TIER 1 ANALYSIS EXAM, JANUARY 2020

Write the solution to each of the following problems on a separate, clearly identified page. Each problem is graded on a scale of zero to ten.

Problem 1. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that $\lim_{n\to\infty} a_n = 0$. Show that there exist infinitely many $n \in \mathbb{N}$ with the following property:

$$a_m \le a_n \text{ for every } m \ge n.$$

Problem 2. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that $\lim_{n\to\infty} a_n = 0$ and

$$|a_n - a_{n+1}| \le \frac{1}{n^2} \text{ for every } n \in \mathbb{N}.$$

Prove that the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges.

Problem 3. Denote by X the collection of all sequences $x = \{x_n\}_{n\in\mathbb{N}}$ with the property that $x_n \in [0,1]$ for every $n \in \mathbb{N}$. Define a metric on X by

$$d(x,y) = \sup_{n\in\mathbb{N}} |x_n - y_n|, \quad x = \{x_n\}_{n\in\mathbb{N}}, y = \{y_n\}_{n\in\mathbb{N}} \in X.$$

Let $f : X \to \mathbb{R}$ be a uniformly continuous function. Show that f is bounded. (NOTE: Take for granted the fact that d is in fact a metric. The conclusion is not correct if f is just continuous.)

Problem 4. Define a sequence $\{a_n\}_{n\in\mathbb{N}}$ as follows:

$$a_1 = 1, a_2 = \sqrt{2}, a_3 = \sqrt{2\sqrt{3}}, \ldots, a_n = \sqrt{2\sqrt{3\sqrt{\cdots\sqrt{n}}}}, \quad n \ge 3.$$

Show that the sequence converges in $\mathbb{R}$.

Problem 5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function such that $f(0,0) = 0$. Show that the improper integral

$$\iint_{x^2+y^2\le 1} \frac{f(x,y)}{(x^2+y^2)^{4/3}}\, dxdy$$

converges, that is,

$$\lim_{\varepsilon\downarrow 0} \iint_{\varepsilon\le x^2+y^2\le 1} \frac{f(x,y)}{(x^2+y^2)^{4/3}}\, dxdy$$

exists.

Problem 6. Let $f : \mathbb{R} \to (0,+\infty)$ be a differentiable function such that $f'(x) > f(x)$ for every $x \in \mathbb{R}$.

(1) Show that there exists a constant $k > 0$ such that

$$\lim_{x\to\infty} f(x)e^{-kx} = +\infty. \tag{0.1}$$

(2) Find the least upper bound of the numbers k for which (0.1) can be proved.

Problem 7. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is uniformly continuous and $\lim_{x\to+\infty} f(x) = 2020$. Does the limit $\lim_{x\to+\infty} f'(x)$ necessarily exist? (NOTE: Prove if true, provide an example if false.)

Problem 8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x + 1) = f(x)$ for every $x \in \mathbb{R}$. Define functions $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, as follows:

$$f_1(x) = f(x), f_n(x) = \frac{1}{2}(f_{n-1}(x - 2^{-n}) + f_{n-1}(x + 2^{-n})), \quad x \in \mathbb{R}, n \geq 2.$$

Show that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on $\mathbb{R}$.

Problem 9. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable function. Suppose that the Jacobian determinant $\det Df(0, 0)$ is equal to zero. Show that for every $\varepsilon > 0$ there exist $M, \delta > 0$ with the following property:

> If B_r is the closed disk of radius $r < \delta$ centered at $(0, 0)$, then $f(B_r)$ is contained in a rectangle with sides Mr and εr.