TIER I ANALYSIS EXAMINATION
August 2018

Instructions: There are ten problems, each of equal value. Show your work, justifying all steps by direct calculation or by reference to an appropriate theorem.

Notation: For \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \), \( |x| = \sqrt{x_1^2 + \cdots + x_n^2} \), and \( d(x, y) = |x - y| \).

1. Suppose \( (a_n)_{n=1}^{\infty} \) is a sequence of positive real numbers and \( \sum_{n=1}^{\infty} a_n = \infty \). Prove that there exists a sequence of positive real numbers \( (b_n)_{n=1}^{\infty} \) such that \( \lim_{n \to \infty} b_n = 0 \) and \( \sum_{n=1}^{\infty} a_n b_n = \infty \).

2. Show that \( \sum_{n=1}^{\infty} \frac{\sin(x^n)}{n!} \) converges uniformly for \( x \in \mathbb{R} \) to a \( C^1 \) function \( f : \mathbb{R} \to \mathbb{R} \), and compute an expression for the derivative. Justify this computation.

3. Let \( f : (0, \infty) \to \mathbb{R} \) be differentiable. Show that the intersection of all tangent planes to the surface \( z = x f(x/y) \) \( (x, y \in (0, \infty)) \) is nonempty.

4. For \( x \in \mathbb{R} \), let \( \lfloor x \rfloor \) denote the largest integer that is less than or equal to \( x \). Prove that
   \[
   \sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \sqrt{n} \rfloor}}{n}
   \]
   converges. Suggestion: The inequality
   \[
   \frac{1}{\ell + 1} < \int_{\ell}^{\ell+1} \frac{1}{x} \, dx < \frac{1}{\ell}
   \]
   might be helpful. You do not need to justify this inequality.

5. Let \( B \) be the closed unit ball in \( \mathbb{R}^2 \) with respect to the usual metric, \( d \) (defined above). Let \( \rho \) be the metric on \( B \) defined by
   \[
   \rho(x, y) = \begin{cases} 
   |x - y| & \text{if } x \text{ and } y \text{ are on the same line through the origin,} \\
   |x| + |y| & \text{otherwise,}
   \end{cases}
   \]
   for \( x, y \in B \). (Note that \( \rho(x, y) \) is the minimum distance travelled in the usual metric in going from \( x \) to \( y \) along lines through the origin.) Suppose \( f : B \to \mathbb{R} \) is a function that is uniformly continuous on \( B \) with respect to the metric \( \rho \) on \( B \) and the usual metric on \( \mathbb{R} \). Prove that \( f \) is bounded.

6. Let
   \[
   f(x) := \begin{cases} 
   \sin x + 2x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\
   0 & \text{if } x = 0.
   \end{cases}
   \]
   Prove or disprove: there exists \( \epsilon > 0 \) such that \( f \) is invertible when restricted to \( (-\epsilon, \epsilon) \).
7. Define a sequence of functions \( f_n : [0, 2\pi] \subset \mathbb{R} \to \mathbb{R} \) by
\[
f_n(x) = e^{\sin(nx)},
\]
and define \( F_n(x) = \int_0^x f_n(y) \, dy \). Show that there exists a subsequence \((F_{n_k})_{k=1}^{\infty}\) of \((F_n)_{n=1}^{\infty}\) that converges uniformly on \( x \in [0, 2\pi] \) to a continuous limit \( F_* \).

8. Let a closed curve, \( \gamma \), be parameterized by a function \( f : [0, 1] \to \mathbb{R}^2 \) with a continuous derivative and \( f(0) = f(1) \). Suppose that
\[
\hat{\gamma}(y) = \int_0^y (y^3 \sin^2 x \, dx - x^5 \cos^2 y \, dy) = 0.
\]
Show that there exists a pair \( \{x, y\} \neq \{0, 1\} \) with \( x \neq y \) and \( f(x) = f(y) \). Give an example of a curve satisfying (1) such that the only pairs \( \{x, y\} \) with \( x \neq y \) and \( f(x) = f(y) \) are subsets of \( \{0, 1/2, 1\} \).

9. Fix \( a > 0 \). Let \( S \) be the half-ellipsoid defined by \( S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z/a)^2 = 1 \text{ and } z \geq 0\} \). Let \( v \) be the vector field given by \( v(x, y, z) = (x, y, z + 1) \), and let \( n \) be the outward unit normal field to the ellipsoid \( \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z/a)^2 = 1\} \).
   (a) From the fact that the volume of \( D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1 \text{ and } z \geq 0\} \) is \( 2\pi/3 \), which you may assume without proof, use the change-of-variables formula in \( \mathbb{R}^3 \) to find the volume of \( E := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z/a)^2 \leq 1 \text{ and } z \geq 0\} \).
   (b) Evaluate
\[
\int_S v \cdot n \, dA,
\]
where \( dA \) denotes the surface area element.

10. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be \( C^2 \), let \( I \) denote the \( n \times n \) identity matrix, let
\[
D^2 f(x) = \begin{pmatrix} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \end{pmatrix}_{1 \leq i, j \leq n},
\]
and assume that there exists a positive real number \( a \) such that \( D^2 f(x) - aI \) is positive definite for all \( x \in \mathbb{R}^n \), or equivalently, assume that there exists a positive real number \( a \) such that \( D_u[D_u f](x) \geq a \) for all unit vectors \( u \in \mathbb{R}^n \) and points \( x \in \mathbb{R}^n \), where \( D_u \) denotes the directional derivative in the direction \( u \). (You do not have to prove the equivalence of these two versions of the assumption.)
   (a) Let \( \nabla f \) denote the gradient of \( f \). Show that there exists a point \( x \in \mathbb{R}^n \) such that \( \nabla f(x) = 0 \).
   (b) Show that the map \( \nabla f : \mathbb{R}^n \to \mathbb{R}^n \) is onto.
   (c) Show that the map \( \nabla f : \mathbb{R}^n \to \mathbb{R}^n \) is globally invertible, and the inverse is \( C^1 \).