Tier 1 Exam—Analysis

January 2024

You may answer as many questions as you like and all questions count equally. Show computations and justify your answers; a correct answer without a correct proof earns little credit. Write a solution of each problem on a separate page. Write the problem number on each page. At the end, assemble your solutions with the problems in increasing order. You have 4 hours.

1. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of real numbers such that the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge absolutely. Show that the series $\sum_{n=1}^{\infty} \sqrt{|\sin(a_n b_n)|}$ converges.

2. A decreasing sequence $(a_n)_{n=1}^{\infty}$ of positive numbers is given such that $\lim_{n\to\infty} a_n = 0$ and the series $\sum_{n=1}^{\infty} a_n$ diverges. Define

$$b_n = \begin{cases} -a_n & \text{if } n \text{ is a multiple of 3,} \\ a_n & \text{if } n \text{ is not a multiple of 3.} \end{cases}$$

   Either show that the series $\sum_{n=1}^{\infty} b_n$ necessarily diverges or provide an example where it converges.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) \neq x$ for every $x \in \mathbb{R}$, and let $a_0 \in \mathbb{R}$. Define inductively $a_n = f(a_{n-1})$ for $n \geq 1$. Show that the sequence $(a_n)_{n=0}^{\infty}$ is monotone and unbounded.

4. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of positive numbers, and define

$$A_n = \sum_{k=1}^{n} a_k, \quad B_n = \sum_{k=1}^{n} b_k, \quad n = 1, 2, \ldots$$

   Suppose that the sequence $(a_n/b_n)_{n=1}^{\infty}$ is nondecreasing and bounded.

   (a) Prove that the sequence $(A_n/B_n)_{n=1}^{\infty}$ is also nondecreasing and bounded.

   (b) Show that,

$$\lim_{n\to\infty} \frac{A_n}{B_n} \leq \lim_{n\to\infty} \frac{a_n}{b_n}.$$

5. Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 10\}$, and let $f : D \to [-1, 1]$ be a continuous function such that $|f(x, y)| < 1$ whenever $x^2 + y^2 < 10$. Show that the following limit exists and find its value:

$$\lim_{n \to \infty} \iint_D f(x, y)^n \, dx dy.$$

6. Let $Q \subset \mathbb{R}^2$ be the square $[-2, 2]^2$, and denote by γ its boundary, oriented counterclockwise. Show that the line integral

$$\int_\gamma \frac{x\, dy - y\, dx}{x^2 + y^2}$$

is equal to 2π.

7. Show that the formula

$$f(x) = \sum_{n=0}^{\infty} \frac{\cos(3^n x)}{3^n}$$

defines a continuous function on $\mathbb{R}$ such that f has a maximum at $x = 0$ but $f'(0)$ does not exist.

8. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(x + 1) = g(x)$ for every $x \in \mathbb{R}$. Show that the function

$$f(x) = \int_0^\infty e^{-t} g(tx)\, dt, \quad x \in \mathbb{R}.$$

is uniformly continuous.

9. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable function whose Jacobian determinant is different from zero at every point. Suppose that the preimage $f^{-1}(B)$ is bounded whenever $B \subset \mathbb{R}^2$ is bounded. Show that f must be one-to-one and onto.