ANNOTATED BIBLIOGRAPHY

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This annotated bibliography contains four sections. Section 1 contains an annotated bibliography of my work on representation homology and closely related topics from 2019 onwards. Section 2 contains an annotated bibliography of my work (from 2019 onwards) on derived Poisson structures and string topology. Section 3 contains a reference to a recent (submitted) preprint on spaces of quasi-invariants. Each section begins with a short preamble. Section 4 contains a list of references that includes papers of other authors, as well as papers of mine from *before 2019*, that are cited in this bibliography. These citations use *lettered* references, as opposed to *numbered* references, which are used in the first three sections.

1. Representation homology

There are several (equivalent) ways to define representation homology. Historically the first and (arguably) most appealing definition comes from derived algebraic geometry (see, e.g., [K, PT, PTVV, TV1, TV2]). Let G be an affine algebraic group defined over a field k. Given a pointed connected CW complex X, the classical representation scheme $\mathrm{Rep}_G[\pi_1(X)]$, parametrizing the k-linear representations of the fundamental group of X in G, has a natural derived extension given by a derived affine k-scheme $\mathrm{DRep}_G(X)$. The latter may be defined as the homotopy fibre of the derived mapping stack $\mathbf{Map}(X, BG) \to BG$ parametrizing the flat G-bundles on (unpointed) space X in the Toën-Vezzosi category of derived stacks (see [1, Appendix]). The structure sheaf of $\mathrm{DRep}_G(X)$ can be represented by a simplicial commutative k-algebra whose homotopy groups we denote by $\mathrm{HR}_*(X, G) := \pi_*\mathcal{O}[\mathrm{DRep}_G(X)]$ and call the *representation homology of X in G*. The $\mathrm{HR}_*(X, G)$ is a graded commutative k-algebra, which is naturally a homotopy invariant of X, with $\mathrm{HR}_0(X, G)$ being isomorphic to the affine coordinate ring of $\mathrm{Rep}_G[\pi_1(X)]$. The full representation homology $\mathrm{HR}_*(X, G)$ depends not only on $\pi_1(X)$ but the *entire* homotopy type of X: it thus provides an interesting — and quite nontrivial — refinement of the classical representations variety $\mathrm{Rep}_G[\pi_1(X)]$.

References

[1] *Vanishing theorems for representation homology and the derived cotangent complex*, Algebraic & Geometric Topology **19** (2019), no. 1, 281-339 (with Yu. Berest and W.-K. Yeung).

In this paper, we study the cotangent complex of the derived G-representation scheme $\mathrm{DRep}_G(X)$ of a pointed connected topological space X. We construct an (algebraic version of) unstable Adams spectral sequence relating the cotangent homology of $\mathrm{DRep}_G(X)$ to the representation homology $\mathrm{HR}_*(X, G) := \pi_*[\mathrm{DRep}_G(X)]$ and prove some vanishing theorems for groups and geometrically interesting spaces. Our examples include virtually free groups, Riemann surfaces, link complements in $\mathbb{R}^3$ and generalized lens spaces. In particular, for any f.g. virtually free group Γ, we show that $\mathrm{HR}_i(\mathrm{B}\Gamma, G) = 0$ for all $i > 0$. For a closed Riemann surface Σ_g of genus $g \geq 1$, we have $\mathrm{HR}_i(\Sigma_g, G) = 0$ for all $i > \dim G$. The *sharp* vanishing bounds for Σ_g actually depend on the genus:

Conjecture 1. If $g = 1$, then $\mathrm{HR}_i(\Sigma_g, G) = 0$ for $i > \mathrm{rank}\, G$, and if $g \geq 2$, then $\mathrm{HR}_i(\Sigma_g, G) = 0$ for $i > \dim\, \mathcal{Z}(G)$, where $\mathcal{Z}(G)$ is the center of G.

In particular, for $g \geq 2$ and G complex semisimple, we conjecture that $\mathrm{HR}_i(\Sigma_g, G) = 0$ for $i > 0$. We prove Conjecture 1 locally on the smooth locus of the representation scheme $\mathrm{Rep}_G[\pi_1(\Sigma_g)]$ in the case of complex connected reductive groups. One important consequence of our results is the existence of a well-defined K-theoretic virtual fundamental class for $\mathrm{DRep}_G(X)$ in the sense of Ciocan-Fontanine and Kapranov [CK]. We give a new "Tor formula" for this class in terms of functor homology. This paper ends with an appendix comparing our (relatively elementary) construction of the derived representation scheme $\mathrm{DRep}_G(X)$ in [2] with the Toën-Vezzosi construction $\mathbf{Map}(X, BG)$ of the derived mapping stack of flat G-bundles on X, as well as with Kapranov's original construction $\boldsymbol{R}\mathrm{Loc}_G(X)$ of the derived moduli space of G-local systems on X.

