TIER I ANALYSIS EXAM
AUGUST 2013

Solve each of the following nine problems on a separate and clearly labeled sheet of paper. Fully justify your answers.

Notation:

- $\mathbb{R}$ is the set of real numbers
- $\mathbb{R}^n$ is Euclidean space
- $|x|$ is the Euclidean length of a vector $x \in \mathbb{R}^n$; absolute value when $n = 1$.

(1) Fix positive integers n, N and a bounded set $A \subset \mathbb{R}^n$. We use the notation

$$\overline{B}(a, r) = \{x \in \mathbb{R}^n : |x - a| \leq r\}, \quad a \in \mathbb{R}^n, r \geq 0.$$

Show that there exist $a_1, a_2, \ldots, a_N \in \mathbb{R}^n$ and numbers $r_1, \ldots, r_N \in [0, +\infty)$ such that

$$A \subset \bigcup_{k=1}^{N} \overline{B}(a_k, r_k)$$

and the sum $\sum_{k=1}^{N} r_k^2$ is as small as possible. In other words, the set $\{\sum_{k=1}^{N} r_k^2 : A \text{ can be covered with a collection } (\overline{B}(a_k, r_k))_{k=1}^{N}\}$ has a smallest element.

(2) Is the sequence $(\cos(\pi\sqrt{n^2 + n}))_{n=1}^{\infty}$ convergent?

(3) For which values of $x \in \mathbb{R}$ does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{x + n}$$

converge? Is the convergence uniform on the interval $(-1, 1)$?

(4) Consider the functions $f_n : [0, 1] \to \mathbb{R}$ defined by $f_n(x) = (1 - x^n)^{2^n}$ for $x \in [0, 1]$ and $n \in \mathbb{N}$. Prove that the limit $\lim_{n\to\infty} f_n(x)$ exists for every $x \in [0, 1]$. Is the convergence uniform on $[0, 1]$?

(5) Let $f : [0, 1] \to \mathbb{R}$ be a Riemann integrable function and let $\varepsilon > 0$. Show that there exist continuous functions $g, h : [0, 1] \to \mathbb{R}$ such that $g(x) \leq f(x) \leq h(x)$ for all $x \in [0, 1]$, and

$$\int_0^1 (h(x) - g(x))\, dx < \varepsilon.$$

Is the converse statement true?

(6) Assume the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the property

$$f(x + t) \geq f(x) - t^2$$

for all real values of x and all positive values of t. Prove that f must be nondecreasing.

(7) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be everywhere differentiable, and assume that the Jacobian of f is not singular at any point $x = (x_1, x_2) \in \mathbb{R}^2$. Assume that $|f(x)| \leq 1$ whenever $|x| = 1$, and prove that in fact $|f(x)| \leq 1$ whenever $|x| \leq 1$.

(8) Compute

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-|x-y|^2}}{1+|x+y|^2}\,dx\,dy.$$

(9) Given a positive number $r \neq 1$, set $C_r = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 = r\}$. Calculate the line integral

$$\int_{C_r} \frac{x\,dy - y\,dx}{x^2+y^2},$$

where C_r is oriented counterclockwise relative to $(1,0)$.