1. Let \( \{f_n\} \) be a sequence of nonnegative, continuous, real-valued functions on \([0, 1]\) with the property that \( f_n(x) \leq f_{n+1}(x) \) for all \( x \in [0, 1] \), and \( n \in \mathbb{N} \). Assume that \( \{f_n\} \) converges uniformly on \([0, 1]\) to a function \( f \). Show that
\[
\lim_{n \to \infty} \int_0^1 \left( \sum_{k=1}^n (f_k(x))^n \right)^{1/n} dx = \int_0^1 f(x) dx.
\]

2. Consider the infinite series
\[
\sum_{a,b \geq 0} \frac{1}{p^a q^b} = 1 + \frac{1}{p + 1} + \frac{1}{q + 1} + \frac{1}{p^2} + \frac{1}{pq} + \frac{1}{q^2} + \cdots
\]
where \( p, q \) are distinct primes and the terms are reciprocals of positive integers that are products of powers of \( p \) and powers of \( q \). Thus in the sum \( a, b \) range over all nonnegative integers. Prove that the series converges and find the sum of the series.

3. Let \( \{f_n\} \) be a sequence of continuous, real-valued functions on \([0, 1]\) with the property that for some function \( f \) on \([0, 1]\),
\[
\lim_{n \to \infty} f_n(x_n) = f(x)
\]
for each sequence of points \( \{x_n\} \subset [0, 1] \) with \( \lim_{n \to \infty} x_n = x \), and all \( x \in [0, 1] \). Prove or give a counterexample: \( \{f_n\} \) converges uniformly to \( f \) on \([0, 1]\).
4. Does there exist a sequence \( \{ f_n \} \) of continuously differentiable functions on \( \mathbb{R} \) that converges uniformly to a limit function \( f \) that is not differentiable at 0? Either give an example with full explanations or show that such a sequence cannot exist.

5. Show that
\[
\lim_{n \to \infty} \left( \frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n} - \frac{2}{3} \sqrt{n} \right) = 0.
\]

6. Let \( C \) be a simple closed curve that lies in the plane \( x + y + z = 1 \). Show that the line integral
\[
\int_C z \, dx - 2xy \, dy + 3y \, dz
\]
only depends on the area of the region enclosed by \( C \) and not on the shape of \( C \) or its position in the plane.

7. For a point \( x = (x_1, x_2, \ldots, x_n) \) in the unit cube \([0,1]^n\), let \( A_n(x) = \frac{\sum_{i=1}^{n} x_i}{n} \) be the average value of its coordinates.
   (a) Show that for any \( \delta \in (0, 1) \),
   \[
   \delta^2 \int_{J_\delta} dx_1 \cdots dx_n \leq \frac{1}{12n}
   \]
   where \( J_\delta = \{ x \in [0,1]^n : |A_n(x) - \frac{1}{2}| > \delta \} \).
   (b) Show that for any continuous function \( f \) on the interval \([0,1]\),
   \[
   \lim_{n \to \infty} \int_{[0,1]^n} f(A_n(x)) \, dx_1 \cdots dx_n = f\left(\frac{1}{2}\right).
   \]
   You may use part (a).

8. Consider the function \( f : \mathbb{R}^3 \to \mathbb{R} \) defined by
\[
f(x, y, z) = x^2 y + e^x + z.
\]
   (a) Show that there exists a differentiable function \( \phi \) defined in a neighborhood \( U \) of \((1,-1)\) in \( \mathbb{R}^2 \) such that \( \phi(1,-1) = 0 \) and \( f(\phi(y,z),y,z) = 0 \) for all \((y,z) \in U\).
   (b) Find the values of the gradient \( \nabla \phi(1,-1) \).
9. For $n \geq 2$, let $p : \mathbb{R}^n \to \mathbb{R}$ be the polynomial $p(x_1, ..., x_n) = \sum_{j=1}^{n} x_j^{2j+1}$.

Suppose that $f = (f_1, f_2, ..., f_n) : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function with $p(f(x_1, ..., x_n)) = 0$ for all $(x_1, ..., x_n) \in \mathbb{R}^n$. Show that $\det f'(x_1, ..., x_n) = 0$ for all $(x_1, ..., x_n) \in \mathbb{R}^n$ where

$$
\begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix}
$$