[2] *Representation homology of topological spaces*, International Mathematics Research Notices IMRN 2022, no. 6, 4093-4180 (with Yu. Berest and W.-K. Yeung).

Our starting point in this paper was the simple observation that $\mathrm{HR}_*(X,G)$ can be defined in an elementary way parallel to the usual (simplicial) definition of ordinary homology $\mathrm{H}_*(X;A)$ and the Loday-Pirashvili definition of higher Hochschild homology $\mathrm{HH}_*(X,R)$. Specifically, the coordinate ring $\mathcal{O}(G)$ of the algebraic group G can be viewed as a monoidal functor on the (skeletal) category $\mathfrak{G}$ of f.g. free groups: $\mathbb{F}_n \mapsto \mathcal{O}(G)^{\otimes n}$ that extends naturally to the category of all groups: $\mathsf{Gr} \to \mathsf{Comm}_k$. Combining this last functor with $\mathbb{G}X : \Delta^{\mathrm{op}} \to \mathsf{Gr}$, the classical (Kan) simplicial group model of the space X, we get a simplicial commutative algebra $\mathcal{O}(X,G) : \Delta^{\mathrm{op}} \to \mathsf{Comm}_k$ whose homotopy groups are precisely the representation homology: $\mathrm{HR}_*(X,G) \cong \pi_*\mathcal{O}(X,G)$.

Using our definition of representation homology, we constructed a natural spectral sequence:

$$E^2_{pq} = \mathrm{Tor}^{\mathfrak{G}}_p(\underline{\mathrm{H}}_q(\Omega X;k), \mathcal{O}(G)) \Rightarrow_p \mathrm{HR}_n(X,G)\,,$$

relating representation homology of a space X to the homology of its based loop space. Here, $\mathfrak{G}$ stands for the PROP of finitely generated free groups. This spectral sequence allows us to establish basic properties and compute representation homology in a number of interesting cases. For example, for $X = B\Gamma$, it collapses, giving an isomorphism: $\mathrm{HR}_*(\mathrm{B}\Gamma, G) \cong \mathrm{Tor}^{\mathfrak{G}}_*(k[\Gamma], \mathcal{O}(G))$, where the group algebra $k[\Gamma]$ and the Hopf algebra $\mathcal{O}(G)$ are viewed as monoidal functors $\mathfrak{G} \to \mathtt{Vect}_k$. This 'Tor' formula is remarkable for two reasons: first, it gives a natural interpretation of representation homology in terms of usual (abelian) homological algebra, placing it in one row with other classical invariants such as Hochschild and cyclic homology; second, it provides an efficient tool for computations.

Our elementary definition shows that representation homology may be thought of as a 'multiplicative version' of ordinary homology, where the commutative Hopf algebra $\mathcal{O}(G)$ plays the role of coefficients. In this regard, the $\mathrm{HR}_*(X,G)$ is analogous to the higher Hochschild homology, $\mathrm{HH}_*(X,R)$, which can be viewed as a homology of X with coefficients in a commutative algebra R (see [Pir]). The two homology theories are, in fact, closely related: we show that there is a natural isomorphism

$$\mathrm{HR}_*(\Sigma X_+, G) \cong \mathrm{HH}_*(X, \mathcal{O}(G))$$

for any space X. There is also an important difference: unlike $\mathrm{HH}_*(X,R)$, the $\mathrm{HR}_*(X,G)$ carries a natural algebraic G-action induced by the adjoint action of G. Examples show that this action depends on the space X in a nontrivial way, which makes representation homology a richer and 'more geometric' theory than Hochschild homology. As noted in [3], it turns out to be related to some of the deeper problems in Lie theory and representation theory.

We also compute $\mathrm{HR}_*(X,G)$ explicitly for basic spaces of interest in geometric topology: e.g., the Riemann surfaces, the link complements in $\mathbb{R}^3$ and S^3, and the classical lens spaces). For instance, the representation homology of the complement of (a tubular neighborhood of) a link L in $\mathbb{R}^3$ can be expressed as a Hochschild homology of a commutative algebra with coefficients in an asymmetric bimodule. If L is the closure of a braid β on n strands, we have

$$\mathrm{HR}_\bullet(\mathbb{R}^3 \setminus L, G) \cong \mathrm{HH}_\bullet(\mathcal{O}(G^n), \mathcal{O}(G^n)_\beta)\,,$$

where the left action of $\mathcal{O}(G^n)$ on the coefficient bimodule is by multiplication and the right action is multiplication after applying an automorphism determined by β.

[3] *Representation homology of simply connected spaces*, Journal of Topology **15** (2022), no. 2, 692-744 (with Yu. Berest and W.-K. Yeung).

