Tier 1 Analysis Exam

January 2017

Do all nine problems. They all count equally. Show your work and justify your answers.

1. Define a subset X of $\mathbb{R}^n$ to have property $\mathcal{C}$ if every sequence with exactly one accumulation point in X converges in X. (Recall that x is an accumulation point of a sequence (x_n) if every neighborhood of x contains infinitely many x_n.)

   (a) Give an example of a subset $X \subset \mathbb{R}^n$, for some $n \geq 1$, that does not have property $\mathcal{C}$, together with an example of a non-converging sequence in X with exactly one accumulation point.

   (b) Show that any subset X of $\mathbb{R}^n$ satisfying property $\mathcal{C}$ is compact.

2. Prove that the sequence

$$a_1 = 1, \quad a_2 = \sqrt{7}, \quad a_3 = \sqrt{7\sqrt{7}}, \quad a_4 = \sqrt{7\sqrt{7\sqrt{7}}}, \quad a_5 = \sqrt{7\sqrt{7\sqrt{7\sqrt{7}}}}, \quad \ldots$$

   converges, then find its limit.

3. Given any metric space (X, d) show that $\frac{d}{1+d}$ is also a metric on X, and show that $(X, \frac{d}{1+d})$ shares the same family of metric balls as (X, d).

4. Suppose that a function $f(x)$ is defined as the sum of series

$$f(x) = \sum_{n \geq 3} \Big(\frac{1}{n-1} - \frac{1}{n+1} \Big) \sin(nx).$$

   (a) Explain why $f(x)$ is continuous.

   (b) Evaluate

$$\int_0^{\pi} f(x)\, dx.$$

5. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with $h(0) = 0$, and consider the following system of equations:

$$e^x + h(y) = u^2,$$
$$e^y - h(x) = v^2.$$

   Show that there exists a neighborhood $V \subset \mathbb{R}^2$ of $(1, 1)$ such that for each $(u, v) \in V$ there is a solution $(x, y) \in \mathbb{R}^2$ to this system.

6. Let n be a positive integer. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Assume that $f(\vec{x}) \to 0$ whenever $\|\vec{x}\| \to \infty$. Show that f is uniformly continuous on $\mathbb{R}^n$.

7. Let $f_n(x)$ and $f(x)$ be continuous functions on $[0,1]$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [0,1]$. Answer each of the following questions. If your answer is "yes", then provide an explanation. If your answer is "no", then give a counterexample.

   (a) Can we conclude that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx?$$

   (b) If in addition we assume $|f_n(x)| \leq 2017$ for all n and for all $x \in [0,1]$, can we conclude that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx?$$

8. Evaluate the flux integral $\iint_{\partial V} \vec{F} \cdot \vec{n}\, dS$, where the field $\vec{F}$ is

$$\vec{F}(x,y,z) = (xe^{xy} - 2xz + 2xy\cos^2 z)\,\vec{\imath} + (y^2\sin^2 z - ye^{xy} + y)\,\vec{\jmath} + (x^2+y^2+z^2)\,\vec{k},$$

   and V is the (bounded) solid in $\mathbb{R}^3$ bounded by the xy-plane and the surface $z = 9 - x^2 - y^2$, ∂V is the boundary surface of V, and $\vec{n}$ is the outward pointing unit normal vector on ∂V.

9. A continuously differentiable function f from $[0,1]$ to $[0,1]$ has the properties

   (a) $f(0) = f(1) = 0$;

   (b) $f'(x)$ is a non-increasing function of x.

   Prove that the arclength of the graph of f does not exceed 3.