocyclic set* (resp., a *cocyclic module*) as a covariant functor $\Delta C \to \mathrm{Set}$ (resp, $\Delta C \to \mathrm{Mod}_k$) on ΔC.

Now, if Γ is an ordinary discrete group, there is a natural functor

$$B_*^{\mathrm{cyc}}\Gamma : \Delta C^{\mathrm{op}} \to \mathrm{Set} \tag{3.22}$$

called the cyclic bar construction of Γ that has the property that $k[B_*^{\mathrm{cyc}}\Gamma] \cong C_*(k[\Gamma])$, where $C_*(k[\Gamma])$ is the standard cyclic module associated to $k[\Gamma]$ as an associative k-algebra. Explicitly, the functor (3.22) is defined by (see [47, 7.3.10])

$$\begin{aligned} d_i(g_0, \ldots, g_n) &= \begin{cases} (g_0, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_n) & 0 \leqslant i < n \\ (g_n g_0, g_1, \ldots, g_{n-1}) & i = n \end{cases} \\ s_j(g_0, \ldots, g_n) &= (g_0, \ldots, g_j, 1, g_{j+1}, \ldots, g_n) \\ t_n(g_0, \ldots, g_n) &= (g_n, g_0, g_1, \ldots, g_{n-1}) \end{aligned}$$

where $(g_0, \ldots, g_n) \in \Gamma^{n+1}$. Clearly, $\Gamma \mapsto B_*^{\mathrm{cyc}}\Gamma$ gives a functor $B_*^{\mathrm{cyc}} : \mathrm{Gr} \to \mathrm{Set}^{\Delta C^{\mathrm{op}}}$. If we identify $\mathrm{Gr} = \mathrm{Set}_\otimes^{\mathfrak{G}^{\mathrm{op}}}$ as in (3.4), then it turns out that B_*^{cyc} coincides with the pull-back functor for a certain natural map $\Psi_{\mathrm{cyc}} : \Delta C \to \mathfrak{G}$ in Cat. Specifically,

$$\Psi_{\mathrm{cyc}} : \Delta C \to \mathfrak{G} \tag{3.23}$$

is defined on objects by

$$\Psi_{\rm cyc}([n]) \,:=\, \langle n+1\rangle \,=\, \mathbb{F}\langle x_0,\ldots,x_n\rangle$$

and on morphisms by the following formulas

$$\begin{aligned}
&\Psi_{\rm cyc}(d_n^i) : \langle n\rangle \to \langle n+1\rangle\,, && (x_0,x_1,\ldots,x_{n-1}) \mapsto \begin{cases} (x_0,\ldots,x_{i-1},x_ix_{i+1},\ldots,x_n)\,, & 0\le i<n\\ (x_nx_0,x_1,\ldots,x_{n-1})\,, & i=n\end{cases}\\
&\Psi_{\rm cyc}(s_n^j) : \langle n+2\rangle \to \langle n+1\rangle\,, && (x_0,\ldots,x_{n+1}) \mapsto (x_0,\ldots,x_j,1,x_{j+1},\ldots,x_n)\,,\\
&\Psi_{\rm cyc}(\tau_n) : \langle n+1\rangle \to \langle n+1\rangle\,, && (x_0,x_1,\ldots,x_n) \mapsto (x_n,x_0,x_1,\ldots,x_{n-1})\,.
\end{aligned} \tag{3.24}$$

where $(x_0,x_1,\ldots,x_n)$ is an ordered sequence of generators of the free group $\mathbb{F}\langle x_0,\ldots,x_n\rangle$.

Lemma 3.10. *For any discrete group* Γ *there is a natural isomorphism of cyclic sets*

$$B^{\rm cyc}\Gamma \,\cong\, \Psi^*_{\rm cyc}(\underline{\Gamma})$$

where $\underline{\Gamma} : \mathfrak{G}^{\rm op} \to {\rm Set}$ *is the functor corresponding to* Γ *under the identification* (3.1).

Proof. Straightforward. □

Remark 3.11. The functor (3.23) was defined in [16] on a slightly larger – the so-called epicyclic – category $\Delta\Psi$, which is an extension of ΔC describing the Adams operations on cyclic modules.

Lemma 3.10 motivates the following definition.

Definition 3.12. For a homotopy simplicial group $\Gamma \in {\rm sGr}^h$, we define its *cyclic bar construction* by

$$B^{\rm cyc}\Gamma \,:=\, \Psi^*_{\rm cyc}(\Gamma) :\; \Delta C^{\rm op} \to {\rm sSet} \tag{3.25}$$

and its *cyclic homology* by

$$\mathrm{HC}_*(k[\Gamma]) \,:=\, \mathrm{Tor}_*^{\Delta C^{\rm op}}(k,\, k[B^{\rm cyc}\,\Gamma]) \,\cong\, \mathrm{Tor}_*^{\Delta C}(k[B^{\rm cyc}\,\Gamma],\, k) \tag{3.26}$$

The same argument as in (the proof of) Lemma 3.7 shows that $\mathrm{HC}_*(k[\Gamma])$ depends only on the homotopy type of Γ in the Badzioch model category ${\rm sGr}^h$, and hence, on the homotopy type of its classifying space $B\Gamma$. In view of Lemma 3.10, the above definition of $\mathrm{HC}_*(k[\Gamma])$ for Γ an ordinary discrete group coincides with the classical (Connes') definition of cyclic homology of group algebras (see [47, 6.2.8]).

3.4. **Derived character maps.** Next, we will construct a family of natural transformations relating the cyclic homology to representation homology of a homotopy simplicial group Γ. In the special case when $\mathcal{H} = \mathcal{O}(\mathrm{GL}_n)$, this family contains a distinguished element determined by the usual trace Tr_n that gives the derived character map (1.7) announced in the Introduction. With our current definitions of representation and cyclic homology the construction is actually very simple. It is based on two lemmas. The first one is a standard result of homological algebra that simply exhibits the naturality of derived tensor products (3.13).

Lemma 3.13. *Let* $f : \mathcal{A} \to \mathcal{B}$ *be a functor between small categories. For any complexes* $\mathcal{N} \in \mathrm{Ch}(\mathrm{Mod}_k\,\mathcal{B}^{\rm op})$ *and* $\mathcal{M} \in \mathrm{Ch}(\mathrm{Mod}_k\,\mathcal{B})$, *there is a natural map* $f^*\mathcal{N} \otimes^{\boldsymbol{L}}_{\mathcal{A},k} f^*\mathcal{M} \to \mathcal{N} \otimes^{\boldsymbol{L}}_{\mathcal{B},k} \mathcal{M}$ *in the derived category* $\mathcal{D}(k)$ *of* k*-modules that induces*

$$f^* :\; \mathrm{Tor}_*^{\mathcal{A}}(f^*\mathcal{N},\, f^*\mathcal{M}) \,\to\, \mathrm{Tor}_*^{\mathcal{B}}(\mathcal{N},\, \mathcal{M})$$

To apply this lemma in our situation we recall that every commutative Hopf k-algebra $\mathcal{H}$ defines the covariant functor $\underline{\mathcal{H}} : \mathfrak{G} \to \mathrm{Mod}_k$ by formula (3.15). Restricting this functor via the morphism (3.23) gives rise to a cocyclic k-module that we denote

$$B_{\rm cyc}\mathcal{H} := \Psi^*_{\rm cyc}(\underline{\mathcal{H}}) :\; \Delta C \to \mathrm{Mod}_k$$

On the other hand, by Definition 3.12, $\Psi^*_{\rm cyc}(k[\Gamma]) = k[B^{\rm cyc}(\Gamma)]$ for any homotopy simplicial group Γ. Thus, by Lemma 3.13, the functor $\Psi_{\rm cyc}$ induces a canonical map

$$\Psi^*_{\rm cyc} :\; \mathrm{Tor}_*^{\Delta C}(k[B^{\rm cyc}\,\Gamma],\, B_{\rm cyc}\mathcal{H}) \,\to\, \mathrm{Tor}_*^{\mathfrak{G}}(k[\Gamma],\, \underline{\mathcal{H}}) \tag{3.27}$$

The target of this map is precisely $\mathrm{HR}_*(\Gamma, \mathcal{H})$ (see Definition 3.6), while the domain differs from $\mathrm{HC}_*(k[\Gamma])$ in the second argument of 'Tor' (*cf.* Definition 3.12). To connect the two Tor's we will use the following lemma which we state in the language of affine algebraic groups.

Lemma 3.14. *Let G be an affine algebraic group defined over k, and let $\mathcal{O}(G)$ be its coordinate algebra. There is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Mod}_k(\Delta C)}(k,\ B_{\mathrm{cyc}}[\mathcal{O}(G)]) \cong \mathcal{O}(G)^G \tag{3.28}$$

where $\mathcal{O}(G)^G$ denotes the invariant subalgebra of $\mathcal{O}(G)$ under the adjoint G-action.

Proof. For $m \geq 0$, denote by $\pi_m : [0] \to [m]$ the composition of maps $d^0_m\, d^0_{m-1} \ldots d^0_1$ in ΔC. It follows from (3.24) that $\Psi_{\mathrm{cyc}}(\pi_m) : \langle 1 \rangle \to \langle m+1 \rangle$ is the homomorphism of groups taking the generator x of $\mathbb{F}\langle x \rangle$ to the product of generators $x_0 x_1 \ldots x_m$ in $\mathbb{F}\langle x_0, \ldots, x_m \rangle$. The corresponding map $[B_{\mathrm{cyc}}\mathcal{O}(G)](\pi_m) : \mathcal{O}(G) \to \mathcal{O}(G)^{\otimes(m+1)}$ can thus be identified with the the m-fold coproduct in $\mathcal{O}(G)$:

$$\Delta_G^{(m)} :\ \mathcal{O}(G) \to \mathcal{O}(G^{m+1})\,, \quad P \mapsto \big[(g_0, g_1, \ldots, g_m) \mapsto P(g_0\, g_1\, \ldots\, g_m)\big]\ . \tag{3.29}$$

Now, it is easy to check that, for a fixed $P \in \mathcal{O}(G)^G$, the maps $\Delta_G^{(m)}(P) :\ k \to \mathcal{O}(G^{m+1})$ taking $1 \in k$ to $\Delta_G^{(m)}(P)$ assemble to a morphism of cocylic modules $\Delta_G(P) :\ k \to B_{\mathrm{cyc}}[\mathcal{O}(G)]$, the commutativity with cyclic operators τ_m being a consequence of the G-invariance of P. We claim that the assignment $P \mapsto \Delta_G(P)$ defines a k-linear isomorphism

$$\Delta_G :\ \mathcal{O}(G)^G \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Mod}_k(\Delta C)}(k,\ B_{\mathrm{cyc}}[\mathcal{O}(G)])\ . \tag{3.30}$$

The inverse of (3.30) can be constructed as follows. Let $\varphi \in \mathrm{Hom}_{\mathrm{Mod}_k(\Delta C)}(k,\ B_{\mathrm{cyc}}[\mathcal{O}(G)])$. Note that, for all $[m] \in \Delta C$, its components $\varphi_{[m]} :\ k \to \mathcal{O}(G)^{\otimes(m+1)} \cong \mathcal{O}(G^{m+1})$ are k-linear maps. Define $T\varphi := \varphi_{[0]}(1) \in \mathcal{O}(G)$, where $1 \in k$. Since φ is a natural transformation,

$$\varphi_{[m]}(1) = \{[B_{\mathrm{cyc}}\mathcal{O}(G)](\pi_m)\}(\varphi_{[0]}(1)) = \{[B_{\mathrm{cyc}}\mathcal{O}(G)](\pi_m)\}(T\varphi) = \Delta^m(T\varphi)\ ,$$

where Δ^m is defined in (3.29). Similarly, applying $B_{\mathrm{cyc}}[\mathcal{O}(G)]$ to the cyclic operators τ_m in ΔC, we have

$$\varphi_{[m]}(1) = \{[B_{\mathrm{cyc}}\mathcal{O}(G)](\tau_m)\}(\varphi_{[m]}(1))\,,$$

from which it follows that $T\varphi(g_0 \ldots g_m) = T\varphi(g_m g_0 \ldots g_{m-1})$ for all $g_0, \ldots, g_m \in G$. This is equivalent to the assertion that $T\varphi \in \mathcal{O}(G)^G$. Thus T defines a k-linear map

$$\mathrm{Hom}_{\mathrm{Mod}_k(\Delta C)}(k,\ B_{\mathrm{cyc}}[\mathcal{O}(G)]) \to \mathcal{O}(G)^G\,, \qquad \varphi \mapsto T\varphi\ .$$

It is clear from its construction that the above map is the inverse of (3.30). □

We can now make the following definition.

Definition 3.15. Let $\Gamma \in \mathrm{sGr}^h$ be a homotopy simplicial group. For an affine algebraic group G and an $\mathrm{Ad}\,G$-invariant polynomial $P \in \mathcal{O}(G)^G$, we define the *derived G-character map of Γ associated to P* by

$$\chi_{G,P}(\Gamma)_* :\ \mathrm{HC}_*(k[\Gamma]) \xrightarrow{(\Delta_G P)_*} \mathrm{Tor}_*^{\Delta C}(k[B^{\mathrm{cyc}}\,\Gamma],\, B_{\mathrm{cyc}}[\mathcal{O}(G)]) \xrightarrow{\Psi^*_{\mathrm{cyc}}} \mathrm{HR}_*(\Gamma, G)\ , \tag{3.31}$$

where $(\Delta_G P)_*$ is a linear map induced by the map of cocyclic modules $\Delta_G P :\ k \to B_{\mathrm{cyc}}[\mathcal{O}(G)]$ (see (3.29) and (3.30)), and Ψ^*_{cyc} is the map (3.27) defined for $\mathcal{H} = \mathcal{O}(G)$.

Explicitly, if we choose a projective resolution $Q \xrightarrow{\sim} k[\Gamma]$ of $k[\Gamma]$ in the (abelian) category $\mathrm{Mod}_k(\mathfrak{G}^{\mathrm{op}})$, applying the functor Ψ^*_{cyc} gives a projective resolution $\Psi^*_{\mathrm{cyc}}Q \xrightarrow{\sim} k[B^{\mathrm{cyc}}\,\Gamma]$ of the cyclic module $k[B^{\mathrm{cyc}}\,\Gamma]$ in $\mathrm{Mod}_k(\Delta C^{\mathrm{op}})$. The map (3.31) is then induced by a map of chain complexes

$$\chi_{G,P}(\Gamma)_* :\ (\Psi^*_{\mathrm{cyc}}Q) \otimes_{\Delta C} k \to Q \otimes_{\mathfrak{G}} \mathcal{O}(G) \tag{3.32}$$

which, in turn, is induced by the following map (see (3.12))

$$\bigoplus_{[m] \in \Delta C} Q\langle m+1 \rangle \ \to \bigoplus_{\langle n \rangle \in \mathfrak{G}} Q\langle n \rangle \otimes \mathcal{O}(G)^{\otimes n}\,, \quad v_{m+1} \mapsto v_{m+1} \otimes \Delta_G^{(m)}(P)\ , \tag{3.33}$$

where $v_{m+1} \in Q\langle m+1 \rangle$ and $\Delta_G^{(m)}(P) \in \mathcal{O}(G)^{\otimes(m+1)}$ is defined by (3.29).