In this paper, we use Quillen's rational homotopy theory [Qui69], to compute the representation homology of an arbitrary *simply connected space X over a field k of characteristic zero*. One of our main results is that if X is a simply connected pointed space of finite rational type with Sullivan model $\mathcal{A}_X$, then for any affine algebraic group G with Lie algebra $\mathfrak{g}$, there are isomorphisms

$$\mathrm{HR}_*(X,G) \cong \mathrm{H}^{-*}_{\mathrm{CE}}(\mathfrak{g}(\bar{\mathcal{A}}_X);k) \quad , \quad \mathrm{HR}_*(X,G)^G \cong \mathrm{H}^{-*}_{\mathrm{CE}}(\mathfrak{g}(\mathcal{A}_X),\mathfrak{g};k) \ ,$$

where $\mathrm{H}^{-*}_{\mathrm{CE}}$ denotes the Chevalley-Eilenberg cohomology of the current Lie algebra $\mathfrak{g}(\mathcal{A}_X) = \mathfrak{g}\otimes\mathcal{A}_X$. When X is simply connected (so that $\mathrm{HR}_0(X,G) = k$) and G is reductive over k, it is natural to treat $\mathrm{HR}_*(X,G)$ as an object of representation theory — or even classical invariant theory (in the spirit of [Weyl]) — and ask basic questions about the structure of $\mathrm{HR}_*(X,G)$ as a G-module and its subalgebra $\mathrm{HR}_*(X,G)^G$ of G-invariants. From this perspective, the first basic question is: *When is the algebra $\mathrm{HR}_*(X,G)^G$ free and (locally) finitely generated*? We address this question by constructing a natural map which we call the *Drinfeld homomorphism*:

$$\Psi_G(X):\ \Lambda_k\big[\oplus_{i=1}^{l}\overline{\mathrm{H}}_*^{S^1,(m_i)}(\mathcal{L}X;\,k)\big] \to \mathrm{HR}_*(X,G)^G$$

Here, $m_1,\ldots,m_l$ are the exponents of G and the $\overline{\mathrm{H}}_*^{S^1,(m)}(\mathcal{L}X;\,k)$ are the common eigenspaces of the Frobenius (power) operations on $\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X;k)$. The Drinfeld homomorphism relates the representation homology of a space X to the S^1-equivariant homology of its free loop space $\mathcal{L}X = \mathrm{Map}(S^1,X)$; it is a topological analog of the derived character map of Lie algebras studied in our earlier work [BFPRW]. The above question can then be made more specific: *For which spaces X and reductive groups G is $\Psi_G(X)$ an isomorphism?* We answer it our next main result:

Theorem 1.1. *Assume that the rational cohomology algebra $\mathrm{H}^*(X;\mathbb{Q})$ of X is either generated by one element (in any dimension) or* freely *generated by two elements: one in even and one in odd dimensions. Then $\Psi_G(X)$ is an isomorphism for X and any complex reductive group G.*

Theorem 1.1 can be viewed as a broad topological generalization of the Fishel-Grojnowski-Teleman Theorem [FGT] that settles the so-called Strong Macdonald Conjecture, a celebrated conjecture in representation theory proposed by I. Macdonald, B. Feigin and P. Hanlon in the early 80s (see [M, H1, H2]). To illustrate this result, consider the spaces X with $\mathrm{H}^*(X;\mathbb{Q}) \cong \mathbb{Q}[z]/(z^{r+1})$, where the generator z is in even dimension $d \geq 2$. (The most familiar examples of such X's are the even-dimensional spheres S^{2n} ($r = 1$, $d = 2n$) and the classical projective spaces: the complex ones, $\mathbb{CP}^r$ ($r \geq 1$, $d = 2$), the quaternionic $\mathbb{HP}^r$ ($r \geq 1$, $d = 4$) and the octonionic (Cayley) plane $\mathbb{OP}^2$ ($r = 2$, $d = 8$).) For such spaces, Theorem 1.1 gives

$$\mathrm{HR}_*(X,G)^G \cong \Lambda\,[\xi_1^{(i)},\,\xi_2^{(i)},\,\ldots,\,\xi_r^{(i)}\ :\ i = 1,2,\ldots,l]\,,$$

where the generators $\xi_j^{(i)}$ have degrees $\deg\,\xi_j^{(i)} = (d(r+1)-2)m_i + dj - 1$, with m_i being the exponents of the Lie group G. The original Strong Macdonald Conjecture corresponds to the special case $X = \mathbb{CP}^r$.

[4] *Symmetric homology and representation homology*, Transactions of the American Mathematical Society **376** (2023), no. 9, 6475-6496 (with Yu. Berest).

