Do all nine problems. They all count equally. Show your work and justify your answers.

1. Define a subset $X$ of $\mathbb{R}^n$ to have property $C$ if every sequence with exactly one accumulation point in $X$ converges in $X$. (Recall that $x$ is an accumulation point of a sequence $(x_n)$ if every neighborhood of $x$ contains infinitely many $x_n$.)

(a) Give an example of a subset $X \subset \mathbb{R}^n$, for some $n \geq 1$, that does not have property $C$, together with an example of a non-converging sequence in $X$ with exactly one accumulation point.

(b) Show that any subset $X$ of $\mathbb{R}^n$ satisfying property $C$ is compact.

2. Prove that the sequence 

$$a_1 = 1, \quad a_2 = \sqrt{7}, \quad a_3 = \sqrt{7\sqrt{7}}, \quad a_4 = \sqrt{7\sqrt{7\sqrt{7}}}, \quad a_5 = \sqrt{7\sqrt{7\sqrt{7\sqrt{7}}}}, \ldots$$

converges, then find its limit.

3. Given any metric space $(X, d)$ show that $\frac{d}{1+d}$ is also a metric on $X$, and show that $(X, \frac{d}{1+d})$ shares the same family of metric balls as $(X, d)$.

4. Suppose that a function $f(x)$ is defined as the sum of series

$$f(x) = \sum_{n \geq 3} \left( \frac{1}{n-1} - \frac{1}{n+1} \right) \sin(nx).$$

(a) Explain why $f(x)$ is continuous.

(b) Evaluate

$$\int_0^\pi f(x) \, dx.$$

5. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with $h(0) = 0$, and consider the following system of equations:

$$e^x + h(y) = u^2,$$
$$e^y - h(x) = v^2.$$

Show that there exists a neighborhood $V \subset \mathbb{R}^2$ of $(1, 1)$ such that for each $(u, v) \in V$ there is a solution $(x, y) \in \mathbb{R}^2$ to this system.

6. Let $n$ be a positive integer. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Assume that $f(\bar{x}) \to 0$ whenever $\|\bar{x}\| \to \infty$. Show that $f$ is uniformly continuous on $\mathbb{R}^n$. 
7. Let \( f_n(x) \) and \( f(x) \) be continuous functions on \([0, 1]\) such that \( \lim_{n \to \infty} f_n(x) = f(x) \) for all \( x \in [0, 1] \). Answer each of the following questions. If your answer is “yes”, then provide an explanation. If your answer is “no”, then give a counterexample.

(a) Can we conclude that 
\[
\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 f(x) \, dx.
\]

(b) If in addition we assume \( |f_n(x)| \leq 2017 \) for all \( n \) and for all \( x \in [0, 1] \), can we conclude that 
\[
\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 f(x) \, dx.
\]

8. Evaluate the flux integral 
\[
\iint_{\partial V} \vec{F} \cdot \vec{n} \, dS,
\]
where the field \( \vec{F} \) is
\[
\vec{F}(x, y, z) = (xe^{xy} - 2xz + 2xy \cos^2 z) \hat{i} + (y^2 \sin^2 z - ye^{xy} + y) \hat{j} + (x^2 + y^2 + z^2) \hat{k},
\]
and \( V \) is the (bounded) solid in \( \mathbb{R}^3 \) bounded by the \( xy \)-plane and the surface \( z = 9 - x^2 - y^2 \), \( \partial V \) is the boundary surface of \( V \), and \( \vec{n} \) is the outward pointing unit normal vector on \( \partial V \).

9. A continuously differentiable function \( f \) from \([0, 1]\) to \([0, 1]\) has the properties

(a) \( f(0) = f(1) = 0 \);

(b) \( f'(x) \) is a non-increasing function of \( x \).

Prove that the arclength of the graph of \( f \) does not exceed 3.