Tier I Analysis Exam, August 2014

Try to work all questions. Providing justification for your answers is crucial.

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable with $f(0) = f(1) = 0$ and
$$\{x : f'(x) = 0\} \subset \{x : f(x) = 0\} \ .$$
Show that $f(x) = 0$ for all $x \in [0, 1]$.

2. Let (a_n) be a bounded sequence for $n = 1, 2, \ldots$ such that
$$a_n \geq (1/2)(a_{n-1} + a_{n+1}) \text{ for } n \geq 2 \ .$$
Show that (a_n) converges.

3. Suppose $K \subset \mathbb{R}^n$ is a compact set and $f : K \to \mathbb{R}$ is continuous. Let $\varepsilon > 0$ be given. Prove that there exists a positive number M such that for all x and y in K one has the inequality:
$$|f(x) - f(y)| \leq M \, \|x - y\| + \varepsilon.$$
Here $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. Then give a counter-example to show that the inequality is not in general true if one takes $\varepsilon = 0$.

4. Let $f \colon \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function and let $g \colon \mathbb{R}^n \to \mathbb{R}$ be defined by
$$g(x_1, \ldots, x_n) = x_1^5 + \ldots + x_n^5.$$
Suppose $g \circ f \equiv 0$. Show that $\det Df \equiv 0$.

5. The point $(1, -1, 2)$ lies on both the surface described by the equation
$$x^2(y^2 + z^2) = 5$$
and on the surface described by
$$(x - z)^2 + y^2 = 2.$$
Show that in a neighborhood of this point, the intersection of these two surfaces can be described as a smooth curve in the form $z = f(x)$, $y = g(x)$. What is the direction of the tangent to this curve at $(1, -1, 2)$?

6. For what smooth functions $f\colon \mathbb{R}^3 \to \mathbb{R}$ is there a smooth vector field $\boldsymbol{W} : \mathbb{R}^3 \to \mathbb{R}^3$ such that curl $\boldsymbol{W} = \boldsymbol{V}$, where

$$\boldsymbol{V}(x, y, z) = (y, x, f(x, y, z))?$$

For f in this class, find such a $\mathbf{W}$. Is it unique?

7. For each positive integer n let $f_n : [0, 1] \to \mathbb{R}$ be a continuous function, differentiable on $(0, 1]$, such that

$$|f_n'(x)| \leq \frac{1 + |\ln x|}{\sqrt{x}} \quad \text{for } 0 < x \leq 1.$$

and such that

$$-10 \leq \int_0^1 f_n(x)\, dx \leq 10.$$

Prove that $\{f_n\}$ has a uniformly convergent subsequence on $[0, 1]$.

8. Define for $n \geq 2$ and $p > 0$

$$H_n(p) = \sum_{k=1}^{n} (\log k)^p \text{ and } a_n(p) = \frac{1}{H_n(p)} .$$

For which p does $\sum_n a_n(p)$ converge?

9. Given any continuous, piecewise smooth curve $\gamma : [0, 1] \to \mathbb{R}^2$, consider the following notion of its 'length' $\tilde{L}$ defined through the line integral:

$$\tilde{L}(\gamma) := \int_\gamma |x|\ ds = \int_0^1 |x(t)|\ \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

where a point in $\mathbb{R}^2$ is written as (x, y) and $\gamma(t) = (x(t), y(t))$.

(a) Suppose we define a notion of distance $\tilde{d}$ between two points p_1 and p_2 in $\mathbb{R}^2$ via

$$\tilde{d}(p_1, p_2) := \inf\{\tilde{L}(\gamma) :\ \gamma(0) = p_1,\ \gamma(1) = p_2\}.$$

Working through the definition of metric, determine which properties of a metric hold for $\tilde{d}$, and which, if any, do not.

(b) Determine the value of $\tilde{d}\big((1, 1), (-1, -2)\big)$ and determine a curve achieving this infimum.