Tier I Analysis Exam
August, 2017

- **Be sure to fully justify all answers.**
- **Scoring:** Each problem is worth 10 points.
- **Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.**
- Please be sure that you assemble your test with the problems presented in correct order.

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(1) Let X be the set of all functions $f : \mathbb{N} \to \{0, 1\}$, taking only two values 0 and 1. Define the metric d on X by

$$d(f,g) = \begin{cases} 0 & \text{if } f = g, \\ \dfrac{1}{2^m} & \text{if } m = \min\{n \mid f(n) \neq g(n)\}. \end{cases}$$

(a) **Prove** that (X, d) is compact.
(b) **Prove** that no point in (X, d) is isolated.

(2) Let $C[0,1]$ be the space of all real continuous functions defined on the interval $[0,1]$. Define the distance on $C[0,1]$ by

$$d(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|.$$

Prove that the following set $\mathcal{S} \subset C[0,1]$ is not compact:

$$\mathcal{S} = \{f \in C[0,1] \mid d(f,0) = 1\},$$

where $0 \in C[0,1]$ stands for the constant function with value 0.

(3) Let $F(x,y) = \sum_{n=1}^{\infty} \sin(ny) \cdot e^{-n(x+y)}$. **Prove that** there are a $\delta > 0$ and a unique differentiable function $y = \varphi(x)$ defined on $(1-\delta, 1+\delta)$, such that

$$\varphi(1) = 0, \qquad F(x, \varphi(x)) = 0 \quad \forall x \in (1-\delta, 1+\delta).$$

(4) **Prove or find** a counterexample: if $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with $f(0) = 0$, then there exist continuous functions $g_1, ..., g_n : \mathbb{R}^n \to \mathbb{R}$ with

$$f(x) = x_1 g_1(x_1, ..., x_n) + \cdots + x_n g_n(x_1, ..., x_n).$$

(5) Let $\{f_n\}$ be a sequence of real-valued, concave functions defined on an open interval interval $(-a, a)$ ($-f_n$ is convex). Let $g : (-a, a) \to \mathbb{R}$. Suppose f_n and g are differentiable at 0,

$$\liminf f_n(t) \geq g(t) \text{ for all } t, \text{ and } \lim f_n(0) = g(0).$$

Show that $\lim f_n'(0) = g'(0)$.

(6) Let $f(x, y) = \frac{x^2 y}{x^4+y^2}$ for $(x, y) \neq (0, 0)$.
  (a) Can f be defined at $(0, 0)$ so that $f_x(0, 0)$ and $f_y(0, 0)$ exist? **Justify** your answer.
  (b) Can f be defined at $(0, 0)$ so that f is differentiable at $(0, 0)$? **Justify** your answer.

(7) Let $f : [-1, 1] \to \mathbb{R}$ with f, f', f'', f''' being continuous. **Show** that

$$\sum_{n=2}^{\infty} \left\{ n \left[f\left(\frac{1}{n}\right) - f\left(-\frac{1}{n}\right) \right] - 2f'(0) \right\}$$

converges absolutely.

(8) Let $\{f_n\}$ be a uniformly bounded sequence of continuous real-valued functions on a closed interval $[a, b]$, and let $g_n(x) = \int_a^x f_n(t)\, dt$ for each $x \in [a, b]$. **Show** that the sequence of functions $\{g_n\}$ contains a uniformly convergent subsequence on $[a, b]$.

(9) **Compute** $\int_D x dx dy$, where $D \subset \mathbb{R}^2$ is the region bounded by the curves $x = -y^2$, $x = 2y - y^2$, and $x = 2 - 2y - y^2$. **Show** your work.

(10) Let

$$x_0 > 0, \qquad x_{n+1} = \frac{1}{2}\left(x_n + \frac{4}{x_n}\right), \quad n = 0, 1, 2, 3, \ldots.$$

Show that $x = \lim_{n\to\infty} x_n$ exists, and **find** x.