Symmetric homology (HS) is a natural generalization of cyclic homology (HC) in which symmetric groups replace the cyclic groups. Introduced by Z. Fiedorowicz [F] in the early 90s, this homology is known to be notoriously hard to compute even in most basic cases. The reason seems to be that, unlike cyclic or Hochschild, the HS theory is not easily accessible by algebraic methods and has to be approached topologically. Some 15 years ago, Ault and Fiedorowicz [AF] (see also [Au]) proposed a number of conjectures on symmetric homology of algebras, including the following main one on polynomial rings (see [AF, Conjecture 1]):

$$\mathrm{HS}_*(k[x_1,\ldots,x_n]) \cong \mathrm{H}_*\left(\prod_{i=1}^{n}\mathcal{C}_\infty(S^0)\times\prod_{i=2}^{n}\Omega^\infty\Sigma^\infty(S^{i-1})^{\binom{n}{i}};\ k\right) \tag{1.1}$$

where $\Omega^\infty\Sigma^\infty$ is the stable homotopy functor and $\mathcal{C}_\infty$ is the monad associated to the little ∞-cubes operad, both defined as functors on the category of based topological spaces Top_*. Somewhat surprisingly, for $n \geq 2$, this conjecture remained open (in fact, $\mathrm{HS}_*(k[x_1,\ldots,x_n])$ was not known!) even in the rational case: when k is a field of characteristic zero.

In this paper, we first observe that for any associative k-algebra A, there is a canonical *algebra* map $\mathtt{SR}_*(A) : \mathrm{HS}_*(A) \to \mathrm{HR}_*(A,k)$ induced by the derived character map $\mathrm{Tr}_*(A) : \mathrm{HC}_*(A) \to \mathrm{HR}_*(A,k)$ constructed in [BKR13]. We then show that when k is a field of characteristic zero, the map $\mathtt{SR}_*(A)$ is an isomorphism of graded commutative algebras

$$\mathrm{HS}_*(A) \cong \mathrm{HR}_*(A,k)\ . \tag{1.2}$$

The above result enables us to settle one of the conjectures of Ault and Fiedorowicz over fields of characteristic 0. It further allows us to translate known facts about representation homology to symmetric homology: we show that if $U\mathfrak{a}$ is the universal enveloping algebra of a Lie algebra $\mathfrak{a}$ defined over a field k of characteristic 0, then there is an isomorphism of graded commutative algebras

$$\mathrm{HS}_*(U\mathfrak{a}) \cong \mathrm{Sym}_k(\overline{\mathrm{H}}_{*+1}(\mathfrak{a};k))\,, \tag{1.3}$$

where $\overline{\mathrm{H}}_*(\mathfrak{a};k)$ is the reduced (Chevalley-Eilenberg) homology of $\mathfrak{a}$ with trivial coefficients and Sym stands for graded symmetric algebra. As a special case of (1.3), we obtain for V a finite dimensional k-vector space in homological degree 0,

$$\mathrm{HS}_*(\mathrm{Sym}_k V) \cong \mathrm{Sym}_k\left(\bigoplus_{i=1}^{\dim_k V}\wedge^i V[i-1]\right)\,, \tag{1.4}$$

from which Conjecture (1.1) (over fields of characteristic 0) follows without much difficulty.

[5] *Derived character maps of group representations*, to appear in Algebraic & Geometric Topology, `arXiv:2210.01304` (with Yu. Berest).

The cyclic homology of group algebras has a beautiful topological interpretation that goes back to the work of Burghelea, Goodwillie, Fiedorowicz and others (see, e.g., [L, Chapter 7]). Specifically, for Γ a (homotopy) simplicial group, there is a natural isomorphism

$$\mathrm{HC}_*(k[\Gamma]) \cong \mathrm{H}_*(ES^1\times_{S^1}\mathcal{L}(B\Gamma);k) \tag{1.5}$$

where the right hand side is the S^1-equivariant homology of the free loop space $\mathcal{L}(B\Gamma) := \mathrm{Map}(S^1, B\Gamma)$ of the classifying space of Γ. In fact, (1.5) is one of a list of several classical isomorphisms relating algebraic homotopy theories associated with crossed simplicial groups [FL] to stable homotopy theory. One of these isomorphisms is the isomorphism

$$\mathrm{HS}_*(k[\Gamma]) \cong \mathrm{H}_*(\Omega\,\Omega^\infty\Sigma^\infty(B\Gamma);k) \tag{1.6}$$

for symmetric homology originally discovered by Fiedorowicz [F] but proven more recently by Ault [Au]. The first main result of this paper adds representation homology to the list mentioned above.

