Work all nine problems. They all count equally. Show computations and justify your answers; a correct answer without a correct proof earns little credit. Write a solution of each problem on a separate page. You have 4 hours.

**Notation:** For a function \( f: \mathbb{R}^n \to \mathbb{R} \), denote the partial derivative in the \( i \)-th coordinate direction by \( D_i f \). The partial derivative of \( D_i f \) in the \( j \)-th coordinate direction is likewise denoted by \( D_{ij} f := D_j D_i f \). The expression \( := \) is used to indicate a definition.

1. Let \( B(K) \) denote the set of bounded functions \( f: K \to \mathbb{R} \), where \( K \subset \mathbb{R} \) is compact.
   
   (a) For \( f, g \in B(K) \), define
   \[
   d(f, g) := \sup_{x \in K} |f(x) - g(x)|.
   \]
   Show that \( d: B(K) \times B(K) \to \mathbb{R} \) defines a metric on \( B(K) \).

   (b) Show that the set \( C(K) \) of continuous functions \( f: K \to \mathbb{R} \) is a closed subset of \( B(K) \) in the topology given by this metric.

2. Let \( a_{n,m} \in [0,1] \) for all positive integers \( n \) and \( m \). Suppose that for each \( n \), we have \( \lim_{m \to \infty} a_{n,m} = n/2^n \). For each of the following inequalities, prove that it must hold or prove (with a counterexample) that it need not hold:
   
   (a) \( \lim \inf_{m \to \infty} \sum_{n=1}^{\infty} \frac{a_{n,m}}{n} \geq 1 \);

   (b) \( \lim \sup_{m \to \infty} \sum_{n=1}^{\infty} \frac{a_{n,m}}{n} \leq 1 \).

3. Let \( a_k \geq 0 \) for all nonnegative integers \( k \). Suppose that \( \sum_{k=0}^{\infty} a_k < \infty \). Define \( f(x) := \sum_{k=0}^{\infty} a_k x^k \) for \( x \in [0,1] \). Do not assume that \( b := \sum_{k=0}^{\infty} k a_k \) is finite.
   
   (a) Show that (the left-hand derivative) \( f'(1) = b \).

   (b) Show that \( \lim_{x \to 1^-} f'(x) = b \).

4. Let \( a_k \geq 0 \) for all nonnegative integers \( k \). Suppose that \( \sum_{k=0}^{\infty} a_k = 1, \sum_{k=0}^{\infty} k a_k = 1, \) and \( c := \sum_{k=0}^{\infty} k(k-1) a_k \) is finite. Define \( f(x) := \sum_{k=0}^{\infty} a_k x^k \) for \( x \in [0,1] \). You may assume without proof that \( \lim_{x \to 1^-} f'(x) = 1 \) and \( \lim_{x \to 1^-} f''(x) = c \). (These both follow from problem 3(b).) Define
   \[
   g(x) := \frac{1}{1 - f(x)} - \frac{1}{1 - x}
   \]
   for \( x \in [0,1] \). Show that \( \lim_{x \to 1^-} g(x) = c/2 \).
5. Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose that $D_1 f$ exists at $(0,0)$. Suppose also that $D_2 f$ exists in a neighborhood of $(0,0)$ and is continuous at $(0,0)$. Prove that $f$ is differentiable at $(0,0)$.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable in a neighborhood of $(0,0)$, with $D_2 f(0,0) = 0$ and $D_{22} f(0,0) > 0$.

   (a) Prove that there are $\epsilon, \delta > 0$ such that for each $x \in (-\epsilon, \epsilon)$, the formula
   
   
   \[ g(x) := \min \{ f(x,y) : |y| \leq \delta \} \]
   
   defines a differentiable function $g : (-\epsilon, \epsilon) \to \mathbb{R}$.
   
   (b) Find a formula for $g'(0)$ in terms of $f$ and prove it.

7. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) := x^4 + y^4 - 4xy$.

   (a) Identify and classify all critical points of $f$ on $\mathbb{R}^2$.
   
   (b) Determine the minimum and maximum values of $f$ on the curve $x^4 + y^4 = 32$.

8. Let $\mathbb{Q} \subset \mathbb{R}$ denote the rationals. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

   
   \[
   f(x,y) := \begin{cases} x & \text{if } (x,y) \in \mathbb{Q} \times \mathbb{Q}, \\ y & \text{otherwise} \end{cases}
   \]

   is not Riemann integrable on the unit square $[0,1] \times [0,1] \subset \mathbb{R}^2$.

9. Let $S := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \text{ with } y \geq 0 \text{ and } z \geq 0\}$ be the surface consisting of a quarter of the unit sphere in $\mathbb{R}^3$. Let $f : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable. Define the vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ by $F(x,y,z) := (f(z) + x^2 y, xy^2 + 1, xyz)$. Evaluate the surface integral $\int_S F \cdot \hat{n} \, dS$, where $\hat{n}$ is the unit normal vector field of $S$ pointing away from the origin.