Tier I Analysis

January 3, 2012

Solve all 10 problems, justifying all answers.

1. For $(x, y) \in \mathbb{R}^2$, let

$$f(x,y) = \begin{cases} [(2x^2 - y)(y - x^2)]^{1/4}, & \text{for } x^2 \leq y \leq 2x^2; \\ 0, \text{ otherwise.} \end{cases}$$

   Show that all directional derivatives of f exist at $(0, 0)$, but f is not differentiable at $(0, 0)$.

2. Let $(a_n)_{n=1}^{\infty}$ be a monotonically decreasing sequence of positive real numbers and assume $\sum_{n=1}^{\infty} a_n < \infty$. Show that $\lim_{n\to\infty} na_n = 0$.

3. For $(x, y) \in \mathbb{R}^2$, let $f(x, y) = 5x^2 + xy^3 - 3x^2y$. Find the critical points for f, and for each critical point determine whether it is a local maximum, local minimum or a saddle point.

4. Establish the convergence or divergence of the improper integral

$$\int_0^{\infty} \sin(x^2)\, dx.$$

5. Let $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ be sequences of functions from $\mathbb{R}$ to $\mathbb{R}$. Assume that

   (a) the partial sums $F_n = \sum_{k=1}^{n} f_k$ are uniformly bounded,

   (b) $g_n \to 0$ uniformly,

   (c) $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$, for all $x \in \mathbb{R}$.

   Prove that $\sum_{n=1}^{\infty} f_n g_n$ converges uniformly. *Hint:* Use the fact that

$$\sum_{p}^{q} f_n g_n = \sum_{p}^{q-1} F_n(g_n - g_{n+1}) + F_q g_q - F_{p-1} g_p.$$

   (If you make use of this fact, you are required to prove it.)

6. Let

$$X = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^4 + x_2^4 + x_3^4 + x_4^4 = 64 \text{ and } x_1 + x_2 + x_3 + x_4 = 8\}.$$

For which points $p \in X$ is it possible to find a product of open intervals $V = I_1 \times I_2 \times I_3 \times I_4$ containing p such that $X \cap V$ is the graph of a function expressing two of the variables x_1, x_2, x_3, x_4 in terms of the other two? If there are any points in X where this is not possible, explain why not.

7. Let $\mathbf{F} : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ be given by

$$\mathbf{F}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

and suppose for $j = 1, 2$ we have one-to-one C^1 maps $\gamma_j : [0, 1] \to \mathbb{R}^2$, such that $\gamma_j(0) = p$ and $\gamma_j(1) = q$ for some $p, q \in \mathbb{R}^2 \setminus \{(0,0)\}$. Assume furthermore that $\gamma_j(t) \neq (0,0)$ and $\gamma_j'(t) \neq 0$ for all $t \in [0, 1]$, and $\gamma_1((0,1)) \cap \gamma_2((0,1)) = \emptyset$. Carefully demonstrate that

$$\int_{\Gamma_1} \mathbf{F} \cdot \mathbf{T}_1 \, ds = \int_{\Gamma_2} \mathbf{F} \cdot \mathbf{T}_2 \, ds + 2\pi k, \text{ for either } k = 0, 1 \text{ or } -1,$$

where $\Gamma_j := \gamma_j([0, 1])$, $\mathbf{T}_j$ denotes the unit tangent vector to γ_j and s is the arc length parameter.

8. Suppose $\phi : \mathbb{R}^2 \to \mathbb{R}$ is any C^1 function, and let $g : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be given by $g(x, y) := \ln\left(\sqrt{x^2 + y^2}\right)$. Prove that

$$\lim_{\epsilon \to 0} \int_{\partial B_\epsilon} (\phi \nabla g \cdot \mathbf{n} - g \nabla \phi \cdot \mathbf{n}) \, ds = 2\pi \phi(0, 0),$$

where B_ϵ denotes the disk centered at $(0,0)$ of radius ϵ and $\mathbf{n}$ denotes the outer unit normal to the circle ∂B_ϵ.

9. Let $\alpha \in (0, 1]$. A function $f : [0, 1] \to \mathbb{R}$ is defined to be α-Hölder continuous if

$$N_\alpha(f) := \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} : x, y \in [0, 1], x \neq y \right\} < \infty.$$

   (a) Suppose $(f_n)_{n=1}^\infty$ is a sequence of functions from $[0, 1]$ to $\mathbb{R}$ such that for all $n = 1, 2, \ldots$ we have $N_\alpha(f_n) \leq 1$ and $|f_n(x)| \leq 1$ for all $x \in [0, 1]$. Show that $(f_n)_{n=1}^\infty$ has a uniformly convergent subsequence.

   (b) Show that (a) is false if the condition "$N_\alpha(f_n) \leq 1$" is replaced by "$N_\alpha(f_n) < \infty$."

10. Assume $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that

(a) there exist points x_0 and $x_1 \in \mathbb{R}^n$ with $f(x_0) = 0$ and $f(x_1) = 3$,

(b) there exist positive constants C_1 and C_2 such that $f(x) \geq C_1|x| - C_2$ for all $x \in \mathbb{R}^n$.

Let $S := \{x \in \mathbb{R}^n : f(x) < 2\}$ and let $K := \{x \in \mathbb{R}^n : f(x) \leq 1\}$. Define the distance from K to ∂S (the boundary of S) by the formula

$$\operatorname{dist}(K, \partial S) := \inf_{p \in K, q \in \partial S} |p - q|.$$

Prove that $\operatorname{dist}(K, \partial S) > 0$. Then give an example of a continuous function f satisfying (a), but $\operatorname{dist}(K, \partial S) = 0$.