Tier I ANALYSIS EXAM

August 2019

Try to solve all 9 problems. They each count the same amount. Justify your answers.

1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   (a) Show that the function f has a directional derivative in the direction of any unit vector $\mathbf{v} \in \mathbb{R}^2$ at the origin.

   (b) Show that the function f is not continuous at the origin.

2. (a) Prove that if the infinite series

$$(*) \qquad \sum_{n=1}^{\infty} |a_{n+1} - a_n| \quad \text{converges for some sequence } \{a_n\} \subset \mathbb{R},$$

   then necessarily the sequence $\{a_n\}$ converges as well.

   (b) Give an example of a sequence $\{a_n\}$ such that $(*)$ holds while the series

$$\sum_{n=1}^{\infty} a_n \quad \text{diverges.}$$

3. Let $f : [0,1] \to \mathbb{R}$ be Riemann integrable and continuous at 0. Show that

$$\lim_{n\to\infty} \int_0^1 f(x^n)dx = f(0)\ .$$

4. Let

$$\mathbf{F} = \cos(y^2 + z^2)\mathbf{i} + \sin(z^2 + x^2)\mathbf{j} + e^{x^2+y^2}\mathbf{k}$$

   be a vector field on $\mathbb{R}^3$. Calculate $\int_S \mathbf{F} \cdot d\mathbf{S}$, where the surface S is defined by

$$x^2 + y^2 = e^z \cos z,\ 0 \leq z \leq \pi/2, \qquad \text{and oriented upward.}$$

5. For positive integers n and m suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and suppose $K \subset \mathbb{R}^n$ is compact. Give a proof that $f(K)$ is compact, that is, give a proof of the fact that the image of a compact set in $\mathbb{R}^n$ under a continuous map is compact.

6. Suppose that $f : (0, \infty) \to (0, \infty)$ is a differentiable and positive function. Show that for any constant $a > 1$, it must hold that

$$\liminf_{x \to \infty} \frac{f'(x)}{\big(f(x)\big)^a} \leq 0.$$

Hint: You might consider an argument that proceeds by contradiction.

7. Prove that the following series

$$\sum_{n=1}^{\infty} \frac{3n^2 + x^4 \cos(nx)}{n^4 + x^2}$$

converges to a continuous function $f : \mathbb{R} \to \mathbb{R}$.

8. Consider the two functions

$$F(x, y, z) := xe^{2y} + ye^z - ze^x$$

and

$$G(x, y, z) := \ln(1 + x + 2y + 3z) + \sin(2x - y + z).$$

(a) Argue that in a neighborhood of $(0, 0, 0)$, the set

$$\{(x, y, z) : \; F(x, y, z) = 0\} \cap \{(x, y, z) : \; G(x, y, z) = 0\}$$

can be represented as a continuously differentiable curve parametrized by x.

(b) Find a vector that is tangent to this curve at the origin.

9. Let $\{f_n\}$ be a monotone sequence of continuous functions on $[a, b]$, that is, $f_1(x) \leq f_2(x) \leq f_3(x) \leq \cdots$ for all $x \in [a, b]$. Suppose $\{f_n\}$ converges pointwise to a function f which is also continuous on $[a, b]$, as $n \to \infty$. Show that the convergence is uniform on $[a, b]$.