TIER 1 ANALYSIS EXAM, JANUARY 2011

- Solve the following 10 problems, justifying all answers.
- Write the solution of each problem on a separate, clearly identified page.

(1) In this problem we use the notation

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

for the Euclidean norm of a vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Let $f : [0,1] \to \mathbb{R}^n$ be a continuous function. Show that

$$\left| \int_0^1 f(t)\, dt \right| \leq \int_0^1 |f(t)|\, dt.$$

(2) A sequence

$$A_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}, \quad n \geq 1,$$

of real matrices is said to converge if the sequences $(a_n)_{n=1}^\infty$, $(b_n)_{n=1}^\infty$, $(c_n)_{n=1}^\infty$, $(d_n)_{n=1}^\infty$ converge. Fix a real matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and define

$$A_n = A - \frac{1}{3!}A^3 + \cdots + \frac{(-1)^n}{(2n+1)!}A^{2n+1}, \quad n \geq 1.$$

Show that the sequence $(A_n)_{n=1}^\infty$ converges. (The limit is denoted $\sin(A)$.)

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the following property: for every positive integer n, and every $x, y \in \mathbb{R}$ such that $|x| + |y| > n^2$ and $|x - y| < 1/n^2$, we have $|f(x) - f(y)| < 1/n$. Show that f is uniformly continuous.

(4) Determine the area enclosed by the curve

$$c(t) = (3\cos t - \cos(3t), 3\sin t - \sin(3t)), \quad t \in [0, 2\pi].$$

You can take it for granted that this is a simple curve.

(5) Determine the volume of the solid $\{(x, y, z) : \sqrt{x} + \sqrt{y} + \sqrt{z} \leq 1, x, y, z \geq 0\}$.

(6) Consider a differentiable, strictly decreasing function $f : [0,1] \to [0,1]$, and let $a \in [0,1]$ satisfy $f(a) = a$. (There obviously exists exactly one such point.) Assume that $f'(a) < -1$. Define a sequence $(x_n)_{n=0}^\infty$ by setting $x_0 = 0$ and $x_{n+1} = f(x_n)$ for $n \geq 0$. Show that the sequence $(x_n)_{n=0}^\infty$ does not converge.

(7) Show that there exists a differentiable function $f(x)$ defined in a neighborhood of $x_0 = \sqrt{2}$ such that $x^{f(x)} = f(x)$.

(8) Given a sequence $f_n : [0,1] \to [0,1]$ of continuous functions, define $g_n : [0,1] \to \mathbb{R}$ by setting

$$g_n(x) = \int_0^1 \frac{f_n(t)}{(t - x)^{1/3}}\, dt, \quad x \in [0,1].$$

(Observe that this is an improper Riemann integral.) Show that the sequence $(g_n)_{n \in \mathbb{N}}$ has a uniformly convergent subsequence.

(9) A sequence $(a_n)_{n=1}^{\infty}$ satisfies the inequality $|\sum_{k=1}^{n} a_k| \leq \sqrt{n}$ for all $n \geq 1$. Show that the series

$$\sum_{k=1}^{\infty} \frac{a_k}{k}$$

converges.

(10) Show that there does not exist a sequence $I_n = [a_n, b_n], n = 1, 2, \ldots$ of nonempty, pairwise disjoint intervals such that $\bigcup_n I_n = [0, 1]$.