In the special case when $G = \mathrm{GL}_n(k)$ and $P = \mathrm{Tr}_n \in \mathcal{O}(\mathrm{GL}_n)$ is the usual trace function on $(n \times n)$-matrices, we denote the map (3.31) by

$$\mathrm{Tr}_n(\Gamma)_* : \ \mathrm{HC}_*(k[\Gamma]) \to \mathrm{HR}_*(\Gamma, \mathrm{GL}_n(k))\,, \tag{3.34}$$

and call it the *derived character map of n-dimensional representations of* Γ. In the rest of the paper, we will study the maps $\mathrm{Tr}_n(\Gamma)_*$ in two extreme cases: $n = 1$ and $n = \infty$. In the first case, we will give a topological realization of $\mathrm{Tr}(\Gamma)_* := \mathrm{Tr}_1(\Gamma)_*$ by showing that this map is induced on homology by a natural map of topological spaces; in the second case, we will show that $\mathrm{Tr}_\infty(\Gamma)_* := \varprojlim \mathrm{Tr}_n(\Gamma)_*$ extends to an isomorphism between the graded symmetric algebra generated by $\overline{\mathrm{HC}}_*(k[\Gamma])$ and the GL_∞-invariant subalgebra of the stable representation homology $\mathrm{HR}_*(\Gamma, \mathrm{GL}_\infty(k))$. We close this section with a general remark linking the above construction to earlier work.

Remark 3.16. If Γ is an ordinary discrete or (strict) simplicial group, then $k[\Gamma]$ is naturally a simplicial associative k-algebra. By (a monoidal version of) the classical Dold-Kan correspondence (see [64]), we can therefore view $k[\Gamma]$ as a differential-graded (DG) associative k-algebra. For such algebras (defined over a field k of characteristic 0), the derived character maps of n-dimensional representations were constructed in [9]. One can show that these maps agree with (3.34) in the case of group algebras, although the comparison is not entirely trivial as the methods used in [9] and the present paper are quite different. We will address this question in our forthcoming paper [11] in a greater generality.

4. Topological Realization of Derived Character Maps

In this and next sections, we will prove our main results (Theorem 1.1 and Theorem 1.2) stated in the Introduction. Here we will construct the required spaces and maps simplicially: in terms of homotopy colimits of small diagrams of simplicial sets and associated natural maps. Then, in the next section, we will reproduce these maps in topological terms, using Goodwillie homotopy calculus and topological operads. The connection between the two approaches seems instructive and deserves a further investigation.

4.1. **The space X_Γ.** Recall that $\mathfrak{G}$ denotes the skeleton of the category of finitely generated free groups. There is a natural *abelianization functor*

$$\underline{\mathbb{Z}} : \mathfrak{G} \to \mathrm{Set}\,, \qquad \langle n \rangle \mapsto \mathbb{Z}^n\,, \tag{4.1}$$

that takes the free group $\langle n \rangle = \mathbb{F}_n$ to (the underlying set of) its abelianization $\langle n \rangle_{\mathrm{ab}} = \mathbb{Z}^n$. As in Section 2, we can form the category of elements of (4.1), using the Grothendieck construction:

$$\mathfrak{G}_{\mathbb{Z}} := \mathfrak{G} \textstyle\int \underline{\mathbb{Z}} \tag{4.2}$$

The objects of $\mathfrak{G}_{\mathbb{Z}}$ are given explicitly by

$$\mathrm{Ob}(\mathfrak{G}_{\mathbb{Z}}) = \{(\langle n \rangle; k_1, \ldots, k_n)) : \langle n \rangle \in \mathfrak{G}\,,\, (k_1, \ldots, k_n) \in \mathbb{Z}^n\}$$

and the morphism sets are

$$\mathrm{Hom}_{\mathfrak{G}_{\mathbb{Z}}}((\langle n \rangle; k_1, \ldots, k_n), (\langle m \rangle; l_1, \ldots, l_m)) = \{\varphi \in \mathrm{Hom}_{\mathfrak{G}}(\langle n \rangle, \langle m \rangle) : \ \varphi_{\mathrm{ab}}(k_1, \ldots, k_n) = (l_1, \ldots, l_m)\}$$

Note that the abelianized map $\varphi_{\mathrm{ab}} : \mathbb{Z}^n \to \mathbb{Z}^m$ above is represented by an integral $(m \times n)$-matrix, $\varphi_{\mathrm{ab}} \in \mathbb{M}_{m \times n}(\mathbb{Z})$, and its action on n-tuples of integers is simply given by matrix multiplication. The category (4.2) comes together with the canonical (forgetful) functor

$$p : \mathfrak{G}_{\mathbb{Z}} \to \mathfrak{G}\,, \qquad (\langle n \rangle; k_1, \ldots, k_n) \mapsto \langle n \rangle\,. \tag{4.3}$$

Given a homotopy simplicial group $\Gamma : \mathfrak{G}^{\mathrm{op}} \to \mathrm{sSet}$, we now define

$$X_\Gamma := |\, \mathrm{hocolim}_{\mathfrak{G}_{\mathbb{Z}}^{\mathrm{op}}}(p^*\Gamma)\,|\,, \tag{4.4}$$

where p^* is the pullback functor $\mathrm{sSet}^{\mathfrak{G}^{\mathrm{op}}} \to \mathrm{sSet}^{\mathfrak{G}_{\mathbb{Z}}^{\mathrm{op}}}$ associated to (4.3). The relation of the space (4.4) to representation homology becomes clear from the following observation.

Lemma 4.1. *For any Γ and any commutative ring k, there is a natural isomorphism*

$$\mathrm{H}_*(X_\Gamma, k) \,\cong\, \mathrm{HR}_*(\Gamma,\, k^\times)\,, \tag{4.5}$$

where $k^\times = \mathrm{GL}_1(k)$ denotes the multiplicative group of the ring k.

Proof. We have the sequence of natural isomorphisms

$$H_*(X_\Gamma, k) \,\cong\, \mathrm{Tor}_*^{\mathfrak{G}_{\mathbb{Z}}^{\mathrm{op}}}(k,\, k[p^*\Gamma]) \,\cong\, \mathrm{Tor}_*^{\mathfrak{G}_{\mathbb{Z}}}(k[p^*\Gamma],\, k) \,=\, \mathrm{Tor}_*^{\mathfrak{G}_{\mathbb{Z}}}(p^*k[\Gamma],\, k) \,\cong\, \mathrm{Tor}_*^{\mathfrak{G}}(k[\Gamma],\, k[\mathbb{Z}])\,,$$

where the first two are standard (see, e.g., [47, Appendix C.10]) and the last one follows from the classical Shapiro Lemma (see Corollary 2.9). To complete the proof it remains to note that $k[\mathbb{Z}]$ can be identified with $\mathcal{O}[k^\times]$ as a commutative Hopf algebra. ☐

As in the Introduction, we shorten notation for one-dimensional representation homology, writing

$$\mathrm{HR}_*(k[\Gamma]) \,:=\, \mathrm{HR}_*(\Gamma,\, k^\times)\,. \tag{4.6}$$

Our next goal is to identify the homotopy type of the space X_Γ in terms of the classifying space of Γ. The following theorem is one of the main results of the present paper.

Theorem 4.2. *For any homotopy simplicial group Γ, there is a weak equivalence in Top_*:*

$$X_\Gamma \,\simeq\, \Omega\,\mathrm{SP}^\infty(B\Gamma) \tag{4.7}$$

where $B\Gamma$ is the classifying space of Γ (see Definition 3.4).

Before proving this theorem, we recall a few basic facts about the Dold-Thom space and related constructions (see, e.g., [38, Chapter 4.K]). For any pointed connected CW complex X, the *Dold-Thom space* $\mathrm{SP}^\infty(X)$ is defined as the infinite symmetric product: namely,

$$\mathrm{SP}^\infty(X) \,=\, \varinjlim_n \mathrm{SP}^n(X) \tag{4.8}$$

where $\mathrm{SP}^n(X) := X^n/S_n$ with S_n acting on X^n the natural way (by permuting the factors). The maps $\mathrm{SP}^n(X) \to \mathrm{SP}^{n+1}(X)$ along which the inductive limit (4.8) is taken are induced by the natural inclusion

$$X^n \hookrightarrow X^{n+1}\,, \qquad (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, *)\,,$$

where '$*$' stands for the basepoint of X. The Dold-Thom Theorem asserts that, for all $i \geq 1$, there are isomorphisms of abelian groups

$$\pi_i[\mathrm{SP}^\infty(X)] \,\cong\, H_i(X, \mathbb{Z})\,,$$

that are natural in pointed connected CW complexes X. In fact, this classical theorem provides a topological realization for the Hurewicz homomorphisms in the sense that the natural map of spaces

$$X = \mathrm{SP}^1(X) \hookrightarrow \mathrm{SP}^\infty(X) \tag{4.9}$$

induces the homomorphisms of groups: $\pi_i(X) \to \mathrm{H}_i(X, \mathbb{Z})$ for all $i \geqslant 1$.

Now, let $\mathcal{F}X$ denote the homotopy fibre of the inclusion map (4.9) so that we have a homotopy fibration sequence

$$\mathcal{F}X \to X \to \mathrm{SP}^\infty(X)\,. \tag{4.10}$$

There is an alternative way to obtain this fibration sequence, using Kan's simplicial group model $\mathbb{G}(X)$ of the space[6] X. Namely (see, e.g., [5, Section 7]), (4.10) arises from the short exact sequence of simplicial groups

$$1 \to \mathbb{G}_2(X) \to \mathbb{G}(X) \to \mathbb{A}(X) \to 1 \tag{4.11}$$

by applying the classifying space functor $B = |\bar{W}(-)|$. Here $\mathbb{G}_2(X) := [\mathbb{G}(X), \mathbb{G}(X)]$ denotes the commutator subgroup of the Kan loop group $\mathbb{G}(X)$ and $\mathbb{A}(X)$ its abelianization:

$$\mathbb{A}(X) \,:=\, (\mathbb{G}X)_{\mathrm{ab}} \,:=\, \mathbb{G}(X)/\mathbb{G}_2(X) \tag{4.12}$$

[6] Abusing notation, we will use the same symbol X for a (pointed connected) space and its (reduced) simplicial set model.

Thus, we have $\mathrm{SP}^\infty(X) \simeq B\mathbb{A}(X) = |\bar{W}\mathbb{A}(X)|$, which, by Kan's Theorem (see (3.7)), implies

$$\Omega\,\mathrm{SP}^\infty(X) \,\simeq\, \Omega\,|\bar{W}\mathbb{A}(X)| \,\simeq\, |\mathbb{G}\bar{W}\mathbb{A}(X)| \,\simeq\, |\mathbb{A}(X)|\,. \tag{4.13}$$

Note that for any reduced simplicial set X, $\mathbb{A}(X) \cong \tilde{\mathbb{Z}}[X]$ is just the reduced free simplicial abelian group generated by X. After these preliminary remarks we can proceed with

Proof of Theorem 4.2. As a first step we apply Proposition 2.6 to express the homotopy colimit (4.4) as a homotopy coend:

$$\mathrm{hocolim}_{\mathfrak{G}^{\mathrm{op}}_{\mathbb{Z}}}(p^*\Gamma) \,\simeq\, \int_{\boldsymbol{L}}^{\langle n\rangle\in\mathfrak{G}} \Gamma\langle n\rangle \times \underline{\mathbb{Z}}^n \tag{4.14}$$

Next, observe that the bifunctor

$$\Gamma \times \underline{\mathbb{Z}}:\ \mathfrak{G}^{\mathrm{op}} \times \mathfrak{G} \,\to\, \mathrm{sSet}\,, \quad (\langle n\rangle, \langle m\rangle) \,\mapsto\, \Gamma\langle n\rangle \times \mathbb{Z}^m\,, \tag{4.15}$$

that appears in the homotopy coend (4.14) can be factored as

$$\mathfrak{G}^{\mathrm{op}} \times \mathfrak{G} \xrightarrow{\Gamma\otimes\mathbb{F}} \mathrm{sGr} \xrightarrow{(-)_{\mathrm{ab}}} \mathrm{sAb} \xrightarrow{\text{forget}} \mathrm{sSet}$$

where the first arrow is precisely the bifunctor $\Gamma \otimes \mathbb{F}$ that appears in formula (3.10) of Lemma 3.5, expressing the rigidification functor K. This last bifunctor takes an object $(\langle n\rangle, \langle m\rangle) \in \mathfrak{G}^{\mathrm{op}} \times \mathfrak{G}$ to the simplicial group $\amalg_{\Gamma\langle n\rangle} \mathbb{F}_m$, which is given, in each simplicial degree, by a free product of copies of the free group $\mathbb{F}_m$ indexed by the components of the simplicial set $\Gamma\langle n\rangle$. Hence $\Gamma \otimes \mathbb{F}$ is an objectwise cofibrant diagram in sGr, and therefore

$$\boldsymbol{L}(\Gamma \otimes \mathbb{F})_{\mathrm{ab}} \,\simeq\, (\Gamma \otimes \mathbb{F})_{\mathrm{ab}} \,\cong\, \Gamma \times \underline{\mathbb{Z}}\,, \tag{4.16}$$

where $\boldsymbol{L}(-)_{\mathrm{ab}}$ stands for the (left) derived functor of the abelianization functor $(-)_{\mathrm{ab}} : \mathrm{sGr} \to \mathrm{sAb}$. Since the abelianization functor is left Quillen, its derived functor commutes with homotopy coends (see (2.20)). Hence, combining (3.11) with (4.16), we get

$$\boldsymbol{L}[\boldsymbol{L}K(\Gamma)]_{\mathrm{ab}} \,\simeq\, \int_{\boldsymbol{L}}^{\langle n\rangle\in\mathfrak{G}} \boldsymbol{L}(\Gamma\langle n\rangle \otimes \mathbb{F}\langle n\rangle)_{\mathrm{ab}} \,\simeq\, \int_{\boldsymbol{L}}^{\langle n\rangle\in\mathfrak{G}} (\Gamma\langle n\rangle \,\otimes\, \mathbb{F}\langle n\rangle)_{\mathrm{ab}} \,\simeq\, \int_{\boldsymbol{L}}^{\langle n\rangle\in\mathfrak{G}} \Gamma\langle n\rangle \times \underline{\mathbb{Z}}^n \tag{4.17}$$

On the other hand, $\boldsymbol{L}[\boldsymbol{L}K(\Gamma)]_{\mathrm{ab}} \simeq [\mathbb{G}\bar{W}\boldsymbol{L}K(\Gamma)]_{\mathrm{ab}} = \mathbb{A}(\bar{W}\boldsymbol{L}K(\Gamma))$, hence, by (4.13), we have

$$|\,\boldsymbol{L}[\boldsymbol{L}K(\Gamma)]_{\mathrm{ab}}\,| \,\simeq\, |\,\mathbb{A}(\bar{W}\boldsymbol{L}K(\Gamma))\,| \,\simeq\, \Omega\,\mathrm{SP}^\infty(B\Gamma) \tag{4.18}$$

Combining now (4.14), (4.17) and (4.18), we get the desired equivalence $X_\Gamma \simeq \Omega\,\mathrm{SP}^\infty(B\Gamma)$. □

Note that Theorem 4.2 combined with Lemma 4.1 implies Theorem 1.1 stated in the Introduction.

