Tier 1 Analysis Exam

August 2023

All ten problems count equally. Show computations and justify your answers; a correct answer without a correct proof earns little credit. Write a solution of each problem on a separate page. Write the problem number on each page. At the end, assemble your solutions with the problems in increasing order. You have 4 hours.

Notation: For a function $f\colon \mathbb{R}^n \to \mathbb{R}$, denote the partial derivative in the ith coordinate direction by $D_i f$. The partial derivative of $D_i f$ in the jth coordinate direction is likewise denoted by $D_{ij} f := D_j D_i f$. The expression $:=$ is used to indicate a definition.

1. Fix a positive integer n. Let $A \subseteq \mathbb{R}^n$ be convex. Is the closure of A necessarily convex? Prove that it is or give a counterexample with proof.

2. For $f_n(x) := \prod_{j=1}^n \big(1 + \sin(x/j^2)\big)$, show that:

(a) for each $x \in [0,1]$, there exists a limit $f(x) := \lim_{n\to\infty} f_n(x)$;

(b) f is Riemann integrable on $[0,1]$ and $\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x$.

3. Compute the surface integral

$$I := \iint_{x+y+z=1} f(x,y,z)\,\mathrm{d}S,$$

where

$$f(x,y,z) := \begin{cases} 1 - x^2 - y^2 - z^2 & \text{for } x^2+y^2+z^2 \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

4. Does $\sum_{n=1}^\infty \frac{(-1)^{\lfloor \log_{10} n \rfloor}}{n}$ converge? Here, $\lfloor x \rfloor$ denotes the greatest integer that is at most x.

5. For a positive integer n, let A_n be the arithmetic mean of the collection of $2n-1$ numbers $\sqrt{2n-1}$, $\sqrt{2(2n-2)}$, $\sqrt{3(2n-3)}$, $\ldots$, $\sqrt{(2n-2)2}$, $\sqrt{2n-1}$, that is, $1/(2n-1)$ times their sum. For large n, most of these numbers are of order n, so we would expect A_n also to have order n. Evaluate $\lim_{n\to\infty} A_n/n$.

6. Let (X,d) be a metric space such that $\inf\{d(x,y) \mid x \neq y\} = 0$. Must X contain a Cauchy sequence of pairwise distinct elements? Prove that it must or give a counterexample with proof.

7. Let $F(x,y,z) := 2x^2 + y^2 + z^2 - 2xy - 2x + z - 5$.

(a) Use the implicit function theorem to prove that $F(x,y,z) = 0$ defines a compact two-dimensional surface $\mathcal{S}$ embedded in $\mathbb{R}^3$, in other words, a closed, bounded set $\mathcal{S} \subset \mathbb{R}^3$ that is locally describable near each point in $\mathcal{S}$ in the form of a graph of one of the variables—not necessarily always the same one—as a function of the other two.

(b) Find the points $(x,y,z) \in \mathcal{S}$ with maximum and minimum values of z.

8. Let $u = u(x, y)$ be twice continuously differentiable in a neighborhood of $B_1 := \{(x, y) \mid x^2 + y^2 \leq 1\}$ and satisfy

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{x^2+y^2}.$$

Compute

$$\iint_{B_1} \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y.$$

Hint: Let $S_1 := \{(x, y) \mid x^2 + y^2 = 1\}$. Green's first identity says that if w and v are twice continuously differentiable in a neighborhood of B_1, then

$$\int_{S_1} w\frac{\partial v}{\partial n}\,\mathrm{d}s = \iint_{B_1} \left(\frac{\partial v}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial w}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y + \iint_{B_1} w \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \mathrm{d}x\,\mathrm{d}y,$$

where the derivative with respect to n denotes the derivative with respect to the outer unit normal.

9. Let $u = u(x, y)$ be a twice continuously differentiable function defined on a neighborhood of the origin in $\mathbb{R}^2$. Also assume that $\frac{\partial u}{\partial y} \neq 0$ in a neighborhood of the origin and $u(0, 0) = 0$.

(a) Prove that y is a twice continuously differentiable function of x and u, $y = y(x, u)$, defined in a neighborhood of $(x, u) = (0, 0)$ such that $y(0, 0) = 0$.

(b) Show that under the condition that

$$\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} \cdot \frac{\partial^2 u}{\partial x \partial y} + \left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 u}{\partial y^2} = 0$$

in a neighborhood of the origin, we have $\partial^2 y/\partial x^2 = 0$ in a neighborhood of the origin.

Equivalently, we can rephrase this problem using the following notation. You may solve the problem using either version of the statement. Let f be a C^2 function defined on a neighborhood of the origin in $\mathbb{R}^2$. Also assume that $D_2 f \neq 0$ in a neighborhood of the origin and $f(0, 0) = 0$.

(a) Prove the existence of a C^2 function g in a neighborhood of the origin such that $f(x, g(x, u)) = u$ for (x, u) in a neighborhood of the origin and $g(0, 0) = 0$.

(b) Show that under the condition that

$$(D_2 f)^2 D_{11} f - 2(D_1 f)(D_2 f)(D_{12} f) + (D_1 f)^2 D_{22} f = 0$$

in a neighborhood of the origin, we have $D_{11} g = 0$ in a neighborhood of the origin.

10. Give an example of continuous functions $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f(0, 0) = (0, 0)$, $g(0, 0) = 0$, and both f and g have directional derivatives at $(0, 0)$ in all directions, but $g \circ f$ does not. Note: to say that f has a directional derivative in a given direction means that each of its component functions does.