Tier I Analysis Exam
August, 2017

- Be sure to fully justify all answers.
- **Scoring:** Each problem is worth 10 points.
- Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write a problem number on each sheet of paper.
- Please be sure that you assemble your test with the problems presented in correct order.

(1) Let $X$ be the set of all functions $f : \mathbb{N} \to \{0, 1\}$, taking only two values 0 and 1. Define the metric $d$ on $X$ by

$$d(f, g) = \begin{cases} 0 & \text{if } f = g, \\ \frac{1}{2^m} & \text{if } m = \min \{n \mid f(n) \neq g(n)\}. \end{cases}$$

(a) **Prove** that $(X, d)$ is compact.
(b) **Prove** that no point in $(X, d)$ is isolated.

(2) Let $C[0, 1]$ be the space of all real continuous functions defined on the interval $[0, 1]$. Define the distance on $C[0, 1]$ by

$$d(f, g) = \max_{x \in [0, 1]} |f(x) - g(x)|.$$  

**Prove** that the following set $S \subset C[0, 1]$ is not compact:

$$S = \{f \in C[0, 1] \mid d(f, 0) = 1\},$$

where $0 \in C[0, 1]$ stands for the constant function with value 0.

(3) Let $F(x, y) = \sum_{n=1}^{\infty} \sin(ny) \cdot e^{-n(x+y)}$. **Prove that** there are a $\delta > 0$ and a unique differentiable function $y = \varphi(x)$ defined on $(1 - \delta, 1 + \delta)$, such that

$$\varphi(1) = 0, \quad F(x, \varphi(x)) = 0 \quad \forall x \in (1 - \delta, 1 + \delta).$$

(4) **Prove or find** a counterexample: if $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with $f(0) = 0$, then there exist continuous functions $g_1, \ldots, g_n : \mathbb{R}^n \to \mathbb{R}$ with

$$f(x) = x_1 g_1(x_1, \ldots, x_n) + \cdots + x_n g_n(x_1, \ldots, x_n).$$
(5) Let \( \{f_n\} \) be a sequence of real-valued, concave functions defined on an open interval \((-a, a)\) (-\( f_n \) is convex). Let \( g : (-a, a) \to \mathbb{R} \). Suppose \( f_n \) and \( g \) are differentiable at 0,
\[
\lim \inf f_n(t) \geq g(t) \text{ for all } t, \text{ and } \lim f_n(0) = g(0).
\]
Show that \( \lim f_n'(0) = g'(0) \).

(6) Let \( f(x, y) = \frac{xy}{x^2+y^2} \) for \((x, y) \neq (0, 0)\).
(a) Can \( f \) be defined at \((0, 0)\) so that \( f_x(0, 0) \) and \( f_y(0, 0) \) exist? Justify your answer.
(b) Can \( f \) be defined at \((0, 0)\) so that \( f \) is differentiable at \((0, 0)\)? Justify your answer.

(7) Let \( f : [-1, 1] \to \mathbb{R} \) with \( f, f', f'', f''' \) being continuous. Show that
\[
\sum_{n=2}^{\infty} \left\{ n \left[ f \left( \frac{1}{n} \right) - f \left( -\frac{1}{n} \right) \right] - 2f'(0) \right\}
\]
converges absolutely.

(8) Let \( \{f_n\} \) be a uniformly bounded sequence of continuous real-valued functions on a closed interval \([a,b]\), and let \( g_n(x) = \int_a^x f_n(t) \, dt \) for each \( x \in [a,b] \). Show that the sequence of functions \( \{g_n\} \) contains a uniformly convergent subsequence on \([a,b]\).

(9) Compute \( \int_D x \, dxdy \), where \( D \subset \mathbb{R}^2 \) is the region bounded by the curves \( x = -y^2, x = 2y - y^2, \) and \( x = 2 - 2y - y^2 \). Show your work.

(10) Let
\[
x_0 > 0, \quad x_{n+1} = \frac{1}{2} \left( x_n + \frac{4}{x_n} \right), \quad n = 0, 1, 2, 3, \ldots.
\]
Show that \( x = \lim_{n \to \infty} x_n \) exists, and find \( x \).