Analysis Tier I exam

August 2020

Instructions:

1. Be sure to fully justify all answers.

2. Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.

3. Please assemble your test with the problems in the proper order.

4. Each problem is worth 11 points.

Problem 1. *Let $x_0 > 0$ be a fixed real number and consider the sequence*

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{4}{x_n}\right) , \quad \textit{if} \;\; n = 0, 1, 2, 3, \ldots ,$$

(a) Show that $x_{n+1} \geq 2$, if $n \geq 0$.
(b) Show that $x_{n+1} \leq x_n$, if $n \geq 1$.
(c) Show that $x = \lim_{n\to\infty} x_n$ exists.
(d) Find x.

Problem 2. *Find the value of $\iint_E \mathbf{F} \cdot \mathbf{n} \; dS$ where $\mathbf{F}(x, y, z) = (yz^2, \sin x, x^2)$, E is the upper half of the ellipsoid $\{x^2 + y^2 + 4z^2 = 1,\ 0 \leq z\}$, and $\mathbf{n}$ is the outward pointing unit normal vector on the ellipsoid.*

Problem 3. *Find the value of*

$$\iint_D \frac{1}{4x + y} \exp\left(\frac{2x + y}{4x + y}\right) \; dxdy$$

where D is the quadrilateral with vertices $(1, -2)$, $(1/2, -1)$, $(1, -3)$, $(2, -6)$.

Problem 4. *Find the absolute minimum of the function $f : \mathbb{R}^4 \to \mathbb{R}$ given by*

$$f(x, y, z, w) = x^2y + y^2z + z^2w + w^2x$$

on the set

$$S = \{(x, y, z, w) \in \mathbb{R}^4 : xyzw = 1 \;\; \textit{and} \;\; x > 0,\ y > 0,\ z > 0,\ w > 0\} .$$

Problem 5. *Set $a_0 := 0$ and define for $k \geq 1$*

$$a_k = \sqrt{1 + \frac{1}{2} + \ldots + \frac{1}{k}} \; .$$

Assume furthermore that b_k *is sequence of positive real numbers such that* $\sum_{k=1}^{\infty} b_k^2 < \infty$, *and that* $f : \mathbb{R}^2 \to \mathbb{R}$ *is a continuous, positive valued function so that*

$$f(x) \le b_k \quad \text{when} \quad a_{k-1} \le |x| \le a_k$$

for $k = 1, 2, 3, \ldots$. *Show that the improper integral* $\int_{\mathbb{R}^2} f(x)\, dx$ *exists.*

Problem 6. *Let* f *be a continuous function on* $[0, 1]$ *and twice differentiable on* $(0, 1)$ *such that* $f(0) = f(1) = 0$ *and* $|f''(x)| < 2$ *for all* $x \in (0, 1)$.

(a) Show that $f(x) \ge x^2 - x$ *for all* $x \in [0, 1]$.

(b) Show that

$$\left| \int_0^1 f(x)\ dx \right| \le \frac{1}{6}\ .$$

Problem 7. *Let* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be a differentiable map (but not necessarily continuously differentiable) with component functions* f_1 *and* f_2, *that is* $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$ *for all* $(x_1, x_2) \in \mathbb{R}^2$. *Suppose that for all* $(x_1, x_2) \in \mathbb{R}^2$, *one has*

$$\left| \frac{\partial f_1}{\partial x_1}(x_1, x_2) - 2 \right| + \left| \frac{\partial f_1}{\partial x_2}(x_1, x_2) \right| + \left| \frac{\partial f_2}{\partial x_1}(x_1, x_2) \right| + \left| \frac{\partial f_2}{\partial x_2}(x_1, x_2) - 2 \right| \le \frac{1}{2}$$

Prove that f *is one-to-one*[1] *on* $\mathbb{R}^2$.

Problem 8. *Let* I *be the interval* $[0, 1]$, *and let* $f : I \to \mathbb{R}$ *be a continuous function such that*

$$\int_I f(x) x^n\ dx = 0 \text{ for all } n = 3, 4, 5 \ldots .$$

Show that $f(x) = 0$ *for all* $x \in I$.

Problem 9. *Let* $f_n : [0, 1] \to [0, 1]$ *be a sequence of functions that converge uniformly to a limit function* $f : [0, 1] \to [0, 1]$. *Assume that each* f_n *maps compact sets to compact sets. Is it true that* f *also maps compact sets to compact sets? Note that we do not assume that the* f_n *are continuous. Either give a proof, or provide a detailed counterexample.*

[1] i.e., injective