4.2. **Symmetric homology.** In Section 3.3, we defined cyclic homology of homotopy simplicial groups by associating to each $\Gamma \in \mathrm{sGr}^h$ a cyclic bar construction $B^{\mathrm{cyc}}\Gamma : \Delta C^{\mathrm{op}} \to \mathrm{sSet}$ (see Definition 3.12). In this section, we introduce an analogue of this construction for symmetric groups. Recall that the *symmetric crossed simplicial category* ΔS is defined to be an extension of Δ that has the same objects as Δ (and ΔC) with morphisms characterized by the two properties (*cf.* [47, 6.1.4]):

- (Sym1) For each $n \geqslant 0$, $\mathrm{Aut}_{\Delta S}([n]) \cong S_{n+1}^{\mathrm{op}}$, where S_{n+1} is the $(n+1)$-th symmetric group.
- (Sym2) Any morphism $f : [n] \to [m]$ in ΔS can be factored uniquely as the composite $f = g \circ \sigma$ with $g \in \mathrm{Hom}_\Delta([n], [m])$ and $\sigma \in \mathrm{Aut}_{\Delta S}([n]) \cong S_{n+1}^{\mathrm{op}}$.

There is an inclusion functor (a morphism in Cat):

$$\iota:\ \Delta C^{\mathrm{op}} \stackrel{\sim}{\to} \Delta C \hookrightarrow \Delta S\,, \tag{4.19}$$

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where the first arrow is an isomorphism of categories (called Connes' duality) and the second one is induced by the natural inclusion of groups $C_{n+1} \hookrightarrow S_{n+1}$(*cf.* [47, 6.1.11]). Explicitly, the functor (4.19) is given on objects by $\iota([n]) = [n]$ and on generators by the following formulas

$$\begin{aligned} &\iota(d_i^n) = \begin{cases} s_{n-1}^i \ , & 0 \le i < n \\ s_{n-1}^0 \circ (n, 0, 1, \dots, n-1) \ , & i = n \end{cases} \\ &\iota(s_j^n) = d_{n+1}^{j+1} \\ &\iota(t_n) = (n, 0, 1, \dots, n-1) \end{aligned} \tag{4.20}$$

where $d_i^n : [n] \to [n-1]$, $s_j^n : [n] \to [n+1]$ and $t_n : [n] \to [n]$ denote the generators of ΔC^{op} dual (opposite) to the generators d_n^i, s_n^j and τ_n of ΔC, respectively.

Lemma 4.3. *The functor* $\Psi_{\mathrm{cyc}}^{\mathrm{op}} : \Delta C^{\mathrm{op}} \to \mathfrak{G}^{\mathrm{op}}$ *defined by* (3.23), (3.24) *extends through* ι, *giving a commutative diagram of small categories*

$$\begin{array}{ccc} \Delta C^{\mathrm{op}} & \xrightarrow{\Psi_{\mathrm{cyc}}^{\mathrm{op}}} & \mathfrak{G}^{\mathrm{op}} \\ {\scriptstyle \iota}\big\downarrow & \nearrow_{\Psi_{\mathrm{sym}}} & \\ \Delta S & & \end{array} \tag{4.21}$$

Proof. In order to construct the functor Ψ_{sym} it is convenient to use the following notation for morphisms in ΔS (*cf.* [2, Section 1.1]). Any morphism $f : [n] \xrightarrow{\sigma} [n] \xrightarrow{g} [m]$ in ΔS can be written uniquely as a 'tensor product' of $m+1$ noncommutative monomials $X_0, X_1, \dots, X_m$ in $n+1$ formal variables $\{x_0, x_1, \dots, x_n\}$:

$$f = X_0 \otimes X_1 \otimes \dots \otimes X_m \tag{4.22}$$

where each X_i is the product $x_{i_1} x_{i_2} \dots x_{i_r}$ of $r = |f^{-1}(i)|$ variables whose indices i_k occur in the fibre $f^{-1}(i)$ and that are ordered in the same way as numbers in $\{\sigma(0), \dots, \sigma(n)\}$, i.e. $\sigma(i_1) < \sigma(i_2) < \dots < \sigma(i_r)$. For example, if $f : [4] \to [3]$ is given by the composition $g \circ \sigma$ in ΔS, where $g \in \mathrm{Hom}_{\Delta}([4], [3])$ is defined by $g(0) = g(1) = 0$, $g(2) = g(3) = 1$ and $g(4) = 3$ and $\sigma \in \mathrm{Aut}_{\Delta S}([4]) = S_5^{\mathrm{op}}$ is the permutation

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 4 & 2 & 3 \end{pmatrix}$$

then f is represented by $x_1 x_0 \otimes x_3 x_4 \otimes 1 \otimes x_2$. The composition of morphisms $f_1 \circ f_2$ is defined by a natural substitution rule: for example, if $f_1 : [3] \to [3]$ and $f_2 : [4] \to [3]$ in ΔS are represented by

$$f_1 = 1 \otimes x_0 \otimes 1 \otimes x_3 x_2 x_1 \ , \quad f_2 = x_2 x_1 \otimes x_4 \otimes 1 \otimes x_0 x_3 \ ,$$

then $f_1 \circ f_2 : [4] \to [3]$ can be computed as

$$\begin{aligned} f_1 \circ f_2 &= (1 \otimes X_0 \otimes 1 \otimes X_3 X_2 X_1) \circ (\underbrace{x_2 x_1}_{X_0} \otimes \underbrace{x_4}_{X_1} \otimes \underbrace{1}_{X_2} \otimes \underbrace{x_0 x_3}_{X_3}) \\ &= 1 \otimes x_2 x_1 \otimes 1 \otimes (x_0 x_3) \cdot 1 \cdot (x_4) \ = \ 1 \otimes x_2 x_1 \otimes 1 \otimes x_0 x_3 x_4 \end{aligned}$$

With this notation, we define the functor

$$\Psi_{\mathrm{sym}} : \ \Delta S \to \mathfrak{G}^{\mathrm{op}} \tag{4.23}$$

on objects by

$$\Psi_{\mathrm{sym}}([n]) = \langle n+1 \rangle$$

and on morphisms by the following formula: if $f \in \mathrm{Hom}_{\Delta S}([n], [m])$ is represeented by

$$f = (x_{i_1} \dots x_{i_r}) \otimes \dots \otimes (x_{k_1} \dots x_{k_s}) \ ,$$

then

$$\Psi_{\mathrm{sym}}(f) : \langle m+1 \rangle \to \langle n+1 \rangle \ , \qquad X_0 \mapsto x_{i_1} \dots x_{i_r} \ , \ \dots \ , X_m \mapsto x_{k_1} \dots x_{k_s} \ , \tag{4.24}$$

where $\langle m+1\rangle = \mathbb{F}\langle X_0, \ldots, X_m\rangle$ and $\langle n+1\rangle = \mathbb{F}\langle x_0, \ldots, x_n\rangle$. Note that the maps (4.20) can be rewritten in this tensor notation as

$$\iota(d_i^n) = \begin{cases} x_0 \otimes \ldots \otimes x_{i-1} \otimes x_i x_{i+1} \otimes x_{i+2} \otimes \ldots \otimes x_n\ , & 0 \le i < n \\ x_n x_0 \otimes x_1 \otimes \ldots \otimes x_{n-1}\ , & i = n \end{cases}$$
$$\iota(s_j^n) = x_0 \otimes \ldots \otimes x_j \otimes 1 \otimes x_{j+1} \otimes \ldots \otimes x_n$$
$$\iota(\tau_n) = x_n \otimes x_0 \otimes x_1 \otimes \ldots \otimes x_{n-1}$$

The commutativity of (4.21) can now be checked by a trivial calculation that we leave to the reader. □

Having in hand the functor $\Psi_{\rm sym} :\ \Delta S \to \mathfrak{G}^{\rm op}$, we can now define a symmetric bar construction in the same way as we defined the cyclic bar construction in Definition 3.12.

Definition 4.4. For a homotopy simplicial group $\Gamma \in {\rm sGr}^h$, its *symmetric bar construction* is the functor

$$B_{\rm sym}\Gamma \ :=\ \Psi^*_{\rm sym}\Gamma : \Delta S \to {\rm sSet} \tag{4.25}$$

ans its *symmetric homology* is defined by

$$\mathrm{HS}_*(k[\Gamma]) \ :=\ \mathrm{Tor}_*^{\Delta S}(k, k[B_{\rm sym}\Gamma])\ . \tag{4.26}$$

Remark 4.5. The same argument as (in the proof of) Lemma 3.7 shows that $\mathrm{HS}_*(k[\Gamma])$ depends only on the homotopy type of Γ in ${\rm sGr}^h$ and hence on the homotopy type of the space $B\Gamma$.

Remark 4.6. For Γ an ordinary discrete group, the definition (4.25) agrees with Fiedorowicz's original definition of the symmetric bar construction (see [30] and also [2]). In this case, formula (4.26) defines the symmetric homology of the group algebra $k[\Gamma]$. Note that, unlike $B^{\rm cyc}\Gamma$ (see (3.25)), the functor $B_{\rm sym}\Gamma$: $\Delta S \to {\rm sSet}$ is covariant on ΔS (which we emphasize by writing "sym" as a subscript).

Remark 4.7. To study symmetric homology it is often convenient to work with the *augmented* symmetric category ΔS_+ which is defined by adding to ΔS the initial object $[-1]$ and morphisms $[-1] \to [n]$, one for each $n \geqslant -1$ (see [2]). It is easy to see that the map $\Psi_{\rm sym}$ defined in Lemma 4.3 extends to ΔS_+:

$$\Psi_{{\rm sym},+} :\ \Delta S_+ \to \mathfrak{G}^{\rm op} \tag{4.27}$$

by letting $\Psi_{{\rm sym},+}([-1]) := \langle 0\rangle$. Now, the category ΔS_+ is isomorphic to the category of so-called *finite associative sets*, $\mathcal{F}({\rm as})$, introduced in [57] (see also [62, Section 15.4] for a detailed discussion) . The latter is known to be a permutative category (PROP) that describes the associative unital algebras (see [56] and also [62]). Its opposite category $\mathcal{F}({\rm as})^{\rm op}$ describes the coassociative counital coalgebras. If we identify $\Delta S_+ = \mathcal{F}({\rm as})$, the restriction functor $\Psi^*_{{\rm sym},+} :\ {\rm Mod}_k(\mathfrak{G}) \to {\rm Mod}_k[\mathcal{F}({\rm as})^{\rm op}]$ associated to the opposite of (4.27) takes commutative Hopf algebras viewed as functors (3.15) on $\mathfrak{G}$ to the underlying coassociative coalgebras viewed as functors on $\mathcal{F}({\rm as})^{\rm op}$. In other words, the morphism $\Psi^{\rm op}_{{\rm sym},+}$ is isomorphic to a morphism of PROPs: $\mathcal{F}({\rm as})^{\rm op} \to \mathfrak{G}$ that "forgets" the algebra structure on commutative Hopf algebras.

4.3. **Symmetric homology vs representation homology.** Recall that in Section 3.4, we constructed the derived character map $\mathrm{Tr}(\Gamma)_*$ relating the cyclic homology of Γ to its (one-dimensional) representation homology:

$$\mathrm{Tr}(\Gamma)_* :\ \mathrm{HC}_*(k[\Gamma]) \to \mathrm{HR}_*(k[\Gamma]) \tag{4.28}$$

On the other hand, as a consequence of Lemma 4.3, we have a restriction map

$$\iota^* :\ \mathrm{HC}_*(k[\Gamma]) = \mathrm{Tor}_*^{\Delta C^{\rm op}}(k,\, k[B^{\rm cyc}\Gamma]) \,\to\, \mathrm{Tor}_*^{\Delta S}(k,\, k[B_{\rm sym}\Gamma]) = \mathrm{HS}_*(k[\Gamma]) \tag{4.29}$$

induced by the isomorphism of cyclic spaces

$$B^{\rm cyc}\Gamma \,\cong\, \iota^* B_{\rm sym}\Gamma \tag{4.30}$$

The next proposition shows that the derived character map (4.28) factors through (4.29), thus relating representation homology to symmetric homology.

Proposition 4.8. *For any homotopy simplicial group $\Gamma \in \mathrm{sGr}^h$, there is a natural map*

$$\tilde{\Psi}^*_{\mathrm{sym}} :\ \mathrm{HS}_*(k[\Gamma]) \to \mathrm{HR}_*(k[\Gamma]) \tag{4.31}$$

such that

$$\begin{array}{ccc} \mathrm{HC}_*(k[\Gamma]) & \xrightarrow{\mathrm{Tr}(\Gamma)_*} & \mathrm{HR}_*(k[\Gamma]) \\ & {\scriptstyle \iota^*}\searrow \quad \nearrow{\scriptstyle \tilde{\Psi}^*_{\mathrm{sym}}} & \\ & \mathrm{HS}_*(k[\Gamma]) & \end{array} \tag{4.32}$$

Proof. As our notation suggests, the map (4.31) is actually induced by a morphism $\tilde{\Psi}_{\mathrm{sym}}$ in Cat. We construct $\tilde{\Psi}_{\mathrm{sym}}$ by lifting the functor Ψ_{sym} of Lemma 4.3 to the (opposite) category of elements of the abelianization functor (4.1):

$$\begin{array}{ccccc} & & & & \mathfrak{G}^{\mathrm{op}}_{\mathbb{Z}} \\ & & & \nearrow^{\tilde{\Psi}_{\mathrm{sym}}} & \downarrow{\scriptstyle p^{\mathrm{op}}} \\ \Delta C^{\mathrm{op}} & \xrightarrow{\iota} & \Delta S & \xrightarrow{\Psi_{\mathrm{sym}}} & \mathfrak{G}^{\mathrm{op}} \end{array} \tag{4.33}$$

The existence of such a lifting is a consequence of the following observation. Consider the composition of functors

$$\Delta S^{\mathrm{op}} \xrightarrow{\Psi^{\mathrm{op}}_{\mathrm{sym}}} \mathfrak{G} \xrightarrow{(-)_{\mathrm{ab}}} \mathrm{Ab} \tag{4.34}$$

that takes an object $[n] \in \Delta S$ to the abelian group $\mathbb{Z}^{n+1}$. If we represent a morphism $f : [n] \to [m]$ in ΔS using the tensor notation (4.22), then $\Psi^{\mathrm{op}}_{\mathrm{sym}}(f)_{\mathrm{ab}} : \mathbb{Z}^{m+1} \to \mathbb{Z}^{n+1}$, the value of (4.34) on f, is represented by an integral $(n+1) \times (m+1)$-matrix whose rows are indexed by $0 \leqslant i \leqslant n$ and columns by $0 \leqslant j \leqslant m$, and the j-th column consists entirely of 0's and 1's, with the 1's occurring in positions indicated by the elements of $f^{-1}(j)$. For example, if $f : [4] \to [3]$ in ΔS is represented by the product $x_1x_0 \otimes x_3x_4 \otimes 1 \otimes x_2$, then $\Psi^{\mathrm{op}}_{\mathrm{sym}}(f)_{\mathrm{ab}} : \mathbb{Z}^4 \to \mathbb{Z}^5$ is given by

$$\Psi^{\mathrm{op}}_{\mathrm{sym}}(f)_{\mathrm{ab}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} .$$