To be precise, for any commutative ring k, let $\mathrm{HR}_*(k[\Gamma]) := \mathrm{HR}_*(B\Gamma, \mathbb{G}_m(k))$. We prove that for any (homotopy) simplicial group Γ, there is a natural isomorphism

$$\mathrm{HR}_*(k[\Gamma]) \cong \mathrm{H}_*(\Omega\,\mathrm{SP}^\infty(B\Gamma); k)\ . \tag{1.7}$$

Our other main result is a topological interpretation of the derived character maps

$$\mathrm{Tr}_* : \mathrm{HC}_*(k[\Gamma]) \to \mathrm{HR}_*(k[\Gamma]) \tag{1.8}$$

of one-dimensional representations of $k[\Gamma]$. We show that with the identifications (1.5) and (1.7), the derived character maps (1.8) are induced on homology by a natural transformation

$$\mathtt{CR}_{B\Gamma} : ES^1 \times_{S^1} \mathcal{L}(B\Gamma) \to \Omega\,\mathrm{SP}^\infty(B\Gamma)$$

which, in turn, factors (as a homotopy natural transformation) through the Carlsson-Cohen map $\mathtt{CS}_{B\Gamma} : ES^1 \times_{S^1} \mathcal{L}(B\Gamma) \to \Omega\,\Omega^\infty\Sigma^\infty(B\Gamma)$. The induced map $\mathtt{SR}_{B\Gamma} : \Omega\,\Omega^\infty\Sigma^\infty(B\Gamma) \to \Omega\,\mathrm{SP}^\infty(B\Gamma)$ is the (looped once) canonical natural transformation $\Omega^\infty\Sigma^\infty \to \mathrm{SP}^\infty$ relating stable homotopy to (reduced) singular homology of pointed spaces. This shows that the derived character map (1.8) factors through symmetric homology, and that the induced map $\mathtt{SR}_{B\Gamma,*} : \mathrm{HS}_*(k[\Gamma]) \to \mathrm{HR}_*(k[\Gamma])$ is determined by maps that are well-known in topology. Topological results then allow us to conclude that $\mathtt{SR}_{B\Gamma,*}$ is an isomorphism when $B\Gamma$ is simply connected. These results led to our subsequent work [4] summarized above, where the isomorphism between symmetric homology and one-dimensional representation homology (in characteristic 0) is established in full generality.

2. Derived Poisson structures

In [Cr], Crawley-Boevey proposed a notion of Poisson structure for associative algebras that agrees with the usual definition for commutative algebras and has nice categorical properties. The idea of [Cr] was to find the weakest structure on A that induces natural Poisson structures on the moduli spaces $\mathrm{Rep}_n(A)/\!/\mathrm{GL}(n)$ of finite dimensional semi-simple representations of A. It turns out that such a weak Poisson structure is given by a Lie bracket on $\mathrm{HC}_0(A) = A/[A, A]$ satisfying some extra conditions. It is thus called a H_0-Poisson structure in [Cr]. Derived Poisson structures, introduced in [BCER], are a homological generalization of Crawley-Boevey's Poisson structures. In particular, a derived Poisson structure on a (associative) DG algebra A induces a graded Lie bracket on $\mathrm{HC}_*(A)$ along with compatible graded Poisson structures (see [BCER]) on the representation homologies $\mathrm{HR}_*(A, n)^{\mathrm{GL}}$ (of the associative DG algebra A, see [BKR13]) for all n.

References

[6] *Cyclic pairings and derived Poisson structures*, New York Journal of Mathemetics **25** (2019), 1-44 (with Y. Zhang)

By a fundamental theorem of D. Quillen, there is a natural duality - an instance of general Koszul duality - between differential graded (DG) Lie algebras and DG cocommutative coalgebras defined over a field k of characteristic 0. A cyclic pairing (i.e., an inner product satisfying a natural cyclicity condition) on the cocommutative coalgebra gives rise to a derived Poisson structure on a universal enveloping algebra $U\mathfrak{a}$ of the Koszul dual (DG) Lie algebra $\mathfrak{a}$. This, in turn, yields a graded Lie bracket on the (reduced) cyclic homology $\overline{\mathrm{HC}}_*(U\mathfrak{a})$. A prototypical example is the graded Lie structure with the so-called *string topology bracket* on the (reduced) S^1-equivariant homology $\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X; \mathbb{Q})$ of the free loop space $\mathcal{L}X$ of a simply connected closed oriented manifold X, which arises in this manner from the Poincaré duality pairing on a suitable chain coalgebra of X (see [LS]).

In this paper, we study such derived Poisson structures on $U\mathfrak{a}$, and their relation to the classical Poisson structures on the derived moduli spaces $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})$ of $\mathfrak{a}$ in a finite dimensional reductive Lie algebra $\mathfrak{g}$. Recall that $\overline{\mathrm{HC}}_*(U\mathfrak{a})$ has a canonical direct sum decomposition

$$\overline{\mathrm{HC}}_*(U\mathfrak{a}) \cong \bigoplus_{p=1}^{\infty} \mathrm{HC}_*^{(p)}(\mathfrak{a})\,, \tag{2.1}$$

which is called the *Lie Hodge decomposition* (see [BFPRW, Ka]). The *Drinfeld trace* $\mathrm{Tr}_{\mathfrak{g}}(P,\mathfrak{a}) : \mathrm{HC}_*^{(p)}(\mathfrak{a}) \to \mathrm{HR}_*(\mathfrak{a},\mathfrak{g})$ associated with invariant polynomial $P \in \mathrm{Sym}^p(\mathfrak{g}^*)^{\mathrm{ad}\,\mathfrak{g}}$ of degree p is a certain derived character map with values in the representation homology of $\mathfrak{a}$ in $\mathfrak{g}$ (see [BFPRW]). Extending results obtained in [BRZ], we show that the Drinfeld trace maps intertwine the derived Poisson bracket with the classical Poisson bracket on the representation homology $\mathrm{HR}_*(\mathfrak{a},\mathfrak{g})$.

