Tier 1 August 2022

(i) Be sure to fully justify all answers.

(ii) Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.

(iii) Please assemble your test with the problems in the proper order.

(iv) Each problem is worth 11 points.

1. For the sequence $\{x_n\}$ defined by $0 < x_1 < 1$ and

$$x_{n+1} = 1 - \sqrt{1 - x_n}\ , \quad n = 1, 2, 3, \ldots$$

(a) Prove that the sequence $\{x_n\}$ decreases monotonically to zero as $n \to \infty$.
(b) Show that $\frac{x_{n+1}}{x_n} \to \frac{1}{2}$ as $n \to \infty$.

2. Suppose that $\{f_n\}$ is a sequence of increasing, real-valued functions on $[a, b]$. Just using the definitions, show that if $\{f_n\}$ converges pointwise to a continuous function f on $[a, b]$, then $\{f_n\}$ converges uniformly to f on $[a, b]$.

3. Find the value of $\iint_E \mathbf{F} \cdot \mathbf{n}\ dS$ where $\mathbf{F}(x, y, z) = (x, ze^x, y^2)$, E is the upper hemisphere $\{x^2 + y^2 + z^2 = 1,\ z \geq 0\}$, and $\mathbf{n}$ is the outward pointing unit normal vector on the sphere. [Recall that the volume of a 3D unit ball is $\frac{4}{3}\pi$.]

4.

(a) Suppose $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^2$ with component functions $g_1(x_1, x_2, x_3)$ and $g_2(x_1, x_2, x_3)$ satisfies $\mathbf{G}(x_0, y_0, z_0) = (0, 0)$ for some point $(x_0, y_0, z_0) \in \mathbb{R}^3$. Carefully state under what conditions on $\mathbf{G}$ there exist continuously differentiable functions $\phi : I \to \mathbb{R}$ and $\psi : I \to \mathbb{R}$ defined on some open interval $I \ni x_0$ such that the set of points satisfying $\{(x_1, x_2, x_3) : \mathbf{G}(x_1, x_2, x_3) = (0, 0)\}$ in a neighborhood of (x_0, y_0, z_0) can be expressed as $\{\big(x_1, \phi(x_1), \psi(x_1)\big) : x_1 \in I\}$.

(b) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable, that $f(1, 1) = 1$, and

$$\frac{\partial f}{\partial x_1}(1,1) \neq 0, \qquad \frac{\partial f}{\partial x_2}(1,1) \neq 0, \qquad \left(\frac{\partial f}{\partial x_2}(1,1)\right)^2 \neq 1\ .$$

Show that the system

$$f(x_3, f(x_1, x_2)) = 1$$
$$f(f(x_1, x_3), x_2) = 1$$

defines functions $x_2 = \varphi(x_1)$, and $x_3 = \psi(x_1)$ for x_1 in a neighborhood of 1 satisfying the system

$$f(\psi(x_1), f(x_1, \varphi(x_1))) = 1$$
$$f(f(x_1, \psi(x_1)), \varphi(x_1)) = 1\ .$$

5. Suppose $f\colon [a,b] \to \mathbb{R}$ is Riemann-integrable, and moreover $f(x) \geq \mu$ for all $x \in [a,b]$ for some constant $\mu > 0$. Show that $1/f$ is also Riemann-integrable.

6. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Show that f is continuous at the origin.
(b) Show that f has a directional derivative in any direction at the origin.
(c) Decide whether or not f is differentiable at the origin. If it is differentiable, calculate its derivative.

7. Let $f_n : \mathbb{R}^2 \to \mathbb{R}$, $n = 1, 2, 3\ldots$ be a sequence of continuously differentiable functions that converge pointwise to a continuously differentiable function f. In addition, suppose that for each $n \in \mathbb{N}$, the point $(0,0)$ is a local minimum for f_n. Is it true that $(0,0)$ is a local minimum for f? If so, then prove it, and if not, then provide a counterexample with explanation.

8. Carefully establish either the convergence or divergence of the improper integral

$$\int_3^\infty \frac{\ln x}{x^p \ln(\ln x)}\, dx \quad \text{where } p \text{ is a positive constant.}$$

Note: Your answer may depend on the value of p.

9. Define a function $F : \mathbb{R} \to \mathbb{R}$ by the formula

$$F(x) = \sum_{n=1}^{\infty} n^{-x}.$$

(a) Prove that for any $\delta > 0$ this series converges uniformly on the interval $[1+\delta, \infty)$. Explain why this implies F is continuous on the interval $1 < x < \infty$. Is F continuous for $1 \leq x < \infty$?

(b) Now prove that F is continuously differentiable on the interval $1 < x < \infty$ with

$$F'(x) = -\sum_{n=1}^{\infty} \frac{\ln n}{n^x} \quad \text{on this interval.}$$

You may apply a theorem about the validity of term-wise differentiation of infinite series but be sure to verify its hypotheses.
(*Hint:* Recall that for any positive real number a and any real number b, one can define a^b by the formula $a^b = e^{b \ln a}$.)