TIER I ANALYSIS EXAM, JANUARY 2019

Solve all nine problems. They all count equally. Show all computations.

1. Let $f:\mathbb{R}\to[0,1]$ be continuous. Let $x_1\in(0,1)$. Define x_n via the recurrence

$$x_{n+1}=\frac{3}{4}x_n^2+\frac{1}{4}\int_0^{|x_n|} f, \quad n\geq 1.$$

Prove that x_n is convergent and find its limit.

2. Suppose (X,d) is a compact metric space with an open cover $\{U_a\}$. Show that for some $\epsilon>0$, every ball of radius ϵ is fully contained in at least one of the U_a's.

3. Find

$$\lim_{N\to\infty}\sum_{n=N}^{\infty}\frac{1}{n^{1+\frac{1}{\log N}}}.$$

Here log is the natural logarithm (in base e)

4. (a) Give an example of an everywhere differentiable function $f\colon\mathbb{R}\to\mathbb{R}$ whose derivative $f'(x)$ is not continuous.

(b) Show that when $f,g\colon\mathbb{R}\to\mathbb{R}$ are functions, and for every $\epsilon>0$, there exists a $\delta=\delta(\epsilon)>0$ such that $|h|<\delta$ guarantees

$$\left|\frac{f(x+h)-f(x)}{h}-g(x)\right|<\epsilon$$

for all $x\in\mathbb{R}$, then f' exists and is continuous at every $x\in\mathbb{R}$.

5. (a) Give an example of a continuous function on $(0,1]$ that attains neither a max nor a min on $(0,1]$.

(b) Show that a uniformly continuous function on $(0,1]$ must attain either a max or a min on $(0,1]$.

6. Assume $f:(0,1)^2\to\mathbb{R}$ is continuous and has partial derivative $\frac{\partial f}{\partial x}$ at each point (x,y) satisfying

$$|\frac{\partial f}{\partial x}(x,y)|\geq 1.$$

Consider the set

$$S_\delta=\{(x,y)\in(0,1)^2:\ |f(x,y)|\leq\delta\}.$$

Prove that the area of S_δ is less than or equal to 4δ for each $\delta>0$.

7. Prove that there are real-valued continuously differentiable functions $u(x,y)$ and $v(x,y)$ defined on a neighborhood of the point $(1,2)\in\mathbb{R}^2$ that satisfy the following system of equations,

$$xu^2+yv^2+xy=4$$
$$xv^2+yu^2-xy=1.$$

8. Consider the upper hemi-ellipsoid surface $\Sigma = \left\{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ and } z \geq 0\right\}$ for positive constants $a, b, c \in \mathbb{R}$ and define the vector field $\vec{F} = (\partial_y f, -\partial_x f, 2)$ on Σ for some smooth function $f : \mathbb{R}^3 \to \mathbb{R}$. Evaluate the surface integral $\int_{\Sigma} \vec{F} \cdot \vec{n}\ dS$, where $\vec{n}$ is the upper/outward pointing unit normal field of Σ.

9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous and suppose that for some $R > 0$, $|f(x, y)| < e^{-\sqrt{x^2+y^2}}$ whenever $\sqrt{x^2 + y^2} \geq R$.

(a) Show that the integral

$$g(s, t) = \int\int_{\mathbb{R}^2} f(x, y) \left((x - s)^2 + (y - t)^2\right) dxdy$$

converges for all $(s, t) \in \mathbb{R}^2$

(b) Show that g is continuous on $\mathbb{R}^2$.