[7] *Hodge decomposition of string topology*, Forum of Mathematics, Sigma **9** (2021), Paper No. e33, 31pp (with Yu. Berest and Y. Zhang)

Let X be a simply connected closed oriented manifold, and let $\mathcal{L}X$ denote the free loop space. Chas and Sullivan [CS] showed that the (reduced) rational S^1-equivariant homology of $\mathcal{L}X$ carries a graded Lie algebra structure with the so called *string topology bracket*:

$$\{-,-\} : \overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X;\mathbb{Q}) \times \overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X;\mathbb{Q}) \to \overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X;\mathbb{Q})\ .$$

This bracket is intrinsically related to the geometry of $\mathcal{L}X$, and has many interesting properties that have been extensively studied in recent years.

On the other hand, there is a natural (Hodge) decomposition

$$\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X;\mathbb{Q}) \cong \bigoplus_{p=0}^{\infty} \overline{\mathrm{H}}_*^{S^1,(p)}(\mathcal{L}X;\mathbb{Q}) \tag{2.2}$$

where the direct summands are the common eigenspaces of the Frobenius (power) operations on $\overline{\mathrm{H}}_*^{S^1}(\mathcal{L}X;\mathbb{Q})$. In this paper, we show that the string topology bracket is compatible with the Frobenius operations. More precisely, we prove the following:

Theorem 2.1. *Assume that the manifold X is rationally elliptic as a topolgical space, i.e., $\sum_{i\ge 2}\dim \pi_i(X)\otimes\mathbb{Q} < \infty$. Then*

$$\{\overline{\mathrm{H}}_*^{S^1,(p)}(\mathcal{L}X;\mathbb{Q}), \overline{\mathrm{H}}_*^{S^1,(q)}(\mathcal{L}X;\mathbb{Q})\} \subseteq \overline{\mathrm{H}}_*^{S^1,(p+q-1)}(\mathcal{L}X;\mathbb{Q})\,, \qquad \forall p,q\ge 0\,, p+q\ge 1\ .$$

Theorem 2.1 settles a conjecture of our earlier work [BRZ], albeit under the additional assumption that X is rationally elliptic. This result appears to have been not anticipated in earlier literature in spite of the compatibility of the Hodge decomposition (2.2) with various natural operations, including string topology operations, having been widely studied [FT, FTV, G1, G2, HL, W].

We deduce Theorem 2.1 from an abstract algebraic result on the compatibility of derived Poisson structures on the universal enveloping algebra $U\mathfrak{a}$ of a Lie algebra $\mathfrak{a}$ with Lie-Hodge decomposition. Recall that any DG Lie algebra $\mathfrak{a}$ has a minimal model, which is given by an L_∞-structure on the homology $\mathrm{H}_*(\mathfrak{a})$ together with a L_∞-quasi-isomorphism $\mathfrak{a} \xrightarrow{\sim} \mathrm{H}_*(\mathfrak{a})$. Such a structure is unique upto L_∞-quasi-automorphism of $\mathrm{H}_*(\mathfrak{a})$. We denote this minimal model simply by $\mathrm{H}_*(\mathfrak{a})$.

Theorem 2.2. *Let $\mathfrak{a}$ be a nonegatively graded DG Lie algebra over a field k of characteristic* 0*. Assume that $\mathrm{H}_*(\mathfrak{a})$ is finite-dimensional and nilpotent as a L_∞-algebra. Further assume that $\mathfrak{a}$ has a Koszul dual DG cocommutative coalgebra C that is finite-dimensional. Then, the derived Poisson bracker associated with a(ny) cyclic pairing on C preserves the Lie-Hodge decomposition* (2.1)*, i.e.,*

$$\{\mathrm{HC}_*^{(p)}(\mathfrak{a}), \mathrm{HC}_*^{(q)}(\mathfrak{a})\} \subseteq \mathrm{HC}_*^{(p+q-2)}(\mathfrak{a})\,, \qquad \forall p,q\ge 1\ .$$

Besides Quillen models of rationally elliptic simply connected spaces, Theorem 2.2 applies to an ordinary finite-dimensional nilpotent Lie algebra $\mathfrak{a}$, with derived Poisson bracket coming from the natural pairing on the Chevalley-Eilenberg complex $\mathcal{C}_*(\mathfrak{a};k) = \wedge^*\mathfrak{a}$.

