Try to work all questions. Providing justification for your answers is crucial.

1. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is differentiable with \( f(0) = f(1) = 0 \) and
\[
\{ x : f'(x) = 0 \} \subset \{ x : f(x) = 0 \}.
\]
Show that \( f(x) = 0 \) for all \( x \in [0, 1] \).

2. Let \( (a_n) \) be a bounded sequence for \( n = 1, 2, \ldots \) such that
\[
a_n \geq (1/2)(a_{n-1} + a_{n+1}) \quad \text{for} \quad n \geq 2.
\]
Show that \( (a_n) \) converges.

3. Suppose \( K \subset \mathbb{R}^n \) is a compact set and \( f : K \to \mathbb{R} \) is continuous. Let \( \varepsilon > 0 \) be given.
Prove that there exists a positive number \( M \) such that for all \( x \) and \( y \) in \( K \) one has the inequality:
\[
|f(x) - f(y)| \leq M \|x - y\| + \varepsilon.
\]
Here \( \|\cdot\| \) denotes the Euclidean norm in \( \mathbb{R}^n \). Then give a counter-example to show that the inequality is not in general true if one takes \( \varepsilon = 0 \).

4. Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a smooth function and let \( g : \mathbb{R}^n \to \mathbb{R} \) be defined by
\[
g(x_1, \ldots, x_n) = x_1^5 + \ldots + x_n^5.
\]
Suppose \( g \circ f \equiv 0 \). Show that \( \det Df \equiv 0 \).

5. The point \((1, -1, 2)\) lies on both the surface described by the equation
\[
x^2(y^2 + z^2) = 5
\]
and on the surface described by
\[
(x - z)^2 + y^2 = 2.
\]
Show that in a neighborhood of this point, the intersection of these two surfaces can be described as a smooth curve in the form \( z = f(x), \ y = g(x) \). What is the direction of the tangent to this curve at \((1, -1, 2)\)?
6. For what smooth functions $f: \mathbb{R}^3 \to \mathbb{R}$ is there a smooth vector field $W: \mathbb{R}^3 \to \mathbb{R}^3$ such that $\text{curl } W = V$, where

$$V(x, y, z) = (y, x, f(x, y, z))?$$

For $f$ in this class, find such a $W$. Is it unique?

7. For each positive integer $n$ let $f_n: [0, 1] \to \mathbb{R}$ be a continuous function, differentiable on $(0, 1]$, such that

$$|f'_n(x)| \leq 1 + \frac{|\ln x|}{\sqrt{x}}$$

for $0 < x \leq 1$.

and such that

$$-10 \leq \int_0^1 f_n(x) \, dx \leq 10.$$ 

Prove that $\{f_n\}$ has a uniformly convergent subsequence on $[0, 1]$.

8. Define for $n \geq 2$ and $p > 0$

$$H_n(p) = \sum_{k=1}^{n} (\log k)^p$$

and $a_n(p) = \frac{1}{H_n(p)}$.

For which $p$ does $\sum_n a_n(p)$ converge?

9. Given any continuous, piecewise smooth curve $\gamma: [0, 1] \to \mathbb{R}^2$, consider the following notion of its ‘length’ $\tilde{L}$ defined through the line integral:

$$\tilde{L}(\gamma) := \int_\gamma |x| \, ds = \int_0^1 |x(t)| \sqrt{x'(t)^2 + y'(t)^2} \, dt$$

where a point in $\mathbb{R}^2$ is written as $(x, y)$ and $\gamma(t) = (x(t), y(t))$.

(a) Suppose we define a notion of distance $\tilde{d}$ between two points $p_1$ and $p_2$ in $\mathbb{R}^2$ via

$$\tilde{d}(p_1, p_2) := \inf\{\tilde{L}(\gamma): \gamma(0) = p_1, \gamma(1) = p_2\}.$$ 

Working through the definition of metric, determine which properties of a metric hold for $\tilde{d}$, and which, if any, do not.

(b) Determine the value of $\tilde{d}((1, 1), (-1, -2))$ and determine a curve achieving this infimum.