TIER I ANALYSIS EXAMINATION

August 2018

Instructions: There are ten problems, each of equal value. Show your work, justifying all steps by direct calculation or by reference to an appropriate theorem.

Notation: For $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $|\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2}$, and $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$.

1. Suppose $(a_n)_{n=1}^{\infty}$ is a sequence of positive real numbers and $\sum_{n=1}^{\infty} a_n = \infty$. Prove that there exists a sequence of positive real numbers $(b_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=1}^{\infty} a_n b_n = \infty$.

2. Show that $\sum_{n=1}^{\infty} \sin(x^n)/n!$ converges uniformly for $x \in \mathbb{R}$ to a C^1 function $f : \mathbb{R} \to \mathbb{R}$, and compute an expression for the derivative. Justify this computation.

3. Let $f : (0, \infty) \to \mathbb{R}$ be differentiable. Show that the intersection of all tangent planes to the surface $z = xf(x/y)$ $(x, y \in (0, \infty))$ is nonempty.

4. For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the largest integer that is less than or equal to x. Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \sqrt{n} \rfloor}}{n}$$

converges. Suggestion: The inequality

$$\frac{1}{\ell + 1} < \int_{\ell}^{\ell+1} \frac{1}{x}\, dx < \frac{1}{\ell}$$

might be helpful. You do not need to justify this inequality.

5. Let B be the closed unit ball in $\mathbb{R}^2$ with respect to the usual metric, d (defined above). Let ρ be the metric on B defined by

$$\rho(x, y) = \begin{cases} |\mathbf{x} - \mathbf{y}| & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ are on the same line through the origin,} \\ |\mathbf{x}| + |\mathbf{y}| & \text{otherwise,} \end{cases}$$

for $\mathbf{x}, \mathbf{y} \in B$. (Note that $\rho(x, y)$ is the minimum distance travelled in the usual metric in going from x to y along lines through the origin.) Suppose $f : B \to \mathbb{R}$ is a function that is uniformly continuous on B with respect to the metric ρ on B and the usual metric on $\mathbb{R}$. Prove that f is bounded.

6. Let

$$f(x) := \begin{cases} \sin x + 2x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove or disprove: there exists $\epsilon > 0$ such that f is invertible when restricted to $(-\epsilon, \epsilon)$.

7. Define a sequence of functions $f_n : [0, 2\pi] \subset \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = e^{\sin(nx)},$$

and define $F_n(x) = \int_0^x f_n(y)\, dy$. Show that there exists a subsequence $(F_{n_k})_{k=1}^{\infty}$ of $(F_n)_{n=1}^{\infty}$ that converges uniformly on $x \in [0, 2\pi]$ to a continuous limit F_*.

8. Let a closed curve, γ, be parameterized by a function $f : [0, 1] \to \mathbb{R}^2$ with a continuous derivative and $f(0) = f(1)$. Suppose that

$$\int_{\gamma} (y^3 \sin^2 x\, dx - x^5 \cos^2 y\, dy) = 0. \tag{1}$$

Show that there exists a pair $\{x, y\} \neq \{0, 1\}$ with $x \neq y$ and $f(x) = f(y)$. Give an example of a curve satisfying (1) such that the only pairs $\{x, y\}$ with $x \neq y$ and $f(x) = f(y)$ are subsets of $\{0, 1/2, 1\}$.

9. Fix $a > 0$. Let S be the *half-ellipsoid* defined by $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z/a)^2 = 1 \text{ and } z \geq 0\}$. Let $\mathbf{v}$ be the vector field given by $\mathbf{v}(x, y, z) = (x, y, z + 1)$, and let $\mathbf{n}$ be the outward unit normal field to the ellipsoid $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z/a)^2 = 1\}$.

(a) From the fact that the volume of $D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1 \text{ and } z \geq 0\}$ is $2\pi/3$, which you may assume without proof, use the change-of-variables formula in $\mathbb{R}^3$ to find the volume of $E := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z/a)^2 \leq 1 \text{ and } z \geq 0\}$.

(b) Evaluate

$$\iint_S \mathbf{v} \cdot \mathbf{n}\, dA,$$

where dA denotes the surface area element.

10. Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^2, let I denote the $n \times n$ identity matrix, let

$$D^2 f(x) = \left(\frac{\partial^2 f(x)}{\partial x_i\, \partial x_j} \right)_{1 \leq i,j \leq n},$$

and assume that there exists a positive real number a such that $D^2 f(x) - aI$ is positive definite for all $x \in \mathbb{R}^n$, or equivalently, assume that there exists a positive real number a such that $D_{\mathbf{u}}[D_{\mathbf{u}} f](x) \geq a$ for all unit vectors $\mathbf{u} \in \mathbb{R}^n$ and points $x \in \mathbb{R}^n$, where $D_{\mathbf{u}}$ denotes the directional derivative in the direction $\mathbf{u}$. (You do not have to prove the equivalence of these two versions of the assumption.)

(a) Let ∇f denote the gradient of f. Show that there exists a point $x \in \mathbb{R}^n$ such that $\nabla f(x) = 0$.

(b) Show that the map $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is onto.

(c) Show that the map $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is globally invertible, and the inverse is C^1.