3. Spaces of quasi-invariants of compact Lie groups

Fix a finite Coxeter group W acting in its reflection representation V. Denote by $\mathcal{A} := \{H\}$ the set of reflection hyperplanes of W in V and write $s_H \in W$ for the reflection operator in H. The group W acts naturally on the polynomial algebra $\mathbb{C}[V]$. Note that the invariant polynomials $p \in \mathbb{C}[V]^W$ are determined by the equations

$$s_H(p) = p\ ,\ \ \forall\, H \in \mathcal{A}\ . \tag{3.1}$$

To define quasi-invariants we modify ('weaken') the equations (3.1) in the following way. For each reflection hyperplane $H \in \mathcal{A}$, we choose a linear form $\alpha_H \in V^*$ such that $H = \mathrm{Ker}(\alpha_H)$ and fix a non-negative integer $m_H \in \mathbb{Z}_+$, assuming that $m_{w(H)} = m_H$ for all $w \in W$. In other words, we choose a system of roots of W in V^*, which (abusing notation) we still denote by $\mathcal{A}$, and fix a W-invariant function $m : \mathcal{A} \to \mathbb{Z}_+$, $H \mapsto m_H$, the values of which will be called *multiplicities* of hyperplanes (or roots) in $\mathcal{A}$. Now, with these extra data in hand, we replace the polynomial equations (3.1) by the polynomial congruences

$$s_H(p)\ \equiv\ p\ \bmod \langle\alpha_H\rangle^{2m_H}\ ,\ \ \forall\, H \in \mathcal{A}\ , \tag{3.2}$$

where $\langle\alpha_H\rangle$ denotes the principal ideal in $\mathbb{C}[V]$ generated by the form α_H. Following [CV90], we call $Q_m(W)$ the algebra *W-quasi-invariant polynomials of multiplicity m*. Note that $Q_0(W) = \mathbb{C}[V]$, while for "$m = \infty$", we have $Q_\infty(W) = \varprojlim Q_m(W) = \mathbb{C}[V]^W$. In general, $\mathbb{C}[V]^W \subseteq Q_m(W) \subseteq \mathbb{C}[V]$: thus, for varying m, the quasi-invariants interpolate between the W-invariants and all polynomials. Despite its simple definition, the algebras $Q_m(W)$ have a complicated structure: they do not seem to admit a good combinatorial description, nor do they have a natural presentation in terms of generators and relations. Nevertheless, these algebras possess many remarkable properties, such as Gorenstein duality (see [EG02, BEG03, FV02]), and are closely related to some fundamental objects in representation theory, such as Dunkl operators and double affine Hecke algebras (see [BEG03, BC11]).

References

[8] *Topological realization of algebras of quasi-invariants, I*, Submitted preprint, 2023, `arXiv:2305.10604` (with Yu. Berest).

This is the first in a series of papers, whose goal is to give a topological realization of the algebras of quasi-invariants as (equivariant) cohomology rings of certain spaces naturally attached to compact connected Lie groups. Our main result can be viewed as a generalization of a well-known theorem of A. Borel [Bo53] that realizes the algebra of invariant polynomials of a Weyl group W as the cohomology ring of the classifying space BG of the associated Lie group G. As the algebras $Q_m(W)$ are defined over $\mathbb{C}$, we should clarify what we really mean by "topological realization". It is a fundamental consequence of Quillen's rational homotopy theory [Qui69] that every reduced, locally finite, graded commutative algebra A defined over a field k of characteristic zero is topologically realizable, i.e. $A \cong \mathrm{H}^*(X; k)$ for some (simply-connected) space X. When equipped with cohomological grading, the algebras $Q_m(W)$ have all the above-listed properties; hence, the natural question: For which values of m the $Q_m(W)$'s are realizable, has an immediate answer: for all m. A more interesting (and much less obvious) question is whether one can realize quasi-invariants topologically as a diagram of algebras $\{Q_m(W)\}$ (indexed by m) together with natural structure that these algebras carry (e.g., W-action). To make this precise, equip the set $\mathcal{M}(W)$ of multiplicities on $\mathcal{A}$ a natural partial order, letting $m' \geq m$ iff $m'_\alpha \geq m_\alpha$ for all $\alpha \in \mathcal{A}$. The algebras of W-quasi-invariants of varying multiplicities then form a contravariant diagram of shape $\mathcal{M}(W)$ (i.e., a functor on $\mathcal{M}(W)^{\mathrm{op}}$) that we simply depict as a filtration on $\mathbb{C}[V]$:

$$\mathbb{C}[V] = Q_0(W)\ \supseteq\ \ldots\ \supseteq\ Q_m(W)\ \supseteq\ Q_{m'}(W)\ \supseteq\ \ldots\ \supseteq\ \mathbb{C}[V]^W \tag{3.3}$$

The schemes $V_m(W) = \operatorname{Spec} Q_m(W)$ with natural projections $p_m : V_m(W) \to V/\!/W$ then form a covariant $\mathcal{M}(W)$-diagram (tower) over $V/\!/W$ that is dual to (3.3):

$$V = V_0(W) \to \ldots \to V_m(W) \xrightarrow{\pi_{m,m'}} V_{m'}(W) \to \ldots \to V/\!/W \,. \tag{3.4}$$

The morphisms $\pi_{m,m'}$ in (3.4) have interesting algebro-geometric properties (see [BEG03]). Axiomatizing these in homotopy theoretic terms, we can ask for a topological analog of the tower (3.4), where the schemes $V_m(W)$ are replaced by CW complexes $X_m(G,T)$. We refer to this problem as the **Realization Problem**.

