ANALYSIS TIER 1 EXAM

January 2013

Be sure to fully justify all answers. Each of the 10 problems is worth 10 points. Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write the problem number on each sheet of paper. Please be sure that you assemble your test with the problems presented in the correct order. You have 4 hours.

1. Let X be a bounded closed subset of $\mathbb{R}^4$. Let $f\colon X \to X$ be a homeomorphism. Write f_n for the nth iterate of f if $n > 0$, for the $-n$th iterate of f^{-1} if $n < 0$, and for the identity map if $n = 0$. Thus, $f_{n+1}(x) = f\big(f_n(x)\big)$ for all $n \in \mathbb{Z}$. Write $A(x) := \big\{f_n(x) : n \in \mathbb{Z}\big\}$ for $x \in X$. Suppose that $A(x)$ is dense in X for all $x \in X$. Show that for each given $x \in X$ and all $\epsilon > 0$, there exists $n > 0$ such that for all $y \in X$, there exists $k \in [0, n]$ such that $\|f_k(y) - x\| < \epsilon$.

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function that is differentiable at 0 with $f'(0) \neq 0$. Evaluate

$$\lim_{h\to 0} \frac{f(h^2 + h^3) - f(h)}{f(h) - f(h^2 - h^3)} \,.$$

3. Determine all real x for which the following series converges:

$$\sum_{k=1}^{\infty} \frac{k^k}{k!} x^k \,.$$

You may use the fact that

$$\lim_{k\to\infty} \frac{k!}{\sqrt{2\pi k}(k/e)^k} = 1 \,.$$

4. (a) Prove that for all $a \in \mathbb{R}$,

$$\left|\sum_{n=1}^{\infty} \frac{a}{n^2 + a^2}\right| < \frac{\pi}{2} \,.$$

(b) Determine the least upper bound of the set of numbers

$$\left\{ \left|\sum_{n=1}^{\infty} \frac{a}{n^2 + a^2}\right| : a \in \mathbb{R} \right\} \,.$$

5. Let $f(x)$ be continuous in the interval $I := (0, 1)$. Define

$$D_+f(x_0) := \liminf_{h\to 0^+} \frac{f(x_0 + h) - f(x_0)}{h}\,.$$

Put

$$S := \{x \in I : D_+f(x) < 0\}\,.$$

Suppose that the set $f(I \setminus S)$ does not contain any non-empty open interval. (Note: this is $f(I \setminus S)$, not $I \setminus S$.) Prove that $f(x)$ is non-increasing on I.

6. Let $f\colon (0, 1) \to \mathbb{R}$ be a function satisfying

$$\forall x, y, \theta \in (0, 1) \quad f\big(\theta x + (1 - \theta)y\big) \le \theta f(x) + (1 - \theta)f(y)\,.$$

Prove that f is continuous on $(0, 1)$.

7. Let $f_0\colon \mathbb{R} \to \mathbb{R}$ be the periodic function with period 1 defined on one period by

$$f_0(x) := \begin{cases} x & \text{for } 0 \le x < \dfrac{1}{2}, \\ 1 - x & \text{for } \dfrac{1}{2} \le x \le 1. \end{cases}$$

Let

$$f_k(x) := \frac{1}{10^k} f_0(10^k x) \qquad \text{for } k \in \mathbb{N}$$

and let $s_k := f_0 + f_1 + \cdots + f_k$.

(a) Prove that the sequence $\{s_k\}$ converges uniformly on $\mathbb{R}$ to a continuous function $s\colon \mathbb{R} \to \mathbb{R}$.

(b) Evaluate $\int_0^1 s(x)\,dx$.

8. Let $f\colon [a, b] \to \mathbb{R}$ be a differentiable function.

(a) Prove that if f' is Riemann integrable over $[a, b]$, then

$$\int_a^b f'(x)\,dx = f(b) - f(a)\,.$$

(b) Give an example of f such that f' is not Riemann integrable.

9. Let $A := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 : \mathbf{x} \cdot \mathbf{x} = 1, \mathbf{y} \cdot \mathbf{x} = 0\}$, where "$\cdot$" is the standard dot product in $\mathbb{R}^3$ (note that A can be naturally identified with the set of all tangent vectors to the unit sphere in $\mathbb{R}^3$). Show that, as a subset of $\mathbb{R}^6$, the set A is locally the graph of a C^∞ map $\mathbb{R}^4 \to \mathbb{R}^2$ everywhere, i.e., at every point $p = (a_1, a_2, a_3, a_4, a_5, a_6) \in A$, there exist $1 \le j_1 < j_2 \le 6$ and C^∞ functions f, g defined in a neighborhood of $(a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}) \in \mathbb{R}^4$, where $\{i_1, i_2, i_3, i_4\} = \{1, \ldots, 6\} \setminus \{j_1, j_2\}$, with

$$f(a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}) = a_{j_1}\,,$$

$$g(a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}) = a_{j_2}\,,$$

and such that in a neighborhood of p, the set A is the graph

$$\big(x_{j_1}, x_{j_2}\big) = \big(f(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}), g(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})\big)\,.$$

10. Let $\mathbf{F}$ be the vector field in $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ defined by

$$\mathbf{F}(x, y, z) := \frac{xz\mathbf{j} - xy\mathbf{k}}{(y^2 + z^2)\sqrt{x^2 + y^2 + z^2}}\,.$$

(a) Show that the curl of $\mathbf{F}$ is given by

$$\nabla \times \mathbf{F}\,(x, y, z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}\,.$$

(b) Compute the line integral $\int_C \mathbf{F} \cdot \mathbf{ds}$, where C is the unit circle centered at the point $(1, 1, 1)$ that lies on the plane $x + y + z = 3$ and has the orientation from the point $\left(1 - \frac{1}{\sqrt{6}}, 1 - \frac{1}{\sqrt{6}}, 1 + \frac{2}{\sqrt{6}}\right)$ to $\left(1 - \frac{1}{\sqrt{6}}, 1 + \frac{2}{\sqrt{6}}, 1 - \frac{1}{\sqrt{6}}\right)$ to $\left(1 + \frac{2}{\sqrt{6}}, 1 - \frac{1}{\sqrt{6}}, 1 - \frac{1}{\sqrt{6}}\right)$ and back to $\left(1 - \frac{1}{\sqrt{6}}, 1 - \frac{1}{\sqrt{6}}, 1 + \frac{2}{\sqrt{6}}\right)$.