Observe that for any morphism f in ΔS the matrix $\Psi^{\mathrm{op}}_{\mathrm{sym}}(f)_{\mathrm{ab}}$ thus defined has exactly one nonzero entry in each row and that entry is 1. Hence $\Psi^{\mathrm{op}}_{\mathrm{sym}}(f)_{\mathrm{ab}}$ maps the vector $(1, 1, \ldots, 1)^t \in \mathbb{Z}^{m+1}$ to the vector $(1, 1, \ldots, 1)^t \in \mathbb{Z}^{n+1}$. This shows that there is a well-defined functor

$$\tilde{\Psi}_{\mathrm{sym}} :\ \Delta S \to \mathfrak{G}^{\mathrm{op}}_{\mathbb{Z}} \ , \quad [n] \mapsto (\langle n+1 \rangle; 1, 1, \ldots, 1) \ , \tag{4.35}$$

that makes the diagram (4.33) commutative. It follows from (4.33) that

$$k[B_{\mathrm{sym}}\Gamma] = \Psi^*_{\mathrm{sym}}(k[\Gamma]) = \tilde{\Psi}^*_{\mathrm{sym}}(k[p^*\Gamma])$$

Hence, by Lemma 3.13, the functor (4.35) induces a natural map

$$\mathrm{HS}_*(k[\Gamma]) = \mathrm{Tor}^{\Delta S}_*(k, k[B_{\mathrm{sym}}\Gamma]) \xrightarrow{\tilde{\Psi}^*_{\mathrm{sym}}} \mathrm{Tor}^{\mathfrak{G}^{\mathrm{op}}_{\mathbb{Z}}}_*(k,\ k[p^*\Gamma]) \,. \tag{4.36}$$

We claim that if the target of the map (4.36) is identified with the representation homology of $k[\Gamma]$ via the Shapiro Isomorphism (see Corollary 2.9), then the required factorization property (4.32) holds. To verify this we fix a projective resolution $Q \xrightarrow{\sim} k[\Gamma]$ of $k[\Gamma]$ in $\mathrm{Mod}_k(\mathfrak{G}^{\mathrm{op}})$. Then $p^*(Q) \xrightarrow{\sim} p^*k[\Gamma] = k[p^*\Gamma]$ gives a projective resolution of $k[p^*\Gamma]$ in $\mathrm{Mod}_k(\mathfrak{G}^{\mathrm{op}}_{\mathbb{Z}})$, and the Shapiro Isomorphism

$$\mathrm{Tor}^{\mathfrak{G}_{\mathbb{Z}}}_*(k[p^*\Gamma],\ k) \xrightarrow{\sim} \mathrm{Tor}^{\mathfrak{G}}_*(k[\Gamma],\ p_!(k))$$

is induced by the composition

$$p^*(Q) \otimes_{\mathfrak{G}_{\mathbb{Z}}} k \xrightarrow{\mathrm{Id} \otimes \varepsilon_k} p^*(Q) \otimes_{\mathfrak{G}_{\mathbb{Z}}} p^*p_!(k) \xrightarrow{p^*} Q \otimes_{\mathfrak{G}} p_!(k)$$

where the first map is given by the adjunction unit $\varepsilon : \mathrm{Id} \to p^* p_!$ and the second is the restriction map via p. Explicitly, using the definition (3.12) of functor tensor products, we can represent the above composite map as

$$\bigoplus_{(\langle n\rangle;\, k_1,\ldots,k_n)\in\mathfrak{G}_{\mathbb{Z}}} Q\langle n\rangle \;\to\; \bigoplus_{\langle n\rangle\in\mathfrak{G}} Q\langle n\rangle\otimes k[\mathbb{Z}^n]\ , \quad (v_n)_{(\langle n\rangle;\, k_1,\ldots,k_n)\in\mathfrak{G}_{\mathbb{Z}}} \mapsto (v_n\otimes(k_1,\ldots,k_n))_{\langle n\rangle\in\mathfrak{G}} \tag{4.37}$$

where $v_n \in Q\langle n\rangle$ and the subscripts denote the indices of the corresponding components of direct sums. Now, using the same resolution Q, we can write explicitly the composition of maps (4.29) and (4.36):

$$\Psi^*_{\mathrm{cyc}}(Q)\otimes_{\Delta C} k \xrightarrow{\iota^*} \Psi^*_{\mathrm{sym}}(Q)\otimes_{\Delta S^{\mathrm{op}}} k \xrightarrow{\tilde{\Psi}^*_{\mathrm{sym}}} p^*(Q)\otimes_{\mathfrak{G}_{\mathbb{Z}}} k$$

At the level of chain complexes, this last composition is induced by the map

$$\bigoplus_{[m]\in\Delta C} Q\langle m+1\rangle \;\to\; \bigoplus_{[m]\in\Delta S^{\mathrm{op}}} Q\langle m+1\rangle \;\to\; \bigoplus_{(\langle n\rangle;k_1,\ldots k_n)\in\mathfrak{G}_{\mathbb{Z}}} Q\langle n\rangle \tag{4.38}$$

that takes $(v_{m+1})_{[m]\in\Delta C} \mapsto (v_{m+1})_{[m]\in\Delta S^{\mathrm{op}}} \mapsto (v_{m+1})_{(\langle m+1\rangle;\,1,1,\ldots,1)\in\mathfrak{G}_{\mathbb{Z}}}$. Combining (4.37) and (4.38), we see that the resulting map

$$\bigoplus_{[m]\in\Delta C} Q\langle m+1\rangle \;\to\; \bigoplus_{\langle n\rangle\in\mathfrak{G}} Q\langle n\rangle\otimes k[\mathbb{Z}^n]\ , \quad (v_{m+1})_{[m]\in\Delta C} \;\mapsto\; (v_{m+1}\otimes(1,1,\ldots,1))_{\langle m+1\rangle\in\mathfrak{G}}$$

coincides exactly with the map (3.33) representing the derived character $\chi_{\mathrm{GL}_1,\mathrm{Tr}_1}(\Gamma)_* = \mathrm{Tr}(\Gamma)_*$. This finishes the proof of the proposition. □

Remark 4.9. The proof of Proposition 4.8 shows that, apart from (4.35), any functor of the form

$$\tilde{\Psi}^{(m)}_{\mathrm{sym}} :\ \Delta S \to \mathfrak{G}^{\mathrm{op}}_{\mathbb{Z}}\ , \quad [n]\mapsto (\langle n+1\rangle;\, m, m,\ldots, m)\ , \tag{4.39}$$

where $m\in\mathbb{Z}$ is a fixed integer, satisfies the lifting property (4.33). It is easy to see that there are no other solutions to this lifting problem. Among (4.39) the functor $\tilde{\Psi}^{(0)}_{\mathrm{sym}}$ corresponding to $m = 0$ is the only one that factors through $\mathfrak{G}^{\mathrm{op}}$: $\tilde{\Psi}^{(0)}_{\mathrm{sym}} = s^{\mathrm{op}}\circ\Psi_{\mathrm{sym}}$, where $s : \mathfrak{G} \hookrightarrow \mathfrak{G}_{\mathbb{Z}}$ is the 'zero' section of p.

Next, we observe that the linear maps factoring $\mathrm{Tr}(\Gamma)_*$ in (4.32) arise (on homology) from the natural maps of topological spaces induced by the functors (4.19) and (4.35) (*cf.* Lemma 4.1):

$$|\mathrm{hocolim}_{\Delta C^{\mathrm{op}}}\,(B^{\mathrm{cyc}}\Gamma)| \xrightarrow{\iota^*} |\mathrm{hocolim}_{\Delta S}\,(B_{\mathrm{sym}}\Gamma)| \xrightarrow{\tilde{\Psi}^*_{\mathrm{sym}}} |\mathrm{hocolim}_{\mathfrak{G}^{\mathrm{op}}_{\mathbb{Z}}}\,(p^*\Gamma)| \tag{4.40}$$

(Here, abusing notation, we denote these topological maps by the same symbols as the corresponding linear maps.) By Theorem 4.2, we know that

$$|\mathrm{hocolim}_{\mathfrak{G}^{\mathrm{op}}_{\mathbb{Z}}}(p^*\Gamma)| \;\simeq\; \Omega\,\mathrm{SP}^\infty(B\Gamma) \tag{4.41}$$

On the other hand, by theorems of Goodwillie (see [47, Theorem 7.2.4]) and Fiedorowicz [30] (see [2, Section 5.3]),

$$|\mathrm{hocolim}_{\Delta C^{\mathrm{op}}}(B^{\mathrm{cyc}}\Gamma)| \;\;\simeq\;\; ES^1\times_{S^1}\mathcal{L}(B\Gamma)\,, \tag{4.42}$$

$$|\mathrm{hocolim}_{\Delta S}(B_{\mathrm{sym}}\Gamma)| \;\;\simeq\;\; \Omega\Omega^\infty\Sigma^\infty(B\Gamma)\,, \tag{4.43}$$

where $\mathcal{L}(B\Gamma) := \mathrm{Map}(S^1, B\Gamma)$ and $\Omega^\infty\Sigma^\infty(B\Gamma) := \mathrm{hocolim}_{n\to\infty}\,\Omega^n\Sigma^n(B\Gamma)$ denote the free loop space and the infinite loop space of $B\Gamma$, respectively. Combining (4.40) with equivalences (4.41), (4.42) and (4.43), we can thus refine the result of Proposition 4.8 as follows:

Corollary 4.10. *The derived character map*

$$\mathrm{Tr}(\Gamma)_* : \mathrm{HC}_*(k[\Gamma]) \xrightarrow{\iota^*} \mathrm{HS}_*(k[\Gamma]) \xrightarrow{\tilde{\Psi}^*_{\mathrm{sym}}} \mathrm{HR}_*(k[\Gamma])$$

is induced on homology by a natural map of topological spaces in $\mathtt{Ho}(\mathrm{Top}_*)$*:*

$$ES^1\times_{S^1}\mathcal{L}(B\Gamma) \xrightarrow{\mathtt{CS}_{B\Gamma}} \Omega\Omega^\infty\Sigma^\infty(B\Gamma) \xrightarrow{\mathtt{SR}_{B\Gamma}} \Omega\mathrm{SP}^\infty(B\Gamma) \tag{4.44}$$

In the next section, we will describe the maps $\mathtt{CS}$ and $\mathtt{SR}$ in topological terms in two ways: using the classical 'little cubes' operads and the Goodwillie calculus of homotopy functors.

4.4. **Generalization to monoids.** All results of this section generalize naturally to (simplicial) monoids. We briefly outline this generalization as we will need it in Section 5.3. Instead of $\mathfrak{G}$, we start with the category $\mathfrak{M} \subset \mathrm{Mon}$ whose objects are finitely generated free monoids[7] $\langle n \rangle$, one for each $n \geq 0$. In this case, the abelianization functor reads

$$\underline{\mathbb{N}} : \mathfrak{M} \to \mathrm{Set}\,, \qquad \langle n \rangle \mapsto \mathbb{N}^n$$

where $\mathbb{N}$ is the set of natural numbers, i.e. the underlying set of the free abelian monoid of rank one. The associated category of elements $\mathfrak{M}_{\mathbb{N}} := \mathfrak{M} \smallint \underline{\mathbb{N}}$ has an explicit description similar to that of $\mathfrak{G}_{\mathbb{Z}}$: its objects are $(\langle n \rangle; k_1, \ldots, k_n))$, where $\langle n \rangle$ is the free monoid on n generators and $(k_1, \ldots, k_n) \in \mathbb{N}^n$. Any simplicial monoid M gives a functor $M : \mathfrak{M}^{\mathrm{op}} \to \mathtt{sSet}$ that restricts to $\mathfrak{M}_{\mathbb{N}}^{\mathrm{op}}$ via the canonical projection $p : \mathfrak{M}_{\mathbb{N}} \to \mathfrak{M}$. The analogue (generalization) of Theorem 4.2 says:

Proposition 4.11. *For any simplicial monoid M, there is a weak equivalence in* Top_**:*

$$|\mathrm{hocolim}_{\mathfrak{M}_{\mathbb{N}}^{\mathrm{op}}}(p^* M)| \simeq \Omega\,\mathrm{SP}^{\infty}(BM)\,, \tag{4.45}$$

where BM is the classifying space of M.

Proof. The same argument as in the proof of Theorem 4.2 — based on Proposition 2.6 — shows

$$\mathrm{hocolim}_{\mathfrak{M}_{\mathbb{N}}^{\mathrm{op}}}(p^* M) \simeq \boldsymbol{L}(M)_{\mathrm{ab}}$$

where $\boldsymbol{L}(-)_{\mathrm{ab}}$ denotes the derived abelianization functor on simplicial monoids. To compute this last functor, instead of Kan loop group, we will use the 2-sided (simplicial) bar resolution (5.22): $B_*(\underline{\mathcal{C}}_1, \underline{\mathcal{C}}_1, M) \xrightarrow{\sim} M$ in $\mathtt{sSet}_*$, where $\underline{\mathcal{C}}_1$ is the monad associated to the (simplicial analogue of) little 1-cube operad (see (5.24)). Since $(\underline{\mathcal{C}}_1(X))_{\mathrm{ab}} = \underline{\mathcal{C}}_0(X)$, we have

$$|\boldsymbol{L}(M)_{\mathrm{ab}}| \simeq |B_*(\underline{\mathcal{C}}_1, \underline{\mathcal{C}}_1, M)_{\mathrm{ab}}| \simeq |B_*(\underline{\mathcal{C}}_0, \underline{\mathcal{C}}_1, M)| \simeq \Omega\,\mathrm{SP}^{\infty}(BM)\,,$$

where the last equivalence is a result of Lemma 5.4 below (see (5.27)). □

The relation between monoids and groups is determined by the canonical (group completion) functor $l : \mathfrak{M} \to \mathfrak{G}$. This last functor extends naturally to a functor $\tilde{l} : \mathfrak{M}_{\mathbb{N}} \to \mathfrak{G}_{\mathbb{Z}}$, and the maps $\Psi_{\mathrm{sym}} : \Delta S \to \mathfrak{G}^{\mathrm{op}}$ and $\tilde{\Psi}_{\mathrm{sym}} :\ \Delta S \to \mathfrak{G}_{\mathbb{Z}}^{\mathrm{op}}$ defined by (4.23) and (4.35) factor through l and $\tilde{l}$ respectively, giving the commutative diagram

$$\begin{array}{ccccccc} & & & & \mathfrak{M}_{\mathbb{N}}^{\mathrm{op}} & \xrightarrow{\tilde{l}} & \mathfrak{G}_{\mathbb{Z}}^{\mathrm{op}} \\ & & & \overset{\tilde{\Psi}_{\mathrm{sym}}}{\nearrow} & \downarrow p & & \downarrow p \\ \Delta C^{\mathrm{op}} & \xrightarrow{\iota} & \Delta S & \xrightarrow{\Psi_{\mathrm{sym}}} & \mathfrak{M}^{\mathrm{op}} & \xrightarrow{l} & \mathfrak{G}^{\mathrm{op}} \end{array} \tag{4.46}$$

As a consequence of Proposition 4.11, we get

Corollary 4.12. *For any homotopy simplicial group $\Gamma \in \mathrm{sGr}^h$, there is a weak equivalence*

$$|\mathrm{hocolim}_{\mathfrak{M}_{\mathbb{N}}^{\mathrm{op}}}(p^* l^* \Gamma)| \;\simeq\; \Omega\,\mathrm{SP}^{\infty}(B\Gamma)$$

Proof. Apply Proposition 4.11 to the simplicial group $\boldsymbol{L}K(\Gamma)$ viewed as a simplicial monoid. □

Remark 4.13. Corollary 4.12 can be also deduced from Theorem 4.2 if we notice that the natural map

$$\mathrm{hocolim}_{\mathfrak{M}_{\mathbb{N}}^{\mathrm{op}}}(p^* l^* \Gamma) \xrightarrow{\sim} \mathrm{hocolim}_{\mathfrak{G}_{\mathbb{Z}}^{\mathrm{op}}}(p^* \Gamma)$$

is a weak equivalence for any Γ. This last fact follows from Theorem 2.3, the assumptions of which hold thanks to the known properties of the group completion functor (*cf.* [14, Lemma 3.2]).

