August 2012 Tier 1 Analysis Exam

- Be sure to fully justify all answers.
- Scoring: Each one of the 10 problems is worth 10 points.
- Please write on only one side of each sheet of paper. Begin each problem on a new sheet, and be sure to write the problem number on each sheet of paper.
- Please be sure that you assemble your test with the problems presented in the correct order.

1. Let

$$f_n(x) = \sum_{k=1}^{n} (x^k - x^{2k}).$$

(a) Show that f_n converges pointwise to a function f on $[0, 1]$.

(b) Show that f_n does not converge uniformly to f on $[0, 1]$.

2. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \dfrac{y^3 - \sin^3 x}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.

(a) Compute the directional derivative of f at $(0, 0)$ for an arbitrary direction (u, v).

(b) Determine whether f is differentiable at $(0, 0)$ and prove your answer.

3. Let E be a nonempty subset of a metric space and let $f : E \to \mathbb{R}$ be uniformly continuous on E. Prove that f has a unique continuous extension to the closure of E. That is, there exists a unique continuous function $g : \overline{E} \to \mathbb{R}$ such that $g(x) = f(x)$ for $x \in E$.

4. Let B_r denote the ball $B_r = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < r\}$ and let $f : B_1 \to \mathbb{R}$ be a continuously differentiable function which is zero in the complement of a compact subset of B_1. Show that

$$\lim_{\varepsilon \to 0+} \int_{B_1 \setminus B_\varepsilon} \frac{x_1 f_{x_1} + x_2 f_{x_2}}{|\mathbf{x}|^2} \, dx_1 \, dx_2$$

exists and equals $Cf(\mathbf{0})$ for a constant C which you are to determine.

5. Let E be a nonempty subset of a metric space and assume that for every $\varepsilon > 0$ E is contained in the union of finitely many balls of radius ε. Prove that every sequence in E has a subsequence which is Cauchy.

6. For which exponents $r > 0$ is the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n^2} \frac{n^{r-1}}{n^r + k^r}$$

finite? Prove your answer.

7. Let V be a neighborhood of the origin in $\mathbb{R}^2$, and $f : V \to \mathbb{R}$ be continuously differentiable. Assume that $f(0,0) = 0$ and $f(x,y) \geq -3x + 4y$ for $(x,y) \in V$. Prove that there is a neighborhood U of the origin in $\mathbb{R}^2$ and a positive number ε such that, if $(x_1, y_1), (x_2, y_2) \in U$ and $f(x_1, y_1) = f(x_2, y_2) = 0$, then

$$|y_2 - y_1| \geq \varepsilon |x_2 - x_1|.$$

8.

(a) Find necessary and sufficient conditions on functions $h, k : \mathbb{R}^2 \to \mathbb{R}^2$ such that, given any smooth $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ of the form $\mathbf{F} = (F_1(y,z), F_2(x,z), 0)$ and whose divergence is zero, there is a smooth $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$ of the form $\mathbf{G} = (G_1, G_2, 0)$ such that $\nabla \times \mathbf{G} = \mathbf{F}$ in $\mathbb{R}^3$ and $\mathbf{G} = (h, k, 0)$ on $z = 0$. ($\nabla \times G$ is the curl of the vector field G.)

(b) Let $\mathbf{F}$ be as in (a) and evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dA$$

where S is the hemisphere

$$\{(x,y,z) : \ x^2 + y^2 + z^2 = 1, \ 0 \leq z \leq 1\},$$

$\mathbf{N}$ is the unit normal on S in the positive z-direction, and dA is the surface area element.

9. Let $f = (f^1, \ldots, f^n)$ map an open set U in $\mathbb{R}^n$ into $\mathbb{R}^n$ be C^1 and suppose that, for some $\overline{x} \in U$ the matrix $f'(\overline{x})$ is negative definite (an $n \times n$ matrix A is negative definite if $\xi \cdot A\xi < 0$ for all nonzero $\xi \in \mathbb{R}^n$). Show that there is a positive number ε and a neighborhood V of $\overline{x}$ such that, if $y_1, \ldots, y_n$ are any n points in V and if A is the $n \times n$ matrix whose i-th row is $\nabla f^i(y_i)$, then $\xi \cdot A\xi \leq -\varepsilon |\xi|^2$ for all $\xi \in \mathbb{R}^n$.

10. Let f be a C^1 mapping of an open set $U \subset \mathbb{R}^n$ into $\mathbb{R}^n$ and suppose that $f(\bar{x}) = 0$ for some $\overline{x} \in U$ and that $f'(\bar{x})$ is negative definite. Show that there is a neighborhood W of $\bar{x}$ and a positive number δ such that, if a sequence $\{x_k\}_{k=0}^{\infty}$ is generated from the recursion

$$x_{k+1} = x_k + \delta f(x_k)$$

with $x_0 \in W$, then each x_k is in W and $x_k \to \overline{x}$ as $k \to \infty$. You may use here the result stated in problem 9 without having solved problem 9.