**Analysis Tier I Exam**
**January 2023**

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- **Be sure to fully justify all answers.**
- **Scoring:** Each problem is worth 11 points.
- **Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.**
- Please be sure that you assemble your test with the problems presented in correct order.

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**1.** Let $\{f_n\}$ be a sequence of nonnegative, continuous, real-valued functions on $[0,1]$ with the property that $f_n(x) \leq f_{n+1}(x)$ for all $x \in [0,1]$, and $n \in \mathbb{N}$. Assume that $\{f_n\}$ converges uniformly on $[0,1]$ to a function $f$. Show that

$$\lim_{n\to\infty} \int_0^1 \Big( \sum_{k=1}^{n} (f_k(x))^n \Big)^{1/n} dx = \int_0^1 f(x) dx.$$

**2.** Consider the infinite series

$$\sum_{a,b\geq 0} \frac{1}{p^a q^b} = 1 + \frac{1}{p} + \frac{1}{q} + \frac{1}{p^2} + \frac{1}{p\,q} + \frac{1}{q^2} + \cdots$$

where $p, q$ are distinct primes and the terms are reciprocals of positive integers that are products of powers of $p$ and powers of $q$. Thus in the sum $a, b$ range over all nonnegative integers. Prove that the series converges and find the sum of the series.

**3.** Let $\{f_n\}$ be a sequence of continuous, real-valued functions on $[0,1]$ with the property that for some function $f$ on $[0,1]$,

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for each sequence of points $\{x_n\} \subset [0,1]$ with $\lim_{n\to\infty} x_n = x$, and all $x \in [0,1]$. Prove or give a counterexample: $\{f_n\}$ converges uniformly to $f$ on $[0,1]$.

**4.** Does there exist a sequence $\{f_n\}$ of continuously differentiable functions on $\mathbb{R}$ that converges uniformly to a limit function $f$ that is not differentiable at 0? Either give an example with full explanations or show that such a sequence cannot exist.

**5.** Show that

$$\lim_{n\to\infty}\Big(\frac{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}{n}-\frac{2}{3}\sqrt{n}\Big)=0.$$

**6.** Let $C$ be a simple closed curve that lies in the plane $x+y+z=1$. Show that the line integral

$$\int_C zdx-2xdy+3ydz$$

only depends on the area of the region enclosed by $C$ and *not* on the shape of $C$ or its position in the plane.

**7.** For a point $\mathbf{x}=(x_1,x_2,\cdots,x_n)$ in the unit cube $[0,1]^n$, let $A_n(\mathbf{x})=\frac{\sum_{i=1}^n x_i}{n}$ be the average value of its coordinates.

(a) Show that for any $\delta\in(0,1)$,

$$\delta^2\int_{J_\delta}dx_1\cdots dx_n\le\frac{1}{12n}$$

where $J_\delta=\{\mathbf{x}\in[0,1]^n:\ |A_n(\mathbf{x})-\frac{1}{2}|>\delta\}$.

(b) Show that for any continuous function $f$ on the interval $[0,1]$,

$$\lim_{n\to\infty}\int_{[0,1]^n}f(A_n(\mathbf{x}))\,dx_1\cdots dx_n=f(\frac{1}{2}).$$

You may use part (a).

**8.** Consider the function $f:\mathbb{R}^3\to\mathbb{R}$ defined by

$$f(x,y,z)=x^2y+e^x+z.$$

(a) Show that there exists a differentiable function $\phi$ defined in a neighborhood $U$ of $(1,-1)$ in $\mathbb{R}^2$ such that $\phi(1,-1)=0$ and $f(\phi(y,z),y,z)=0$ for all $(y,z)\in U$.

(b) Find the values of the gradient $\nabla\phi(1,-1)$.

**9.** For $n \geq 2$, let $p : \mathbb{R}^n \to \mathbb{R}$ be the polynomial $p(x_1, ..., x_n) = \sum_{j=1}^{n} x_j^{2j+1}$. Suppose that $\mathbf{f} = (f_1, f_2, ..., f_n) : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function with $p(\mathbf{f}(x_1, ..., x_n)) = 0$ for all $(x_1, ..., x_n) \in \mathbb{R}^n$. Show $\det \mathbf{f}'(x_1, ..., x_n) = 0$ for all $(x_1, ..., x_n) \in \mathbb{R}^n$ where

$$\mathbf{f}' = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & . \; . \; . & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & . \; . \; . & \frac{\partial f_2}{\partial x_n} \\ . & & & . \\ . & & & . \\ . & & & . \\ . & & & . \\ . & & & . \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & . \; . \; . & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$