[7] Abusing notation, we will use the same symbols to denote the objects of $\mathfrak{M}$ and $\mathfrak{G}$.

5. Topological character maps via Goodwillie calculus and operads

In this section, we will describe the maps $\mathtt{CS}$ and $\mathtt{SR}$ explicitly in topological terms, using Goodwillie calculus and classical operads. The latter approach is based on ideas of Fiedorowicz [30] that were developed by Ault in [2]. The former is inspired by results of Biedermann and Dwyer that appeared in [18]. The interpretation in terms of Goodwillie derivatives leads to a natural nonlinear (polynomial) generalization of topological character maps that deserves a further study (see Section 5.4).

5.1. Goodwillie homotopy calculus.

Goodwillie calculus provides a universal approximation ("Taylor decomposition") of basic homotopy functors that arise in topology in terms of polynomial functors. This method — introduced by T. Goodwillie in the series of papers [34, 35, 36] — has been studied extensively in recent years and has found many interesting applications (see, e.g., the survey papers [1] and [45]).

Recall that by a homotopy functor we mean a functor on topological spaces that preserves weak homotopy equivalences. A homotopy functor $F : \mathrm{Top}_* \to \mathrm{Top}_*$ is called *n-excisive* (or *polynomial of degree $\le n$*) if it takes any strongly coCartesian $(n+1)$-dimensional cubical diagram in Top_* to a Cartesian diagram (*cf.* [1, Definition 1.1.2]). For $n = 0$, this simply means that F is homotopically constant: i.e. $F(X) \simeq F(*)$ for any $X \in \mathrm{Top}_*$. For $n = 1$, this is the usual Mayer-Vietoris property: a functor F is 1-excisive if and only if it maps homotopy pushout squares to homotopy pullback squares in Top_* (see [1, Example 1.1.4]). For $n > 1$, F enjoys a higher dimensional version of the Mayer-Vietoris property that reduces to the usual one inductively in n.

The main construction of Goodwillie calculus can be described as follows (*cf.* [36, Theorem 1.8]).

Theorem 5.1 (Goodwillie)**.** *For any homotopy functor $F : \mathrm{Top}_* \to \mathrm{Top}_*$ on pointed spaces, there exists a natural tower of functors (fibrations) under F:*

(5.1)
$$
\begin{array}{ccc}
 & & \vdots \\
 & & \downarrow{\scriptstyle p_3} \\
 & & P_2F(X) \\
 & {\scriptstyle \delta_2}\nearrow & \downarrow{\scriptstyle p_2} \\
 & & P_1F(X) \\
 & {\scriptstyle \delta_1}\nearrow & \downarrow{\scriptstyle p_1} \\
F(X) & \xrightarrow{\delta_0} & P_0F(X)
\end{array}
$$

satisfying the following properties: for all $n \ge 0$,

1. *$P_nF : \mathrm{Top}_* \to \mathrm{Top}_*$ is an n-excisive functor,*
2. *$\delta_n : F \to P_nF$ is the universal weak natural transformation to an n-excisive functor.*

The last property needs an explanation. By a *weak* natural transformation $\delta : F \to P$ one means a pair ('zig-zag') of natural transformations $F \xrightarrow{\delta'} G \xleftarrow{\delta''} P$, where δ'' is a natural weak equivalence, i.e. $\delta''_X : G(X) \xleftarrow{\sim} P(X)$ is a weak homotopy equivalence for all spaces $X \in \mathrm{Top}_*$. Note that if F and P are homotopy functors, a weak natural transformation $\delta : F \to P$ induces a well-defined natural tranformation between the corresponding functors on the homotopy category $\mathtt{Ho}(\mathrm{Top}_*)$. Property (2) of Theorem 5.1 then says that the weak natural transformation $\delta_n : F \to P_nF$ is homotopically initial among all natural transformations from F to n-excisive functors.

Given a homotopy functor $F : \mathrm{Top}_* \to \mathrm{Top}_*$, we define its *$n$-th layer* to be the homotopy fibre

$$D_nF(X) := \mathrm{hofib}\{P_nF(X) \xrightarrow{p_n} P_{n-1}F(X)\} \tag{5.2}$$

where p_n is the canonical projection at the n-th stage of the Goodwillie tower (5.1). A remarkable fact discovered in [36] (see [1, Example 1.2.4]) is that, all layers of a homotopy functor F are naturally infinite loop spaces. More precisely, for each $n \geqslant 0$, there is a spectrum $\partial_n F$ equipped with a (naïve) action of the symmetric group S_n such that

$$D_n F(X) \simeq \Omega^\infty(\partial_n F \wedge (\Sigma^\infty X)^{\wedge n})_{hS_n}\,, \tag{5.3}$$

where $(\Sigma^\infty,\, \Omega^\infty)$ are the suspension spectrum and the infinite loop space functors, respectively. The spectrum $\partial_n F$ is called the *n-th Goodwillie derivative of* F (at the basepoint $*$).

5.2. **The map** $\texttt{CS}$. Recall that, by Corollary 4.10, the derived character map $\mathrm{Tr}(\Gamma)_*$ is induced by the composition of natural maps in $\mathtt{Ho}(\mathrm{Top}_*)$:

$$ES^1 \times_{S^1} \mathcal{L}(X) \xrightarrow{\texttt{CS}_X} \Omega\Omega^\infty\Sigma^\infty(X) \xrightarrow{\texttt{SR}_X} \Omega\mathrm{SP}^\infty(X) \tag{5.4}$$

where $X = B\Gamma$. Since the classifying space functor on homotopy simplicial groups induces an equivalence $\mathtt{Ho}(\mathrm{sGr}^h) \cong \mathtt{Ho}(\mathrm{Top}_{0,*})$, the maps (5.4) are defined on (the homotopy types of) all pointed connected spaces. To analyze these maps we introduce the notation:

$$\Theta(X) := ES^1 \times_{S^1} \mathcal{L}(X) = ES^1 \times_{S^1} \mathrm{Map}(S^1, X)$$

and define $\bar{\Theta} : \mathrm{Top}_* \to \mathrm{Top}_*$ by

$$\bar{\Theta}(X) := \Theta(X)/\Theta(*) \cong ES^1 \times_{S^1} \mathcal{L}(X)/BS^1 \cong ES^1_+ \wedge_{S^1} \mathcal{L}(X) \tag{5.5}$$

Note that (5.5) is a *reduced* homotopy functor, so that $P_0\bar{\Theta}(X) \simeq \bar{\Theta}(*) = \{*\}$ and $P_1\bar{\Theta}(X) \cong D_1\bar{\Theta}(X)$ for any space $X \in \mathrm{Top}_*$ (see (5.2)).

The next proposition shows that the natural transformation $\texttt{CS}$ in (5.4), relating cyclic to symmetric homology, essentially coincides with the first Goodwillie layer of the functor (5.5). We deduce this from results of Carlsson and Cohen [21] by elaborating on a remark of Fiedorowicz [30].

Proposition 5.2. *The map* $\texttt{CS}$ *in* (5.4) *is represented by*

$$\begin{array}{ccccc}
ES^1 \times_{S^1} \mathcal{L}(X) & & \xrightarrow{\quad \texttt{CS}_X \quad} & & \Omega\,\Omega^\infty\Sigma^\infty(X) \\
\Vert & & & & \downarrow \wr \\
\Theta(X) & \xrightarrow{\ \mathrm{can}\ } & \bar{\Theta}(X) & \xrightarrow{\ \delta_{1,X}\ } & D_1\bar{\Theta}(X)
\end{array}$$

where the right vertical arrow is a natural weak equivalence and δ_1 *is the 1-st layer of the functor* (5.5)*.*

Proof. As noticed in [30, Remark 1.4], the map $\texttt{CS}_X$ factors in the homotopy category as

$$ES^1 \times_{S^1} \mathcal{L}(X) \xrightarrow{\mathrm{can}} ES^1_+ \wedge_{S^1} \mathcal{L}(X) \xrightarrow{f_X} \Omega\Omega^\infty\Sigma^\infty(X) \tag{5.6}$$

where f_X is a certain natural map constructed in [21]. We review the construction of f_X and compare it to a well-known general formula for the first Goodwillie layer of a reduced homotopy functor.

First, we recall a standard stabilization construction due to Waldhausen [72]. For a pointed space X, denote by $CX = X \wedge I$ and $\Sigma X = X \wedge S^1$ the reduced cone and the reduced suspension of X, respectively. The latter can be obtained by glueing two copies of the former along a common base which is identified with X: this yields the natural pushout square in Top_*

$$\begin{array}{ccc}
X & \longrightarrow & CX \\
\downarrow & & \downarrow j_0 \\
CX & \xrightarrow[j_1]{} & \Sigma X
\end{array} \tag{5.7}$$

Applying the given functor F to (5.7) and taking the homotopy pullback along the maps j_0 and j_1 induces a natural map

$$F(X) \to \mathrm{holim}\,[\,F(CX) \to F(\Sigma X) \leftarrow F(CX)\,] \tag{5.8}$$

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Since the functor F is homotopic and reduced, we have $F(CX) \simeq F(*) \simeq \{*\}$, which implies that the homotopy colimit in (5.8) is equivalent to $\Omega F(\Sigma X)$. In this way, we get a natural map $s : F(X) \to \Omega F(\Sigma X)$. This last map can be iterated any number of times:

$$s_n : \ F(X) \to \Omega^n F(\Sigma^n X) \ , \quad n \geq 0 \ , \tag{5.9}$$

and eventually stabilized, defining the map

$$s_\infty : \ F(X) \to \varinjlim_n \Omega^n F(\Sigma^n X) = \Omega^\infty F \Sigma^\infty(X) \tag{5.10}$$

In particular, (5.10) exists for our functor $F = \bar{\Theta}$, see (5.5).

Next, for each $n \geqslant 0$, define $\Sigma^n X \to \bar{\Theta}\Sigma^n(\Sigma X)$ to be the composition of the following natural maps

$$\Sigma^n X \xrightarrow{\varepsilon} \Omega\Sigma(\Sigma^n X) = \Omega(\Sigma^{n+1}X) \hookrightarrow \mathcal{L}(\Sigma^{n+1}X) \simeq ES^1 \times \mathcal{L}(\Sigma^{n+1}X) \twoheadrightarrow ES^1_+ \wedge_{S^1} \mathcal{L}(\Sigma^{n+1}X) = \bar{\Theta}\Sigma^n(\Sigma X)\,,$$

where $\varepsilon : \mathrm{Id} \to \Omega\Sigma$ is the adjunction unit of (Σ, Ω). Looping n times then yields an inductive system of maps

$$i_n : \Omega^n \Sigma^n X \to \Omega^n \bar{\Theta}\Sigma^n(\Sigma X) \ , \quad \forall\, n \geq 0 \ , \tag{5.11}$$

which, by [21, Lemma 4.1], induce in the limit a homotopy equivalence

$$i_\infty : \ \Omega^\infty\Sigma^\infty X \xrightarrow{\sim} \Omega^\infty \bar{\Theta}\Sigma^\infty(\Sigma X) \tag{5.12}$$

Finally, we note the following canonical identifications

$$\begin{aligned} \Omega^\infty\bar{\Theta}\Sigma^\infty(X) &:= \varinjlim_n \Omega^n\bar{\Theta}\Sigma^n(X) = \varinjlim_n \Omega^{n+1}\bar{\Theta}\Sigma^{n+1}(X) = \\ &= \varinjlim_n \Omega[\Omega^n\bar{\Theta}\Sigma^n(\Sigma X)] \cong \Omega \varinjlim_n [\Omega^n\bar{\Theta}\Sigma^n(\Sigma X)] = \Omega\,\Omega^\infty\bar{\Theta}\Sigma^\infty(\Sigma X) \end{aligned} \tag{5.13}$$

The Carlsson-Cohen map f_X that appears in (5.6) can now be represented by the zig-zag of natural transformations

$$\bar{\Theta}(X) \xrightarrow{s_\infty} \Omega^\infty\bar{\Theta}\Sigma^\infty(X) \overset{(5.13)}{\cong} \Omega\,\Omega^\infty\bar{\Theta}\Sigma^\infty(\Sigma X) \xleftarrow{\Omega i_\infty} \Omega\,\Omega^\infty\Sigma^\infty(X)\,, \tag{5.14}$$

where the leftmost arrow is the Waldhausen stabilization map (5.10) for $\bar{\Theta}$ and the rightmost arrow is a natural weak equivalence induced by (5.12). To complete the proof it remains to note that $P_1(F) \simeq \Omega^\infty F \Sigma^\infty$ for any reduced homotopy functor F, and the universal natural transformation $\delta_1 : F \to P_1 F = D_1 F$ coincides (up to homotopy) with the stabilization map (5.10) (see, e.g. [45, Example 5.3]). ☐

5.3. **The map $\mathtt{SR}$.** We now turn to the second map $\mathtt{SR}_X$ in (5.4) that relates symmetric homology to representation homology. In this section, we construct this map topologically by a method similar to that of Proposition 5.2; its relation to Goodwillie calculus will be discussed in Section 5.4. Our starting point is the well-known fact that the Dold-Thom functor $\mathrm{SP}^\infty : \mathrm{Top}_* \to \mathrm{Top}_*$ factors through the category of abelian topological monoids: in fact, $\mathrm{SP}^\infty(X)$ is the free abelian topological monoid generated by the space X (see, e.g., [53]). This implies that SP^∞ is a linear (i.e., 1-excisive) functor. The latter can be seen directly as follows. Consider the natural maps (5.9) for the functor $F = \mathrm{SP}^\infty$ constructed in the proof of Proposition 5.2:

$$s_n : \ \mathrm{SP}^\infty(X) \to \Omega^n \mathrm{SP}^\infty \Sigma^n(X) \ , \quad n \geq 0 \ , \tag{5.15}$$

The maps (5.15) are all weak equivalences, which follows immediately from the commutative diagrams

$$\begin{array}{ccc} \pi_i\,\mathrm{SP}^\infty(X) & \xrightarrow{\pi_i(s_n)} & \pi_i\,\Omega^n \mathrm{SP}^\infty\Sigma^n(X) \\ \wr\downarrow & & \downarrow\wr \\ \tilde{H}_i(X) & \xrightarrow{\sim} & \tilde{H}_{i+n}(\Sigma^n X) \end{array}$$

where the vertical arrows are isomorphisms by the Dold-Thom Theorem. Thus, in the limit, we get

$$s_\infty : \ \mathrm{SP}^\infty(X) \xrightarrow{\sim} \Omega^\infty\,\mathrm{SP}^\infty\,\Sigma^\infty(X) \ , \tag{5.16}$$

showing that $\mathrm{SP}^\infty \simeq P_1(\mathrm{SP}^\infty) \simeq D_1(\mathrm{SP}^\infty)$, whence the linearity of SP^∞.

