Problem 1. Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. Let \(f : X \to Y\) be surjective such that 
\[
\frac{1}{2} d_X(x, y) \leq d_Y(f(x), f(y)) \leq 2d_X(x, y)
\]
for all \(x, y \in X\). Show that if \((X, d_X)\) is complete, then also \((Y, d_Y)\) is complete.

Problem 2. Show that 
\[
\lim_{n \to \infty} \left( 2\sqrt{n} - \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \right)
\]
exists.

Problem 3. Assume that \textit{bitter} is a property of subsets of \([0, 1]\) such that the union of two bitter sets is bitter. Subsets of \([0, 1]\) that are not bitter are called \textit{sweet}. Thus every subset of \([0, 1]\) is either bitter or sweet. A \textit{sweet spot} of a set \(A \subset [0, 1]\) is a point \(x_0 \in [0, 1]\) such that for every open set \(U \subset \mathbb{R}\) that contains \(x_0\), the set \(A \cap U\) is sweet. Show that if \(A \subset [0, 1]\) is sweet, then \(A\) has a sweet spot.

Problem 4. Let \(f\) and \(g\) be periodic functions defined on \(\mathbb{R}\), not necessarily with the same period. Suppose that 
\[
\lim_{x \to \infty} f(x) - g(x) = 0 .
\]
Show that \(f(x) = g(x)\) for all \(x\).

Problem 5. Let \(0 < x_n < 1\) be an infinite sequence of real numbers such that for all \(0 < r < 1\) 
\[
\sum_{x_n < r} \log \frac{r}{x_n} \leq 1 .
\]
Show that 
\[
\sum_{n=1}^{\infty} (1 - x_n) < \infty .
\]

Problem 6. Suppose that the series \(\sum_{n=1}^{\infty} a_n\) converges conditionally. Show that the series
\[
\sum_{n=3}^{\infty} n(\log n)(\log \log n)^2 a_n
\]
diverges.

Problem 7. Find the absolute minimum of the function \(f(x, y, z) = xy + yz + zx\) on the set \(g(x, y, z) = x^2 + y^2 + z^2 = 12\).

Problem 8. Let \(f : \mathbb{R}^2 \to \mathbb{R}^2\) be a \(C^1\) map such that \(f^{-1}(y)\) is a finite set for all \(y \in \mathbb{R}^2\). Show that the determinant \(\det df(x)\) of the Jacobi matrix of \(f\) cannot vanish on an open subset of \(\mathbb{R}^2\).
Problem 9. A regular surface is given by a continuously differentiable map $f : \mathbb{R}^2 \to \mathbb{R}^3$ so that the differential $df_x : \mathbb{R}^2 \to \mathbb{R}^3$ has rank 2 for all $x \in \mathbb{R}^2$. The tangent plane $T_x$ is the 2-dimensional subspace $df_x(\mathbb{R}^2) \subset \mathbb{R}^3$. Assume that a vector field $X$ in $\mathbb{R}^3$ is orthogonal to $T_x$ for all $x$, i.e. $X(f(x)) \cdot Y = 0$ for all $x \in \mathbb{R}^2$ and all $Y \in T_x$. Show that $X \cdot (\nabla \times X) = 0$ at all points $f(x)$.

Problem 10. Let $f(x, y)$ be a function defined on $\mathbb{R}^2$ such that
- For any fixed $x$, the function $y \mapsto f(x, y)$ is a polynomial in $y$;
- For any fixed $y$, the function $x \mapsto f(x, y)$ is a polynomial in $x$.
Show that $f$ is a polynomial, i.e.

$$f(x, y) = \sum_{i,j=0}^{N} a_{ij} x^i y^j$$

with suitable $a_{i,j} \in \mathbb{R}$, $i, j = 0, \ldots, N$. 