In this paper, we solve our realization problem in the rank one case using the Ganea fiber-cofiber construction (see [Gan65]). It can be briefly described as follows. Starting with a (homotopy) fibration sequence (well-pointed) spaces: $F \xrightarrow{i} X \xrightarrow{p} B$, one can construct a new fibration sequence on the same base: $F_1 \xrightarrow{i_1} X_1 \xrightarrow{p_1} B$ by taking $X_1 := \mathrm{hocof}_*(i) \cong X \cup C_*(F)$ to be the homotopy cofibre of the fibre inclusion $i : F \to X$ and defining $F_1 := \mathrm{hofib}_*(p_1)$. The map $p_1 : X_1 \to B$ (called 'whisker map') is obtained by extending $p : X \to B$ to X_1 so that the cone $C_*(F)$ erected over X contracts to the basepoint of B. This homotopy-theoretic construction can be iterated *ad infinitum*, producing a tower of fibrations over B:

$$\begin{array}{ccccccc} F & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & \ldots \\ {\scriptstyle i}\downarrow & & {\scriptstyle i_1}\downarrow & & {\scriptstyle i_2}\downarrow & & \\ X & \xrightarrow{\pi_0} & X_1 & \xrightarrow{\pi_1} & X_2 & \longrightarrow & \ldots \\ {\scriptstyle p}\downarrow & & {\scriptstyle p_1}\downarrow & & {\scriptstyle p_2}\downarrow & & \\ B & = & B & = & B & = & \ldots \end{array} \tag{3.5}$$

Now, given a compact connected Lie group G with maximal torus T, we can apply the above construction to the fundamental Borel fibration sequence of classifying spaces

$$G/T \xrightarrow{i} BT \xrightarrow{p} BG \,, \tag{3.6}$$

Theorem 3.1. *For $G = SU(2)$, the diagram of spaces* (3.5) *obtained by the successive application of the fibre-cofibre construction to the fibration sequence* (3.6) *solves the Realization Problem. Moreover, the resulting spaces $X_m(G,T)$ are the unique, up to rational homotopy equivalence, solution to the Realization Problem.*

The spaces $X_m(G,T)$ in Theorem 3.1 can be naturally realized as Borel homotopy quotients of the spaces $F_m(G,T) := G/T * E_{m-1}G$ (equipped with the diagonal G-action). Here, $*$ stands for join and $E_{m-1}G$ is Milnor's model for the $(m-1)$-universal G-bundle. Hence, for $G = SU(2)$

$$\mathrm{H}^*_G(F_m(G,T);\mathbb{C}) \cong \mathrm{H}^*(X_m(G,T);\mathbb{C}) \cong Q_m(W) \,.$$

Here, $W = \mathbb{Z}/2\mathbb{Z}$ is the Weyl group of G. We call the spaces $F_m(G,T)$ the *m-quasi-flag manifolds* and their Borel homotopy quotients $X_m(G,T)$ the *spaces of m-quasi-invariants.* We further extend our construction of spaces quasi-invariants to a large class of finite loop spaces ΩB of homotopy type of S^3, originally introduced Rector [Rec71], called the *Rector spaces* (or fake Lie groups of type $SU(2)$). In addition, we compute the G-equivariant K-theory and G-equivariant (complex analytic) elliptic cohomology of the m-quasi-flag manifolds $F_m(G,T)$ (for $G = SU(2)$), identifying them with exponential quasi-invariants $\mathcal{Q}_m(W)$ (for $W = \mathbb{Z}/2\mathbb{Z}$) and elliptic quasi-invariants for $W = \mathbb{Z}/2\mathbb{Z}$ respectively. In the case of elliptic cohomology, we express the result in two ways: geometrically (as coherent sheaves on a given Tate elliptic curve E) and analytically (in terns of Θ-functions and q-difference equations). Finally, we study the cochain spectra $C^*(X_m(G,T),k)$ associated to the spaces of quasi-invariants and show that these are Gorenstein commutative ring spectra in the sense of Dwyer, Greenlees and Iyengar [DGI06].

4. External references

The following list of references includes papers of other authors, as well as papers of mine from before 2019, that are cited in this annotated bibliography.

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