On the other hand, for all $n \ge 0$, we have canonical maps $\Sigma^n X \to \mathrm{SP}^\infty(\Sigma^n X)$ inducing the Hurewicz homomorphisms, see (4.9). Applying loop functors to these maps yield an inductive system of maps

$$i_n : \Omega^n\Sigma^n(X) \to \Omega^n\mathrm{SP}^\infty\Sigma^n(X)\,, \quad n \ge 0\,, \tag{5.17}$$

which in the limit, induces

$$i_\infty : \Omega^\infty\Sigma^\infty(X) \to \Omega^\infty\mathrm{SP}^\infty\Sigma^\infty(X) \tag{5.18}$$

Unlike the analogous map (5.12) for the functor $\bar{\Theta}$, (5.18) is not a weak equivalence in general. Nevertheless, looping it once and combining with (5.16), we get the pair of natural transformations

$$\Omega\Omega^\infty\Sigma^\infty(X) \xrightarrow{\Omega i_\infty} \Omega\Omega^\infty\mathrm{SP}^\infty\Sigma^\infty(X) \xleftarrow{\Omega s_\infty} \Omega\mathrm{SP}^\infty(X) \tag{5.19}$$

where the rightmost one is a natural weak equivalence. Our goal is to prove the following analogue of Proposition 5.2.

Proposition 5.3. *The map* $\mathtt{SR}$ *is represented by the weak natural transformation* (5.19). *Thus, in the homotopy category,* $\mathtt{SR}_X$ *is equivalent to the map*

$$(\Omega s_\infty)^{-1}\,(\Omega i_\infty) : \Omega\Omega^\infty\Sigma^\infty(X) \to \Omega\mathrm{SP}^\infty(X) \tag{5.20}$$

which is the (*looped once*) *canonical natural transformation relating stable homotopy to* (*reduced*) *singular homology of pointed spaces.*

To prove this proposition we will reinterpret the map (5.20) in terms of (topological) operads. The standard reference for the background material that we need is [52] (for a brief introduction, see also [62, Chapter 12]). Recall that an *operad* $\mathcal{C}$ in Top_* is a collection of pointed spaces $\{\mathcal{C}(j)\}_{j\ge 0}$ with $\mathcal{C}(0) := \{*\}$ such that each $\mathcal{C}(j)$ carries a right S_j-action and there are composition laws $\mathcal{C}(k) \times \mathcal{C}(j_1) \times \ldots \times \mathcal{C}(j_k) \to \mathcal{C}(j_1 + \ldots + j_k)$ satisfying natural associativity and unitality conditions. If $\mathcal{C}$ is an operad, a $\mathcal{C}$*-space* is a pointed space X equipped with an action of $\mathcal{C}$, which is given by a sequence of S_j-equivariant maps $\theta_j : \mathcal{C}(j) \times X^j \to X$, with $\theta_0 : \mathcal{C}(0) \hookrightarrow X$ being the basepoint inclusion, that satisfy associativity and unitality conditions compatible with those of $\mathcal{C}$. Every operad $\mathcal{C}$ determines a monad $\underline{\mathcal{C}}$ on Top_* (i.e., a monoid with respect to '$\circ$' in the category of endofunctors $\mathrm{Top}_* \to \mathrm{Top}_*$) in such a way that the notion of a $\mathcal{C}$-space is equivalent to that of $\underline{\mathcal{C}}$-algebra. Explicitly, given an operad $\mathcal{C}$, the corresponding monad $\underline{\mathcal{C}} : \mathrm{Top}_* \to \mathrm{Top}_*$ is defined by

$$\underline{\mathcal{C}}(X) := \coprod_{j\ge 0} \left(\mathcal{C}(j) \times_{S_j} X^j\right) / \sim \tag{5.21}$$

where the equivalence relation is of the form

$$(c, x_1, \ldots, x_{i-1}, *, x_{i+1}, \ldots, x_j) \sim (\sigma_i(c), x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_j)$$

for certain natural maps $\sigma_i : \mathcal{C}(j) \to \mathcal{C}(j-1)$ (see [52, Construction 2.4]). A $\underline{\mathcal{C}}$*-algebra* is then defined to be a space $A \in \mathrm{Top}_*$ with an action map $\xi : \underline{\mathcal{C}}(A) \to A$ satisfying natural associativity and unitality conditions. Opposite to the notion of a $\underline{\mathcal{C}}$-algebra is that of a $\underline{\mathcal{C}}$*-functor*, which is a functor F on Top_* equipped a morphism $F \circ \underline{\mathcal{C}} \to F$ defining a right action of $\underline{\mathcal{C}}$ on F. Associated to a triple $(F, \underline{\mathcal{C}}, A)$, there is a *two-sided bar construction* $B(F, \underline{\mathcal{C}}, A)$ defined as the geometric realization of a simplicial space $B_*(F, \underline{\mathcal{C}}, A) \in s\mathrm{Top}_*$ with components

$$B_n(F, \underline{\mathcal{C}}, A) := F\underline{\mathcal{C}}^n(A)\,, \quad n \ge 0\,, \tag{5.22}$$

where the faces $d_i : B_n \to B_{n-1}$ and degeneracies $s_j : B_n \to B_{n+1}$ are determined by the structure maps of A and F (see [52, Construction 9.6]).

Now, our main examples will be the so-called *little cubes operads* $\{\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots\}$ originally introduced by Boardman and Vogt (see [52, Section 4]). The $\mathcal{C}_0$ and $\mathcal{C}_1$ are discrete operads[8] defined by $\mathcal{C}_0(j) := \{*\}$ and $\mathcal{C}_1(j) := S_j$ for all $j \ge 0$, with S_j-action being trivial in the former case and induced by multiplication in S_j

[8]These operads are denoted in [52] by $\mathfrak{N}$ and $\mathfrak{M}$, respectively.

in the latter. A $\mathcal{C}_0$-space is just an abelian monoid in Top_*, and the monad associated to $\mathcal{C}_0$ is precisely the Dold-Thom functor:

$$\underline{\mathcal{C}}_0(X) \cong \mathrm{SP}^\infty(X) \tag{5.23}$$

A $\mathcal{C}_1$-space is just a monoid in Top_* (i.e., an associative H-space with 1), and the monad associated to $\mathcal{C}_1$ yields the classical James functor:

$$\underline{\mathcal{C}}_1(X) \cong J(X) \tag{5.24}$$

where $J(X) = (\amalg_{n\geq 0} X^n)/\sim$ is the free topological monoid generated by X. For $n \geq 2$, the operad $\mathcal{C}_n$ is not discrete: for $j \geq 1$, the space $\mathcal{C}_n(j)$ can be represented by the j-tuples of 'little n-cubes' (i.e. linear embeddings $I^n \hookrightarrow I^n$ with parallel axes and disjoint interiors) with natural (permutation) S_j-action. Thus, for $n \geq 2$, each $\mathcal{C}_n(j)$ is homotopy equivalent to $\mathrm{conf}_j(\mathbb{R}^n)$, the configuration space of unordered j-tuples of points in $\mathbb{R}^n$ equipped with canonical free S_j-action. Natural inclusions of cubes $I^n \hookrightarrow I^{n+1}$ induce the embeddings of spaces $\mathcal{C}_n(j) \hookrightarrow \mathcal{C}_{n+1}(j)$, and hence the maps of operads $\mathcal{C}_n \hookrightarrow \mathcal{C}_{n+1}$ for all $n \geq 2$. This allows one to define the operad $\mathcal{C}_\infty := \varinjlim_n \mathcal{C}_n$. Since $\pi_i[\mathcal{C}_n(j)] \cong \pi_i[\mathrm{conf}_j(\mathbb{R}^n)] = 0$ for $i \leq n-2$, each component $\mathcal{C}_\infty(j)$ of $\mathcal{C}_\infty$ is contractible, and as the S_j-action on $\mathcal{C}_\infty(j)$ (induced from $\mathcal{C}_n(j)$) is free, $\mathcal{C}_\infty$ is an E_∞-operad. Finally, we recall May's Approximation Theorem (see [52, Theorem 2.7]) that asserts that the natural map of monads $\alpha_n : \underline{\mathcal{C}}_n(X) \to \underline{\mathcal{C}}_n \Omega^n\Sigma^n(X) \to \Omega^n\Sigma^n(X)$ gives a homotopy equivalence

$$\underline{\mathcal{C}}_n(X) \simeq \Omega^n\Sigma^n(X)\ , \quad \forall\, n = 1, 2, \ldots, \infty\,, \tag{5.25}$$

whenever X is connected.

We can now state the following result which is probably well known to experts.

Lemma 5.4 (*cf.* [30]). *For any topological monoid M, there are natural homotopy equivalences*

$$B(\underline{\mathcal{C}}_\infty, \underline{\mathcal{C}}_1, M) \simeq \Omega\,\Omega^\infty\Sigma^\infty(BM)\,, \tag{5.26}$$

$$B(\underline{\mathcal{C}}_0, \underline{\mathcal{C}}_1, M) \simeq \Omega\,\mathrm{SP}^\infty(BM)\,, \tag{5.27}$$

and the map (5.20) *for $X = BM$ is equivalent to the map*

$$B(\underline{\mathcal{C}}_\infty, \underline{\mathcal{C}}_1, M) \to B(\underline{\mathcal{C}}_0, \underline{\mathcal{C}}_1, M) \tag{5.28}$$

induced by the canonical (*unique*) *morphism of operads $\mathcal{C}_\infty \to \mathcal{C}_0$.*

Proof. The equivalence (5.26) was originally proved by Fiedorowicz (see [30, Proposition 1.7] and also [2, Lemma 39]); the proof of (5.27) is similar. We describe these equivalences in both cases. First,

$$B(\underline{\mathcal{C}}_\infty, \underline{\mathcal{C}}_1, M) \simeq B(\Omega^\infty\Sigma^\infty, \underline{\mathcal{C}}_1, M) \simeq$$
$$B(\Omega\,\Omega^\infty\Sigma^\infty\Sigma, \underline{\mathcal{C}}_1, M) \simeq \Omega\,\Omega^\infty\Sigma^\infty B(\Sigma, \underline{\mathcal{C}}_1, M) \simeq \Omega\,\Omega^\infty\Sigma^\infty(BM)\,,$$

where the first equivalence is induced by (5.25), the second is obvious, the third is a formal property of the bar construction (see [52, Lemma 9.7]), and the last one follows from a theorem of Fiedorowicz (see [29, Corollary 9.7]) that yields $B(\Sigma, \underline{\mathcal{C}}_1, M) \simeq BM$ for any topological monoid M. Similarly,

$$B(\underline{\mathcal{C}}_0, \underline{\mathcal{C}}_1, M) \cong B(\mathrm{SP}^\infty, \underline{\mathcal{C}}_1, M) \simeq B(\Omega\,\mathrm{SP}^\infty\Sigma, \underline{\mathcal{C}}_1, M) \simeq \Omega\,\mathrm{SP}^\infty B(\Sigma, \underline{\mathcal{C}}_1, M) \simeq \Omega\,\mathrm{SP}^\infty(BM)\,,$$

where the first identification follows from (5.23), the second is induced by the equivalence (5.15), which is a consequence of the Dold-Theorem, the third follows from [52, Lemma 9.7], and the last one is [29, Corollary 9.7]. The last statement of the lemma is now deduced by comparing the above equivalences with the construction of the map (5.20) given in the beginning of Section 5.3. □

Proof of Proposition 5.3. For any topological monoid M, consider the following diagram of spaces

$$\begin{array}{ccccc} |\mathrm{hocolim}_{\Delta S_+}(B_{\mathrm{sym}}M)| & \xrightarrow{f_\infty} & B(\underline{\mathcal{C}}_\infty, \underline{\mathcal{C}}_1, M) & \xrightarrow{(5.26)} & \Omega\,\Omega^\infty\Sigma^\infty(BM) \\ \downarrow{\scriptstyle \tilde{\Psi}^*_{\mathrm{sym}}} & & \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle (5.20)} \\ |\mathrm{hocolim}_{\mathfrak{M}^{\mathrm{op}}_{\mathbb{N}}}(p^*M)| & \xrightarrow{f_0} & B(\underline{\mathcal{C}}_0, \underline{\mathcal{C}}_1, M) & \xrightarrow{(5.27)} & \Omega\,\mathrm{SP}^\infty(BM) \end{array} \tag{5.29}$$

In this diagram all horizontal maps are natural weak equivalences: f_∞ is the equivalence constructed by Fiedorowicz in [30] (see [2, Theorem 38]), f_0 is the equivalence (4.45) of Proposition 4.11, and (5.26), (5.27) are the equivalences described in Lemma 5.4. The map $\tilde{\Psi}^*_{\mathrm{sym}}$ is induced by the functor $\tilde{\Psi}_{\mathrm{sym}}$ defined in (4.46). To prove the proposition we need to show that the diagram (5.29) commutes. By Lemma 5.4, we already know that the rightmost square of (5.29) commutes; thus it suffices to prove the commutativity of the leftmost square. For this, we shall describe the maps f_∞ and f_0 explicitly.

The map f_∞ is explicitly constructed in the proof of [2, Lemma 36]. As in *loc. cit.*, we let $\mathcal{N} : \mathrm{Top}_* \to \mathrm{Top}_*$ denote the functor defined as the coend

$$\mathcal{N}(X) := \int^{[n]\in\Delta S_+} N([n]\downarrow \Delta S_+) \times B_{\mathrm{sym}} J(X)[n]\ .$$

By [2, Lemma 36], there is an equivalence of functors $\Theta : \mathcal{N} \simeq \underline{\mathcal{C}}_\infty$, inducing an equivalence of bar constructions $B(\mathcal{N}, \underline{\mathcal{C}}_1, M) \simeq B(\underline{\mathcal{C}}_\infty, \underline{\mathcal{C}}_1, M)$. The identification $|\mathrm{hocolim}_{\Delta S_+}(B_{\mathrm{sym}} M)| \simeq B(\mathcal{N}, \underline{\mathcal{C}}_1, M)$ by [52, Lemma 9.7] then yields f_∞.

