TIER 1 ANALYSIS EXAMINATION
JANUARY 4, 2022

The complete solution to each of the problems below is worth 10 points, so 90 is the maximum score. Please write your solutions on separate sheets, use only one side of each sheet, and make sure each page is labeled with a problem number.

(1) Define continuous functions $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \frac{1 + x^n}{1 + 2^{-n}}, \quad x \in \mathbb{R}, n \in \mathbb{N}.$$

Show that the sequence $(f_n)_{n\in\mathbb{N}}$ is not equicontinuous on $[0, 1]$.

(2) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that

$$\sum_{n=1}^{\infty} |a_n - a_{n+1}| < +\infty.$$

Show that $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence.

(3) Let $f : \mathbb{R} \to [0, +\infty)$ be a differentiable function such that both f and $-f'$ are nonincreasing on $\mathbb{R}$. Prove that

$$\lim_{x\to+\infty} f'(x) = 0.$$

(4) Let $G \subset \mathbb{R}^5$ be the set of vectors $A = (a_0, a_1, a_2, a_3, a_4)$ with the property that the quintic polynomial

$$P_A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + x^5$$

has five distinct real roots. Prove that G is an open set.

(5) Does the improper integral

$$\int_0^\infty \cos(x^{2/3})\, dx$$

converge? Justify your answer.

(6) Let t_0 be an arbitrary real number. Define a sequence $(t_n)_{n\in\mathbb{N}}$ by setting $t_n = \sin(\cos(t_{n-1}))$ for $n \geq 1$. Prove that this sequence converges and that the limit does not depend of t_0.

(7) Suppose that $(a_n)_{n\in\mathbb{N}}$ is an unbounded, increasing sequence of positive numbers. Show that the series

$$\sum_{n=1}^{\infty} \frac{a_{n+1} - a_n}{a_n}$$

diverges.

(8) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous, compactly supported function. Define a new function $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x) = \int_{\mathbb{R}^2} \frac{f(y)}{|x-y|}\,dy, \quad x \in \mathbb{R}^2.$$

Prove that the improper integral does in fact converge and that the function g is continuous. (Here $x = (x_1, x_2)$, $y = (y_1, y_2)$, $|x| = \sqrt{x_1^2 + x_2^2}$, and $dy = dy_1 dy_2$. Convergence of the improper integral means that the Riemann integral

$$\int_{\varepsilon < |x-y| < 1/\varepsilon} \frac{f(y)}{|x-y|}\,dy$$

has a limit as $\varepsilon \downarrow 0$.)

(9) Let $F_1(x, y, z) = 6yz$, $F_2(x, y, z) = 2xz$, $F_3(x, y, z) = 4xy$, and let $\alpha, \gamma : [-\pi, \pi] \to \mathbb{R}^3$ be defined by

$$\alpha(t) = (\cos(t), \sin(t), 0),$$
$$\gamma(t) = (\cos(t), \sin(t), 4 + (\sin(t))(\cos(t^3))).$$

(a) Apply Stokes' Theorem on the surface $S = \{(\cos(t), \sin(t), z) : -\pi \le t \le \pi, 0 \le z \le 4 + (\sin(t))(\cos(t^3))\}$ to express

$$\int_\gamma (F_1 dx + F_2 dy + F_3 dz)$$

in terms of

$$\int_\alpha (F_1 dx + F_2 dy + F_3 dz).$$

(b) Use (a) to evaluate the first integral.