Tier I Analysis

January 3, 2012

Solve all 10 problems, justifying all answers.

1. For \((x, y) \in \mathbb{R}^2\), let
   \[
   f(x, y) = \begin{cases} 
   [(2x^2 - y)(y - x^2)]^{1/4}, & \text{for } x^2 \leq y \leq 2x^2; \\
   0, & \text{otherwise.}
   \end{cases}
   \]
   Show that all directional derivatives of \(f\) exist at \((0, 0)\), but \(f\) is not differentiable at \((0, 0)\).

2. Let \((a_n)_{n=1}^{\infty}\) be a monotonically decreasing sequence of positive real numbers and assume \(\sum_{n=1}^{\infty} a_n < \infty\). Show that \(\lim_{n \to \infty} n a_n = 0\).

3. For \((x, y) \in \mathbb{R}^2\), let \(f(x, y) = 5x^2 + xy^3 - 3x^2y\). Find the critical points for \(f\), and for each critical point determine whether it is a local maximum, local minimum or a saddle point.

4. Establish the convergence or divergence of the improper integral
   \[
   \int_{0}^{\infty} \sin(x^2) \, dx.
   \]

5. Let \((f_n)_{n=1}^{\infty}\) and \((g_n)_{n=1}^{\infty}\) be sequences of functions from \(\mathbb{R}\) to \(\mathbb{R}\). Assume that
   (a) the partial sums \(F_n = \sum_{k=1}^{n} f_k\) are uniformly bounded,
   (b) \(g_n \to 0\) uniformly,
   (c) \(g_1(x) \geq g_2(x) \geq g_3(x) \geq \ldots\), for all \(x \in \mathbb{R}\).
   Prove that \(\sum_{n=1}^{\infty} f_n g_n\) converges uniformly. \textbf{Hint:} Use the fact that
   \[
   \sum_{p=q}^{q-1} f_p g_{n+1} + F_q g_q - F_{q-1} g_p.
   \]
   (If you make use of this fact, you are required to prove it.)
6. Let

\[ X = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^4 + x_2^4 + x_3^4 + x_4^4 = 64 \text{ and } x_1 + x_2 + x_3 + x_4 = 8\}. \]

For which points \( p \in X \) is it possible to find a product of open intervals \( V = I_1 \times I_2 \times I_3 \times I_4 \) containing \( p \) such that \( X \cap V \) is the graph of a function expressing two of the variables \( x_1, x_2, x_3, x_4 \) in terms of the other two? If there are any points in \( X \) where this is not possible, explain why not.

7. Let \( F : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \) be given by

\[ F(x,y) = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right) \]

and suppose for \( j = 1, 2 \) we have one-to-one \( C^1 \) maps \( \gamma_j : [0,1] \to \mathbb{R}^2 \), such that \( \gamma_j(0) = p \) and \( \gamma_j(1) = q \) for some \( p, q \in \mathbb{R}^2 \setminus \{(0,0)\} \). Assume furthermore that \( \gamma_j(t) \neq (0,0) \) and \( \gamma_j'(t) \neq 0 \) for all \( t \in [0,1], \) and \( \gamma_1((0,1)) \cap \gamma_2((0,1)) = \emptyset \). Carefully demonstrate that

\[ \int_{\Gamma_1} F \cdot T_1 \, ds = \int_{\Gamma_2} F \cdot T_2 \, ds + 2\pi k, \text{ for either } k = 0, 1 \text{ or } -1, \]

where \( \Gamma_j := \gamma_j([0,1]) \), \( T_j \) denotes the unit tangent vector to \( \gamma_j \) and \( s \) is the arc length parameter.

8. Suppose \( \phi : \mathbb{R}^2 \to \mathbb{R} \) is any \( C^1 \) function, and let \( g : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R} \) be given by \( g(x,y) := \ln(\sqrt{x^2 + y^2}) \). Prove that

\[ \lim_{\epsilon \to 0} \int_{\partial B_\epsilon} (\phi \nabla g \cdot \mathbf{n} - g \nabla \phi \cdot \mathbf{n}) \, ds = 2\pi \phi(0,0), \]

where \( B_\epsilon \) denotes the disk centered at \((0,0)\) of radius \( \epsilon \) and \( \mathbf{n} \) denotes the outer unit normal to the circle \( \partial B_\epsilon \).

9. Let \( \alpha \in (0,1] \). A function \( f : [0,1] \to \mathbb{R} \) is defined to be \( \alpha \)-Hölder continuous if

\[ N_\alpha(f) := \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} : x, y \in [0,1], x \neq y \right\} < \infty. \]

(a) Suppose \( (f_n)_{n=1}^\infty \) is a sequence of functions from \([0,1]\) to \( \mathbb{R} \) such that for all \( n = 1, 2, \ldots \) we have \( N_\alpha(f_n) \leq 1 \) and \( |f_n(x)| \leq 1 \) for all \( x \in [0,1] \). Show that \( (f_n)_{n=1}^\infty \) has a uniformly convergent subsequence.

(b) Show that (a) is false if the condition “\( N_\alpha(f_n) \leq 1 \)” is replaced by “\( N_\alpha(f_n) < \infty \)”.
10. Assume \( f : \mathbb{R}^n \to \mathbb{R} \) is a continuous function such that

(a) there exist points \( x_0 \) and \( x_1 \in \mathbb{R}^n \) with \( f(x_0) = 0 \) and \( f(x_1) = 3 \),
(b) there exist positive constants \( C_1 \) and \( C_2 \) such that \( f(x) \geq C_1|x| - C_2 \) for all \( x \in \mathbb{R}^n \).

Let \( S := \{ x \in \mathbb{R}^n : f(x) < 2 \} \) and let \( K := \{ x \in \mathbb{R}^n : f(x) \leq 1 \} \). Define the distance from \( K \) to \( \partial S \) (the boundary of \( S \)) by the formula

\[
\text{dist}(K, \partial S) := \inf_{p \in K, q \in \partial S} |p-q|.
\]

Prove that \( \text{dist}(K, \partial S) > 0 \). Then give an example of a continuous function \( f \) satisfying (a), but \( \text{dist}(K, \partial S) = 0 \).