The map f_0 can be constructed in a similar way. Let $\mathcal{P} : \mathrm{Top}_* \to \mathrm{Top}_*$ denote the functor

$$\mathcal{P}(X) := \int^{(\langle n\rangle;k_1,\ldots,k_n)\in\mathfrak{M}_{\mathbb{N}}^{\mathrm{op}}} N((\langle n\rangle; k_1,\ldots,k_n)\downarrow \mathfrak{M}_{\mathbb{N}}^{\mathrm{op}}) \times p^* J(X)(\langle n\rangle; k_1,\ldots,k_n)\ .$$

Identifying $J(X)(\langle n\rangle) = \mathrm{Hom}_{\mathrm{Mon}}(\langle n\rangle, J(X))$ and recalling that $\underline{\mathcal{C}}_0(X) = \mathrm{SP}^\infty(X)$ is the abelianization of $J(X)$, we note that the map

$$\coprod N((\langle n\rangle; k_1,\ldots,k_n)\downarrow \mathfrak{M}_{\mathbb{N}}^{\mathrm{op}}) \times p^* J(X)(\langle n\rangle; k_1,\ldots,k_n) \to \underline{\mathcal{C}}_0(X) = \mathrm{SP}^\infty(X)\,, \qquad y\times\varphi \mapsto \varphi_{\mathrm{ab}}(k_1,\ldots,k_n)$$

descends to the coend to yield a natural equivalence

$$\Lambda : \mathcal{P}(X) \simeq \underline{\mathcal{C}}_0(X)\,,$$

which, in turn, yields an equivalence of bar constructions $B(\mathcal{P}, \underline{\mathcal{C}}_1, M) \simeq B(\underline{\mathcal{C}}_0, \underline{\mathcal{C}}_1, M)$. Composing this with the identification $|\mathrm{hocolim}_{\mathfrak{M}_{\mathbb{N}}^{\mathrm{op}}}(p^* M)| \simeq B(\mathcal{P}, \underline{\mathcal{C}}_1, M)$ by [52, Lemma 9.7] yields the map f_0.

It can be easily verified that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\Theta} & \underline{\mathcal{C}}_\infty \\ {\scriptstyle \tilde{\Psi}^*_{\mathrm{sym}}}\downarrow & & \downarrow{\scriptstyle \mathrm{can}} \\ \mathcal{P} & \xrightarrow{\Lambda} & \underline{\mathcal{C}}_0 \end{array}$$

It follows that the first square in the diagram (5.29) commutes. Finally, we note that in the case when $M = \Gamma$, a simplicial group the map $\tilde{\Psi}^*_{\mathrm{sym}}$ in the diagram (5.29) may be identified with the corresponding map in (4.40) by Corollary 4.12 (also see Remark 4.13). This completes the proof ofthe desired proposition. ☐

Corollary 5.5. *Let Γ be a (homotopy) simplicial group such that $X = B\Gamma$ has homotopy type of a simply-connected CW complex, which is of (locally) finite rational type. If k is a field of characteristic zero, then the map* $\mathtt{SR}_X$ *induces an isomorphism*

$$\mathrm{HS}_*(k[\Gamma]) \cong \mathrm{HR}_*(k[\Gamma])\,.$$

Proof. As mentioned above, the natural map $i_\infty : \Omega^\infty\Sigma^\infty(X) \to \mathrm{SP}^\infty(X)$ (defined by composing (5.18) with the inverse of (5.16) in $\mathtt{Ho}(\mathrm{Top}_*)$) is not an equivalence in general. However, it is known that for any connected CW complex X, this map induces an isomorphism of cohomology rings

$$i^*_\infty :\ H^*(\mathrm{SP}^\infty(X),\,k) \xrightarrow{\sim} H^*(\Omega^\infty\Sigma^\infty(X),\,k) \tag{5.30}$$

provided the coefficients are taken in a field k of characteristic zero (see, e.g., [22, Section 7.3]). Now, under our assumption on X, both $\mathrm{SP}^\infty(X)$ and $\Omega^\infty\Sigma^\infty(X)$ are simply-connected spaces of finite rational type. Hence, there is a natural ('Cotor') spectral sequence with E_2-term $E_2^{*,*}(Z) = \mathrm{Ext}^*_{H^*(Z,\,k)}(k,\,k)$ that converges to $H_*(\Omega Z,\,k)$ for any simply-connected space Z (see, e.g., [22, Section 5.5, (5.13)]). By naturality, the map (5.30) induces an isomorhism $E_2^{*,*}(\Omega^\infty\Sigma^\infty X) \xrightarrow{\sim} E_2^{*,*}(\mathrm{SP}^\infty X)$ of such spectral sequences for $Z = \Omega^\infty\Sigma^\infty(X)$ and $Z = \mathrm{SP}^\infty(X)$. This last isomorphism is compatible with the map Ωi_∞ :

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$H_*(\Omega\,\Omega^\infty\Sigma^\infty(X),\,k)\;\to\;H_*(\Omega\,\mathrm{SP}^\infty(X),\,k)$ which, by Proposition 5.3, coincides with $\mathtt{SR}_X:\mathrm{HS}_*(k[\Gamma])\to\mathrm{HR}_*(k[\Gamma])$ for $X=B\Gamma$. Thus, by Comparison Theorem for spectral sequences (see [73, Theorem 5.2.12]), we conclude that $\mathtt{SR}_X$ is an isomorphism. □

Remark 5.6. We expect that the result of Corollary 5.5 holds for *any* homotopy simplicial group Γ, including the usual (discrete) groups, for which $B\Gamma$ is a $K(1,\Gamma)$-space, i.e. certainly not simply connected.

5.4. **Polynomial extensions.** There is a natural way to describe and generalize the map $\mathtt{SR}$ via Goodwillie calculus. As we have seen above, the Dold-Thom functor SP^∞ is 1-excisive, hence there is a canonical (up to homotopy) natural transformation $\beta_1:\,P_1(\mathrm{Id})\to\mathrm{SP}^\infty$, where $P_1(\mathrm{Id})=D_1(\mathrm{Id})$ is the 1-st layer of the functor Id. The latter is known to be the stable homotopy functor $P_1(\mathrm{Id})\simeq\Omega^\infty\Sigma^\infty$ and $\beta_1\simeq i_\infty$. Thus $\mathtt{SR}\simeq\Omega\beta_1$. It turns out that the map β_1 can be extended naturally to higher layers — and in fact, to the entire Goodwillie tower of the functor Id. This is based on results of the paper [18] that compares the Goodwillie tower of the identity with the lower central series of the Kan loop group.

Recall that, for any connected space X, we can identify $\mathrm{SP}^\infty(X)\simeq B[\mathbb{A}(X)]$, where $\mathbb{A}(X):=\mathbb{G}(X)_{\mathrm{ab}}$ is the abelianization of the Kan loop group $\mathbb{G}(X)$ of (a reduced simplicial set representing) X, see (4.12). Now, instead of just abelianization, consider the lower central series of $\mathbb{G}(X)$:

$$\ldots\to\mathbb{G}(X)/\mathbb{G}_{n+1}(X)\;\to\;\mathbb{G}(X)/\mathbb{G}_n(X)\;\to\;\ldots\;\to\;\mathbb{G}(X)/\mathbb{G}_2(X)=\mathbb{A}(X)$$

where $\mathbb{G}_n(X)$ are the simplicial subgroups of $\mathbb{G}(X)$ defined inductively by

$$\mathbb{G}_1(X):=\mathbb{G}(X)\quad\text{and}\quad\mathbb{G}_{n+1}(X):=[\mathbb{G}(X),\mathbb{G}_n(X)]\,,\;n\geq 1\,.$$

It is shown in [18] that the functor $X\mapsto B[\mathbb{G}(X)/\mathbb{G}_{n+1}(X)]$ is n-excisive for each $n\geq 1$, and there exists a canonical (up to homotopy) morphism of towers

$$\begin{array}{ccccccc}\ldots\longrightarrow & P_n(\mathrm{Id})(X) & \longrightarrow & P_{n-1}(\mathrm{Id})(X) & \longrightarrow\ldots\longrightarrow & P_1(\mathrm{Id})(X)\\ & \downarrow{\scriptstyle\beta_n} & & \downarrow{\scriptstyle\beta_{n-1}} & \ldots & \downarrow{\scriptstyle\beta_1}\\ \ldots\longrightarrow & B[\mathbb{G}(X)/\mathbb{G}_{n+1}(X)] & \longrightarrow & B[\mathbb{G}(X)/\mathbb{G}_n(X)] & \longrightarrow\ldots\longrightarrow & B[\mathbb{A}(X)]\end{array}\tag{5.31}$$

where the rightmost vertical arrow is precisely the map $\beta_1:P_1(\mathrm{Id})\to\mathrm{SP}^\infty$. This morphism induces natural maps on the layers of the Goodwillie tower

$$\beta_n:\;D_n(\mathrm{Id})(X)\;\to\;B[\mathbb{G}_n(X)/\mathbb{G}_{n+1}(X)]\;,\;n\geq 1\;.\tag{5.32}$$

that we can describe in explicit terms. First of all, by a theorem of B. Johnson [41] (*cf.* [1, Example 1.2.5]), all Goodwillie derivatives of the identity functor are known: for $n\geq 1$, the spectrum $\partial_n(\mathrm{Id})$ is equivalent to a wedge of $(n-1)!$ copies of the $(1-n)$-sphere spectrum $\mathbb{S}^{1-n}=\Sigma^{1-n}(\mathbb{S}^0)$. Hence, by formula (5.3), we have

$$D_n(\mathrm{Id})(X)\;\simeq\;\Omega^\infty\Big(\bigvee_{(n-1)!}\Sigma^{1-n}\,(\Sigma^\infty X)^{\wedge n}\Big)_{hS_n}\tag{5.33}$$

On the other hand, the Kan simplicial group $\mathbb{G}(X)$ is (degreewise) free for any X. Hence, by classic PBW Theorem (see, e.g., [65, I.4.3]), for all $n\geq 1$, there are natural isomorphisms of simplicial abelian groups

$$\mathbb{G}_n(X)/\mathbb{G}_{n+1}(X)\;\cong\;\mathrm{Lie}_n(\mathbb{A}X)\,,\tag{5.34}$$

where Lie_n denotes (the simplicial extension of) the degree n graded component of the free graded Lie algebra functor $\mathrm{Lie}_*(A)=\bigoplus_{n\geq 1}\mathrm{Lie}_n(A)$ on abelian groups A. Thus, with identifications (5.33) and (5.34), the morphism of towers (5.31) (looped once) induces on layers natural maps

$$\mathtt{SR}_X^{(n)}:\;\Omega\,\Omega^\infty\Big(\bigvee_{(n-1)!}\Sigma^{1-n}\,(\Sigma^\infty X)^{\wedge n}\Big)_{hS_n}\;\to\;|\,\mathrm{Lie}_n(\mathbb{A}X)\,|\;,\quad n\geq 1\,.\tag{5.35}$$

These can be viewed as nonlinear (polynomial) extensions of our topological trace map $\mathtt{SR}$: in fact, for $n=1$, the map (5.35) coincides with (5.20) under identification (4.13):

$$\mathtt{SR}_X^{(1)}:\;\Omega\,\Omega^\infty\Sigma^\infty(X)\;\to\;|\mathbb{A}(X)|\;\simeq\;\Omega\,\mathrm{SP}^\infty(X)\,,$$

while for $n = 2$, it reads

$$\mathtt{SR}_X^{(2)} : \ \Omega\,\Omega^\infty \Sigma^{-1}(\Sigma^\infty X \wedge \Sigma^\infty X)_{h\mathbb{Z}_2} \ \to \ |\,\mathrm{Lie}_2(\mathbb{A}X)\,|$$

since the $\mathbb{Z}_2$-action on the spectrum $\partial_2(\mathrm{Id}) \simeq \mathbb{S}^{-1}$ is known to be trivial (see [1, Example 1.2.5]). It would be interesting to see whether the maps (5.35) for $n \geq 2$ can be naturally represented by homotopy colimits (similar to $\mathtt{SR}_{B\Gamma} \simeq \tilde{\Psi}^*_{\mathrm{sym}}$ for $n = 1$, see (4.40)), and, in particular, whether the induced maps $\mathtt{SR}^{(n)}_{B\Gamma,*}$ can be described in terms of functor homology (extending the result of Corollary 4.10). The existence of such a description might lead to an interesting link between Goodwillie calculus and homological algebra of polynomial functors (as developed recently in [26, 28, 27, 71]).

6. Stable character maps and derived Poisson brackets

In this section, we study the behavior of the derived character maps (1.7) in the limit as $n \to \infty$. We show that, on simply connected spaces, these maps stabilize, inducing an isomorphism between the graded symmetric algebra generated by the S^1-equivariant homology of the free loop space of $X = B\Gamma$ and the invariant part of the representation homology in the projective limit $\varprojlim \mathrm{HR}_*(\Gamma, \mathrm{GL}_n)^{\mathrm{GL}_n}$. This result is a topological counterpart of a stabilization theorem proved for representation homology of algebras in [10]. In case when X represents a closed manifold, so that its S^1-equivariant homology carries the Chas-Sullivan bracket, we show that the stable character map is an isomorphism of Lie algebras, where the Lie bracket on representation homology is induced by a natural derived Poisson structure on the Quillen model of X.

6.1. **Stabilization of derived character maps.** For this section, let k be a field of characteristic 0. The (homotopy) group homomorphism $\Gamma \to \{1\}$ (resp., $\{1\} \to \Gamma$) induces a morphism of cyclic modules $k[B^{\mathrm{cyc}}\,\Gamma] \to k[B^{\mathrm{cyc}}\,\{1\}] = k$ (resp., $k = k[B^{\mathrm{cyc}}\,\{1\}] \to k[B^{\mathrm{cyc}}\,\Gamma]$). In this way, the trivial cyclic module k is a direct summand of $k[B^{\mathrm{cyc}}\,\Gamma]$ yielding a direct sum decomposition

$$k[B^{\mathrm{cyc}}\,\Gamma] \ \cong \ k \oplus k[\overline{B^{\mathrm{cyc}}\,\Gamma}]\ .$$

The *reduced* cyclic homology $\overline{\mathrm{HC}}_*(k[\Gamma])$ is defined by

$$\overline{\mathrm{HC}}_*(k[\Gamma]) \ := \ \mathrm{Tor}^{\Delta C}_*(k[\overline{B^{\mathrm{cyc}}\,\Gamma}], k)\,,$$

so that

$$\mathrm{HC}_*(k[\Gamma]) \ \cong \ \mathrm{HC}_*(k) \oplus \overline{\mathrm{HC}}_*(k[\Gamma])\ .$$

On the other hand, the homomorphism of group schemes $\mathrm{GL_n} \hookrightarrow \mathrm{GL}_{n+1}$ (given by padding with 1 on the bottom right corner) induces a morphism of commutative Hopf algebras $\mathcal{O}(\mathrm{GL}_{n+1}) \to \mathcal{O}(\mathrm{GL}_n)$, and hence, a morphism of left $\mathfrak{G}$-modules $\underline{\mathcal{O}}(\mathrm{GL}_{n+1}) \to \underline{\mathcal{O}}(\mathrm{GL}_n)$. This induces a morphism on representation homologies

$$\mu_{n+1,n} :\ \mathrm{HR}_*(\Gamma, \mathrm{GL}_{n+1}) = \mathrm{Tor}^{\mathfrak{G}}_*(k[\Gamma], \underline{\mathcal{O}}(\mathrm{GL}_{n+1})) \to \mathrm{HR}_*(\Gamma, \mathrm{GL}_n) = \mathrm{Tor}^{\mathfrak{G}}_*(k[\Gamma], \underline{\mathcal{O}}(\mathrm{GL}_n))\ . \tag{6.1}$$

