Tier 1 Analysis Exam

August 2024

You may answer as many questions as you like and and each question is worth 10 points total. Show computations and justify your answers; a correct answer without a correct proof earns little credit. Write a solution of each problem on a separate page. Write the problem number on each page. At the end, assemble your solutions with the problems in increasing order. There are eight questions. You have 4 hours.

1. Let $a_n = \frac{1}{n(\log n)^{3/2}}$. Determine whether $\Sigma_{n=2}^{\infty} a_n$ is convergent or divergent and explain your answer.

2. Let A and B be two compact sets in $\mathbb{R}^n$. Define

$$A + B := \{a + b : a \in A, b \in B\}.$$

Prove that $A + B$ is also a compact set in $\mathbb{R}^n$.

3. Let

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2+xy}}, \text{ if } (x,y) \neq (0,0); \\ 0, \text{ if } (x,y) = (0,0). \end{cases}$$

Prove that f has a partial derivative in the x direction at $(0,0)$ and a partial derivative in the y direction at $(0,0)$, but f is not differentiable at $(0,0)$.

4. Recall that a collection $\mathcal{F}$ of functions from $\mathbb{R}$ to $\mathbb{R}$ is said to be *equicontinuous* at a point $x_0 \in \mathbb{R}$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ for all $f \in \mathcal{F}$ whenever $|x - x_0| < \delta$. Suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$ such that $(f_n(q))_{n=1}^{\infty}$converges at each point q of the set $\mathbb{Q}$ of rationals. If the set $\mathcal{F} = \{f_n\}_{n=1}^{\infty}$ is equicontinuous at each point of $\mathbb{R}$, show that the sequence $(f_n(a))_{n=1}^{\infty}$ is actually convergent at each point a of $\mathbb{R}$.

5. Let

$$f(x,y,z) = (x+z)^2 + \exp[x^2 - \cos y], \text{for } (x,y,z) \in \mathbb{R}^3.$$

Find the critial points of f and determine whether each is a local minimum, local maximum, or saddle point. (Recall that a *saddle point* is a critical point that is not a local extremum.)

6. Let f be a Riemann integrable function on $[0, 2\pi]$ satisfying $m \leq f \leq M$, where m and M are two constants. Prove that

$$\left| \int_0^{2\pi} f(x) \sin x \, dx \right| \leq 2\,(M - m)\,.$$

7. Suppose S is a bounded connected smooth surface in the upper half space $z > 0$ in $\mathbb{R}^3$ (with the standard Euclidean coordinates (x, y, z)), whose boundary is a simple closed smooth curve C lying in the xy-plane (that is, the $z = 0$ plane). Then C bounds a region D in the xy-plane, and D together with S bounds a solid region E in $\mathbb{R}^3$. Let $\mathbf{F}$ be a vector field defined by

$$\mathbf{F}(x, y, z) = (x + 2y - z^2)\mathbf{i} + (x - 2y + z^2)\mathbf{j} + (z + 1)\mathbf{k}.$$

If S is oriented with outward-pointing normal, show that

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

is equal to the area of D.

8. Let

$$A = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + \sin y = w^2 - \sin z \text{ and } z^2 e^x + y e^w = 0\}.$$

Use the implicit function theorem to show there exists an open neighborhood N of $(1, 0, 0, 1)$ in $\mathbb{R}^4$ such that $A \cap N$ is the graph of a smooth function. Specify which variable(s) among x, y, z, w is/are the independent variable(s) for this function. The other variable(s) among x, y, z, w is/are the dependent variable(s) for the function. The choice of independent variable(s) might not be unique. You only need to specify one choice that works and explain why it works.