Analysis Tier I exam
August 2020

Instructions:
1. Be sure to fully justify all answers.
2. Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.
3. Please assemble your test with the problems in the proper order.
4. Each problem is worth 11 points.

Problem 1. Let $x_0 > 0$ be a fixed real number and consider the sequence

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{4}{x_n} \right), \text{ if } n = 0, 1, 2, 3, \ldots,$$

(a) Show that $x_{n+1} \geq 2$, if $n \geq 0$.
(b) Show that $x_{n+1} \leq x_n$, if $n \geq 1$.
(c) Show that $x = \lim_{n \to \infty} x_n$ exists.
(d) Find $x$.

Problem 2. Find the value of $\iint_E \mathbf{F} \cdot \mathbf{n} \, dS$ where $\mathbf{F}(x, y, z) = (yz^2, \sin x, x^2)$, $E$ is the upper half of the ellipsoid $\{x^2 + y^2 + 4z^2 = 1, 0 \leq z\}$, and $\mathbf{n}$ is the outward pointing unit normal vector on the ellipsoid.

Problem 3. Find the value of

$$\iint_D \frac{1}{4x + y} \exp \left( \frac{2x + y}{4x + y} \right) \, dxdy$$

where $D$ is the quadrilateral with vertices $(1, -2)$, $(1/2, -1)$, $(1, -3)$, $(2, -6)$.

Problem 4. Find the absolute minimum of the function $f : \mathbb{R}^4 \to \mathbb{R}$ given by

$$f(x, y, z, w) = x^2 y + y^2 z + z^2 w + w^2 x$$

on the set

$$S = \{(x, y, z, w) \in \mathbb{R}^4 : xyzw = 1 \text{ and } x > 0, y > 0, z > 0, w > 0\}.$$

Problem 5. Set $a_0 := 0$ and define for $k \geq 1$

$$a_k = \sqrt{1 + \frac{1}{2} + \ldots + \frac{1}{k}}.$$
Assume furthermore that $b_k$ is sequence of positive real numbers such that $\sum_{k=1}^{\infty} b_k^2 < \infty$, and that $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous, positive valued function so that

$$f(x) \leq b_k \quad \text{when} \quad a_{k-1} \leq |x| \leq a_k$$

for $k = 1, 2, 3, \ldots$. Show that the improper integral $\int_{\mathbb{R}^2} f(x) \, dx$ exists.

**Problem 6.** Let $f$ be a continuous function on $[0, 1]$ and twice differentiable on $(0, 1)$ such that $f(0) = f(1) = 0$ and $|f''(x)| < 2$ for all $x \in (0, 1)$.

(a) Show that $f(x) \geq x^2 - x$ for all $x \in [0, 1]$.

(b) Show that

$$\left| \int_0^1 f(x) \, dx \right| \leq \frac{1}{6}.$$

**Problem 7.** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable map (but not necessarily continuously differentiable) with component functions $f_1$ and $f_2$, that is $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$ for all $(x_1, x_2) \in \mathbb{R}^2$. Suppose that for all $(x_1, x_2) \in \mathbb{R}^2$, one has

$$\left| \frac{\partial f_1}{\partial x_1} (x_1, x_2) - 2 \right| + \left| \frac{\partial f_1}{\partial x_2} (x_1, x_2) \right| + \left| \frac{\partial f_2}{\partial x_1} (x_1, x_2) \right| + \left| \frac{\partial f_2}{\partial x_2} (x_1, x_2) - 2 \right| \leq \frac{1}{2}$$

Prove that $f$ is one-to-one\(^1\) on $\mathbb{R}^2$.

**Problem 8.** Let $I$ be the interval $[0, 1]$, and let $f : I \to \mathbb{R}$ be a continuous function such that

$$\int_I f(x)x^n \, dx = 0 \quad \text{for all} \quad n = 3, 4, 5 \ldots .$$

Show that $f(x) = 0$ for all $x \in I$.

**Problem 9.** Let $f_n : [0, 1] \to [0, 1]$ be a sequence of functions that converge uniformly to a limit function $f : [0, 1] \to [0, 1]$. Assume that each $f_n$ maps compact sets to compact sets. Is it true that $f$ also maps compact sets to compact sets? Note that we do not assume that the $f_n$ are continuous. Either give a proof, or provide a detailed counterexample.

\(^1\)i.e., injective