TIER 1 ANALYSIS EXAM, JANUARY 2020

Write the solution to each of the following problems on a separate, clearly identified page. Each problem is graded on a scale of zero to ten.

Problem 1. Let \( \{a_n\}_{n \in \mathbb{N}} \) be a sequence of nonnegative real numbers such that \( \lim_{n \to \infty} a_n = 0 \). Show that there exist infinitely many \( n \in \mathbb{N} \) with the following property:

\[
a_m \leq a_n \text{ for every } m \geq n.
\]

Problem 2. Let \( \{a_n\}_{n \in \mathbb{N}} \) be a sequence of nonnegative real numbers such that \( \lim_{n \to \infty} a_n = 0 \) and

\[
|a_n - a_{n+1}| \leq \frac{1}{n^2} \text{ for every } n \in \mathbb{N}.
\]

Prove that the alternating series \( \sum_{n=1}^{\infty} (-1)^{n-1} a_n \) converges.

Problem 3. Denote by \( X \) the collection of all sequences \( x = \{x_n\}_{n \in \mathbb{N}} \) with the property that \( x_n \in [0,1] \) for every \( n \in \mathbb{N} \). Define a metric on \( X \) by

\[
d(x,y) = \sup_{n \in \mathbb{N}} |x_n - y_n|, \quad x = \{x_n\}_{n \in \mathbb{N}}, y = \{y_n\}_{n \in \mathbb{N}} \in X.
\]

Let \( f : X \to \mathbb{R} \) be a uniformly continuous function. Show that \( f \) is bounded. (NOTE: Take for granted the fact that \( d \) is in fact a metric. The conclusion is not correct if \( f \) is just continuous.)

Problem 4. Define a sequence \( \{a_n\}_{n \in \mathbb{N}} \) as follows:

\[
a_1 = 1, a_2 = \sqrt{2}, a_3 = \sqrt{2\sqrt{3}}, \ldots, a_n = \sqrt{2 \sqrt{3 \cdots \sqrt{n}}}, \quad n \geq 3.
\]

Show that the sequence converges in \( \mathbb{R} \).

Problem 5. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a differentiable function such that \( f(0,0) = 0 \). Show that the improper integral

\[
\int \int_{x^2 + y^2 \leq 1} \frac{f(x,y)}{(x^2 + y^2)^{4/3}} \, dx \, dy
\]

converges, that is,

\[
\lim_{\epsilon \downarrow 0} \int \int_{\epsilon \leq x^2 + y^2 \leq 1} \frac{f(x,y)}{(x^2 + y^2)^{4/3}} \, dx \, dy
\]

exists.

Problem 6. Let \( f : \mathbb{R} \to (0, +\infty) \) be a differentiable function such that \( f'(x) > f(x) \) for every \( x \in \mathbb{R} \).

1. Show that there exists a constant \( k > 0 \) such that

\[
\lim_{x \to \infty} f(x)e^{-kx} = +\infty.
\]

2. Find the least upper bound of the numbers \( k \) for which (0.1) can be proved.
Problem 7. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'$ is uniformly continuous and $\lim_{x \to +\infty} f(x) = 2020$. Does the limit $\lim_{x \to +\infty} f'(x)$ necessarily exist? (NOTE: Prove if true, provide an example if false.)

Problem 8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x + 1) = f(x)$ for every $x \in \mathbb{R}$. Define functions $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, as follows:

$$f_1(x) = f(x),
\quad f_n(x) = \frac{1}{2}(f_{n-1}(x - 2^{-n}) + f_{n-1}(x + 2^{-n})), \quad x \in \mathbb{R}, n \geq 2.$$

Show that the sequence $\{f_n\}_{n \in \mathbb{N}}$ converges uniformly on $\mathbb{R}$.

Problem 9. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable function. Suppose that the Jacobian determinant $\det Df(0,0)$ is equal to zero. Show that for every $\varepsilon > 0$ there exist $M, \delta > 0$ with the following property:

If $B_r$ is the closed disk of radius $r < \delta$ centered at $(0,0)$, then $f(B_r)$ is contained in a rectangle with sides $Mr$ and $\varepsilon r$. 