It is not difficult to verify that (6.1) restricts to a morphism on the invariant part of the representation homologies

$$\mu_{n+1,n}\ :\ \mathrm{HR}_*(\Gamma, \mathrm{GL}_{n+1})^{\mathrm{GL_{n+1}}} \to \mathrm{HR}_*(\Gamma, \mathrm{GL}_n)^{\mathrm{GL}_n}\ . \tag{6.2}$$

Lemma 6.1. *The following diagram commutes for all n :*

$$\begin{array}{ccc}
\overline{\mathrm{HC}}_*(k[\Gamma]) & \xrightarrow{\mathrm{Tr}_{n+1}(\Gamma)} & \mathrm{HR}_*(\Gamma, \mathrm{GL}_{n+1})^{\mathrm{GL}_{n+1}} \\
 & \searrow^{\mathrm{Tr}_n(\Gamma)} & \downarrow{\scriptstyle \mu_{n+1,n}} \\
 & & \mathrm{HR}_*(\Gamma, \mathrm{GL}_n)^{\mathrm{GL}_n}
\end{array}$$

Proof. Since any homotopy simplicial group is weakly equivalent to a cofibrant strict simplicial group, we may assume without loss of generality that Γ is a cofibrant strict simplicial group. Continuing to denote the

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map $\overline{k[\Gamma]} \otimes_{\Delta C} k \to k[\Gamma] \otimes_{\mathfrak{G}} \underline{\mathcal{O}}(\mathrm{GL}_n)$ induced by $\Delta_{\mathrm{GL}_n}\mathrm{tr}$ by $\mathrm{Tr}_n(\Gamma)$, we then need to verify that the following diagram commutes

$$
\begin{array}{ccc}
\overline{k[\Gamma]} \otimes_{\Delta C} k & \xrightarrow{\mathrm{Tr}_{n+1}(\Gamma)} & k[\Gamma] \otimes_{\mathfrak{G}} \underline{\mathcal{O}}(\mathrm{GL}_{n+1}) \\
& \searrow^{\mathrm{Tr}_n(\Gamma)} & \downarrow{\scriptstyle \mu_{n+1,n}} \\
& & k[\Gamma] \otimes_{\mathfrak{G}} \underline{\mathcal{O}}(\mathrm{GL}_n)
\end{array}
\tag{6.3}
$$

By (the proof of) [44, Theorem 4.1], $\mathrm{Tr}_n(\Gamma_m)$ is induced (in each simplicial degree m) by the composite map

$$\Gamma_m \xrightarrow{\rho_n} \mathrm{GL}_n(\mathcal{O}[\mathrm{Rep}_n(\Gamma_m)]) \hookrightarrow \mathbb{M}_n(\mathcal{O}[\mathrm{Rep}_n(\Gamma_m)]) \xrightarrow{\mathrm{Tr}} \mathcal{O}[\mathrm{Rep}_n(\Gamma_m)] \cong k[\Gamma_m] \otimes_{\mathfrak{G}} \underline{\mathcal{O}}(\mathrm{GL_n}) \, ,$$

where ρ_n denotes the universal n-dimensional representation. A similar argument shows that the following diagram commutes:

$$
\begin{array}{ccc}
\Gamma_m & \xrightarrow{\rho_{n+1}} & \mathrm{GL}_{n+1}(\mathcal{O}[\mathrm{Rep}_{n+1}(\Gamma_m)]) \\
{\scriptstyle \rho_n}\downarrow & & \downarrow{\scriptstyle \mu_{n+1,n}} \\
\mathrm{GL}_n(\mathcal{O}[\mathrm{Rep}_n(\Gamma_m)]) & \hookrightarrow & \mathrm{GL}_{n+1}(\mathcal{O}[\mathrm{Rep}_n(\Gamma_m)])
\end{array}
$$

Here, the lower horizontal arrow is given by padding by '1' on the bottom right. It follows that

$$\mathrm{Tr}_n(\Gamma_m)(\langle\gamma\rangle - 1) \,=\, \mu_{n+1,n} \circ \mathrm{Tr}_{n+1}(\Gamma_m)(\langle\gamma\rangle - 1)$$

for every conjugacy class $\langle\gamma\rangle$ in Γ_m. This shows commutativity of the diagram (6.3) in every simplicial degree, proving the desired lemma. □

By Lemma 6.1, the family of maps $\{\mathrm{Tr}_n(\Gamma)\}_{n\geqslant 1}$ yields a k-linear map

$$\mathrm{Tr}_\infty(\Gamma) : \overline{\mathrm{HC}}_*(k[\Gamma]) \to \mathrm{HR}_*(\Gamma, \mathrm{GL}_\infty)^{\mathrm{GL}_\infty} := \varprojlim_n \mathrm{HR}_*(\Gamma, \mathrm{GL}_n)^{\mathrm{GL}_n} \, , \tag{6.4}$$

where the inverse limit is taken along the maps (6.2). The map $\mathrm{Tr}_\infty(\Gamma)$, which we call the *stable character map*, induces a morphism of graded commutative k-algebras

$$\Lambda\mathrm{Tr}_\infty(\Gamma) \,:\, \Lambda_k[\overline{\mathrm{HC}}_*(k[\Gamma])] \to \mathrm{HR}_*(\Gamma, \mathrm{GL}_\infty)^{\mathrm{GL}_\infty} \, . \tag{6.5}$$

Next, recall that a simplicial group Γ is said to be a *simplicial group model* of a pointed, connected topological space X if Γ maps to X under (3.8), i.e. $|\bar{W}(\Gamma)|$ is weakly equivalent to X. In this case, it is well known that

$$\mathrm{HC}_*(k[\Gamma]) \,\cong\, \mathrm{H}_*^{S^1}(\mathcal{L}X; k) \tag{6.6}$$

where $\mathcal{L}X$ is the free loop space of X, and the representation homology $\mathrm{HR}_*(\Gamma, G)$, which is an invariant of (the homotopy type of) X by Lemma 3.7 is denoted by $\mathrm{HR}_*(X, G)$. The isomorphism (6.6) restricts to an isomorphism of graded k-modules

$$\overline{\mathrm{HC}}_*(k[\Gamma]) \,\cong\, \overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X; k) \, . \tag{6.7}$$

Here, $\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X; k)$ stands for the *reduced* S^1-equivariant homology of $\mathcal{L}X$, i.e.

$$\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}_X; k) \,:=\, \mathrm{Ker}[\,\pi_* : \mathrm{H}_*^{S^1}(\mathcal{L}X) \to \mathrm{H}_*^{S^1}(\mathrm{pt})\,] \, .$$

The map π_* is induced on S^1-equivariant homology by the map $\mathcal{L}X \to \mathrm{pt}$. The derived character map $\mathrm{Tr}_n(X) := \mathrm{Tr}_n(\Gamma)$ is thus morphism of graded k-vector spaces

$$\mathrm{Tr}_n(X) \,:\, \overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X; k) \to \mathrm{HR}_*(X, \mathrm{GL}_n)^{\mathrm{GL}_n} \, , \tag{6.8}$$

and the stable character map becomes

$$\mathrm{Tr}_\infty(X) \,:\, \overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X; k) \to \mathrm{HR}_*(X, \mathrm{GL}_\infty)^{\mathrm{GL}_\infty} \, . \tag{6.9}$$

The following theorem is the main result of this section.

Theorem 6.2. *Let X be a simply connected space of finite (rational) type. The stable character map* (6.9) *induces an isomorphism of graded commutative algebras*

$$\Lambda\mathrm{Tr}_\infty(X) \,:\, \Lambda_k[\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X;k)] \xrightarrow{\sim} \mathrm{HR}_*(X,\mathrm{GL}_\infty)^{\mathrm{GL}_\infty}\,.$$

If, moreover, X is a simply connected manifold of dimension d then $\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X;k)$ is equipped with the *Chas-Sullivan bracket* (also called the *string topology bracket*), a graded Lie bracket of (homological) degree $2-d$. This Lie bracket arises out of a derived Poisson structure (in the sense on [15, Section 3.1]) on an algebra weakly equivalent to $k[\Gamma]$. On the other hand, the representation homologies $\mathrm{HR}_*(X,\mathrm{GL}_n)^{\mathrm{GL}_n}$ are equipped with graded, $((2-d)$-shifted) Poisson structures arising from the Poincaré duality pairing on the cohomology of X. Passing to the inverse limit, one obtains a graded $((2-d)$-shifted) Poisson structure on $\mathrm{HR}_*(X,\mathrm{GL}_\infty)^{\mathrm{GL}_\infty}$. As an application of Theorem 6.2, we obtain the following corollary which allows us to express the Chas-Sullivan bracket in terms of a graded Poisson bracket.

Corollary 6.3. *The map*

$$\Lambda\mathrm{Tr}_\infty(X) \,:\, \Lambda_k[\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X;k)\,] \xrightarrow{\sim} \mathrm{HR}_*(X,\mathrm{GL}_\infty)^{\mathrm{GL}_\infty}$$

is an isomorphism of graded $(2-d)$-shifted Poisson algebras.

6.2. **Proofs of Theorem 6.2 and Corollary 6.3.** The shortest way to prove Theorem 6.2 and Corollary 6.3 is to apply the results of the paper [10] that deals with stabilization of representation homology and derived character maps for (augmented) associative algebras. These results being applicable in our case follows from Remark 3.16. In what follows we outline key steps and necessary modifications of the arguments of [10], leaving details for interested readers.

Sketch of proof of Theorem 6.2. Let L_X denote a (cofibrant) Quillen model of X. Since X is of finite rational type, L_X may be chosen to be semi-free, and finitely generated in each homological degree. By Remark 3.16, if suffices to prove the assertions of this theorem working with UL_X instead of $k[\Gamma]$. Further, since X is simply connected, the generators of L_X are in positive homological degree. Theorem 6.2 follows from (a minor modification of the proof of) [10, Theorem 7.8]. Indeed, since $R = UL_X$ is freely generated by finitely many generators in each homological degree, and since all its generators are in positive homological degree, the arguments of [10, Section 7.4] go through to show that for each $k > 0$, the map

$$\tilde{\mu}_{n+1,n} : R_{n+1}^{\mathrm{GL},\leqslant k} \to R_n^{\mathrm{GL},\leqslant k} \tag{6.10}$$

is an isomorphism for n sufficiently large (i.e., for all $n > N(k)$ for some $N(k)$ which possibly depends on k). Here R_n^{GL} is the representation DG algebra as in [10, formula (2.10)], whose homology is isomorphic to $\mathrm{HR}_*(R,n)^{\mathrm{GL}_n} \cong \mathrm{HR}_*(X,\mathrm{GL}_n)^{\mathrm{GL}_n}$ and $R_n^{\mathrm{GL},\leqslant k}$ stands for the (brutal) truncation of R_n^{GL} to homological degrees $\leqslant k$. The map (6.10) is defined as in [10, Section 4] (where it is denoted by $\mu_{n+1,n}$). On homologies, (6.10) induces the map $\mu_{n+1,n} : \mathrm{HR}_*(X,\mathrm{GL}_{n+1})^{\mathrm{GL}_{n+1}} \to \mathrm{HR}_*(X,\mathrm{GL}_n)^{\mathrm{GL}_n}$. As in the proof of [10, Theorem 7.8] (see also Proposition 7.5 of [10], which is the crux thereof), it then follows that the map

$$\Lambda\mathrm{Tr}_\infty(X) \,:\, \Lambda_k[\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X;k)] \to \mathrm{H}_*[R_\infty^{\mathrm{GL}}]$$

is an isomorphism of graded commutative algebras where $R_\infty^{\mathrm{GL}} = \varprojlim_n R_n^{\mathrm{GL}}$. The desired verification is thus complete once we check that $\mathrm{H}_*[R_\infty^{\mathrm{GL}}] \cong \varprojlim_n \mathrm{H}_*[R_n^{\mathrm{GL}}]$. By (6.10), the inverse system $\{R_n^{\mathrm{GL}}\}$ is Mittag-Leffler. (6.10) further implies that for each k, the inverse system $\{\mathrm{H}_{k+1}(R_n^{\mathrm{GL}})\}$ stabilizes, i.e. becomes constant for large n, and is thus Mittag-Leffler. It follows that $\lim_n^1 \mathrm{H}_{k+1}(R_n^{\mathrm{GL}}) = 0$. That $\mathrm{H}_*[R_\infty^{\mathrm{GL}}] \cong \varprojlim_n \mathrm{H}_*[R_n^{\mathrm{GL}}]$, as desired, then follows from [73, Theorem 3.5.8]. This outlines the proof of Theorem 6.2. □

Sketch of proof of Corollary 6.3. Moreover (see [15, Section 4.2] for example), L_X may be chosen so that its universal enveloping algebra UL_X is equipped with a derived Poisson structure inducing the Chas-Sullivan bracket on its (reduced) cyclic homology (which is isomorphic to $\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X;k)$). More precisely, L_X may be chosen to be Koszul dual to the (graded linear dual of) the Lambrechts-Stanley model of X (see [46]), which is equipped with a cyclic pairing. Now, if Γ is a simplicial group model of X, then $k[\Gamma]$ is weakly equivalent to UL_X. By Remark 3.16, if suffices to prove the assertions of this theorem working with UL_X instead of $k[\Gamma]$. In this setting, it follows immediately from [15, Theorem 5.1] (also see [6, Theorem 2] and *loc. cit.*, Theorem 3.1) that the cyclic pairing on (the graded linear dual of) the Lambrechts-Stanley model of X yields a graded ($(2-d)$-shifted) Poisson structure on $\mathrm{HR}_*(X, \mathrm{GL}_n)^{\mathrm{GL}_n}$ such that the derived character map $\mathrm{Tr}_n : \overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X; k) \to \mathrm{HR}_*(X, \mathrm{GL}_n)^{\mathrm{GL}_n}$ is a homomorphism of graded Lie algebras. Moreover, the maps $\mu_{n+1,n} : \mathrm{HR}_*(UL_X, n+1)^{\mathrm{GL}_{n+1}} \to \mathrm{HR}_*(UL_X, n)^{\mathrm{GL}_n}$ are easily seen to be homomorphisms of graded Poisson algebras in the setting of [15, Section 5]. Hence, $\mathrm{HR}_*(X, \mathrm{GL}_\infty)^{\mathrm{GL}} \cong \mathrm{HR}_*(UL_X, \infty)^{\mathrm{GL}}$ acquires the structure of a graded Poisson algebra. It follows that $\mathrm{Tr}_\infty(X) : \overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X; k) \to \mathrm{HR}_*(X, \mathrm{GL}_\infty)^{\mathrm{GL}_\infty}$ is a homomorphism of grade Lie algebras, which implies that $\Lambda\mathrm{Tr}_\infty(X) : \Lambda_k[\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X; k)] \to \mathrm{HR}_*(X, \mathrm{GL}_\infty)^{\mathrm{GL}_\infty}$ is a homomorphism of graded Poisson algebras, where the Poisson structure in the left-hand side is obtained by extending the Chas-Sullivan bracket using the Leibniz rule. That it is an *isomorphism* of graded Poisson algebras then follows from Theorem 6.2. □

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7 Nov 2023 14:19:46 PST
230102-Berest Version 2 - Submitted to Algebr. Geom. Topol.