Tier I Analysis January 2018

Problem 1. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be surjective such that

$$\frac{1}{2} d_X(x, y) \le d_Y(f(x), f(y)) \le 2 d_X(x, y)$$

for all $x, y \in X$. Show that if (X, d_X) is complete, then also (Y, d_Y) is complete.

Problem 2. Show that

$$\lim_{n\to\infty} \left(2\sqrt{n} - \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \right)$$

exists.

Problem 3. Assume that *bitter* is a property of subsets of $[0, 1]$ such that the union of two bitter sets is bitter. Subsets of $[0, 1]$ that are not bitter are called *sweet*. Thus every subset of $[0, 1]$ is either bitter or sweet. A *sweet spot* of a set $A \subset [0, 1]$ is a point $x_0 \in [0, 1]$ such that for every open set $U \subset \mathbb{R}$ that contains x_0, the set $A \cap U$ is sweet. Show that if $A \subset [0, 1]$ is sweet, then A has a sweet spot.

Problem 4. Let f and g be periodic functions defined on $\mathbb{R}$, not necessarily with the same period. Suppose that

$$\lim_{x\to\infty} f(x) - g(x) = 0 \ .$$

Show that $f(x) = g(x)$ for all x.

Problem 5. Let $0 < x_n < 1$ be an infinite sequence of real numbers such that for all $0 < r < 1$

$$\sum_{x_n < r} \log \frac{r}{x_n} \le 1 \ .$$

Show that

$$\sum_{n=1}^{\infty} (1 - x_n) < \infty \ .$$

Problem 6. Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges conditionally. Show that the series

$$\sum_{n=3}^{\infty} n(\log n)(\log\log n)^2 a_n$$

diverges.

Problem 7. Find the absolute minimum of the function $f(x, y, z) = xy + yz + zx$ on the set $g(x, y, z) = x^2 + y^2 + z^2 = 12$.

Problem 8. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map such that $f^{-1}(y)$ is a finite set for all $y \in \mathbb{R}^2$. Show that the determinant $\det df(x)$ of the Jacobi matrix of f cannot vanish on an open subset of $\mathbb{R}^2$.

Problem 9. A *regular surface* is given by a continuously differentiable map $f : \mathbb{R}^2 \to \mathbb{R}^3$ so that the differential $df_x : \mathbb{R}^2 \to \mathbb{R}^3$ has rank 2 for all $x \in \mathbb{R}^2$. The *tangent plane* T_x is the 2-dimensional subspace $df_x(\mathbb{R}^2) \subset \mathbb{R}^3$. Assume that a vector field X in $\mathbb{R}^3$ is orthogonal to T_x for all x, i.e. $X(f(x)) \cdot Y = 0$ for all $x \in \mathbb{R}^2$ and all $Y \in T_x$. Show that $X \cdot (\nabla \times X) = 0$ at all points $f(x)$.

Problem 10. Let $f(x, y)$ be a function defined on $\mathbb{R}^2$ such that

- For any fixed x, the function $y \mapsto f(x, y)$ is a polynomial in y;
- For any fixed y, the function $x \mapsto f(x, y)$ is a polynomial in x.

Show that f is a polynomial, i.e.

$$f(x, y) = \sum_{i,j=0}^{N} a_{ij} x^i y^j$$

with suitable $a_{i,j} \in \mathbb{R}$, $i, j = 0, \ldots, N$.