Department of Mathematics–Tier 1 Analysis Examination

January 7, 2010

Notation: In problems 2, 3, and 9 the notation ∇f denotes the $n-$tuple of first-order partial derivatives of a function f mapping an open set in $\mathbf{R}^n$ into $\mathbf{R}$.

1. Let E be a closed and bounded set in $\mathbf{R}^n$ and let $f : E \to \mathbf{R}$. Suppose that for each $x \in E$ there are positive numbers r and M depending on x such that $f(y) \geq -M$ for all $y \in E$ satisfying $|y - x| < r$. Prove that there is a positive number $\overline{M}$ such that $f(y) \geq -\overline{M}$ for all $y \in E$.

2. Let V be a convex open set in $\mathbf{R}^2$and let $f : V \to \mathbf{R}$ be continuously differentiable in V. Show that if there is a positive number M such that $|\nabla f(x)| \leq M$ for all $x \in V$, then there is a a positive number L such that
$$|f(x) - f(y)| \leq L|x - y|$$
for all $x, y \in V$.

   Is this result still true if V is instead assumed to be open and connected? Prove or disprove with a counterexample.

3. Let f be a C^2 mapping of a neighborhood of a point $x_0 \in \mathbf{R}^n$ into $\mathbf{R}$. Assume that x_0 is a critical point of f and that the second derivative matrix $f''(x_0)$ is positive definite. Prove that there is a neighborhood V of x_0 such that zero is an interior point of the set $\{\nabla f(y) : y \in V\}$.

4. Suppose that F and G are differentiable maps of a neighborhood V of a point $x_0 \in \mathbf{R}^n$ into $\mathbf{R}$ and that $F(x_0) = G(x_0)$. Next let $f : V \to \mathbf{R}$ and suppose that $F(x) \leq f(x) \leq G(x)$ for all $x \in V$. Prove that f is differentiable at $x = x_0$.

5. Let $\{g_k\}_{k=1}^{\infty}$ be a sequence of continuous real-valued functions on $[0, 1]$. Assume that there is a number M such that $|g_k(x)| \leq M$ for every k and every $x \in [0, 1]$ and also that there is a continuous real-valued function g on $[0, 1]$ such that
$$\int_0^1 g_k(x)p(x)dx \to \int_0^1 g(x)p(x)dx \quad \text{as } k \to \infty$$
for every polynomial p. Prove that $|g(x)| \leq M$ for every $x \in [0, 1]$ and that
$$\int_0^1 g_k(x)f(x)dx \to \int_0^1 g(x)f(x)dx$$
for every continuous f.

6. Let $\{a_k\}$ be a sequence of positive numbers converging to a positive number a. Prove that $(a_1a_2\cdots a_k)^{1/k}$ also converges to a.

7. Compute rigorously $\displaystyle\lim_{n\to\infty} \left[\frac{1}{n + \sqrt{n}} \sum_{k=1}^{n} \sin\left(\frac{k}{n}\right)\right]$.

8. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of numbers satisfying $|a_k| \leq k^2/2^k$ for all k and let $f : [0, 1] \times \mathbf{R} \to \mathbf{R}$ be continuous. Prove that the following limit exists:
$$\lim_{n\to\infty} \int_0^1 f\big(x, \sum_{k=1}^{n} a_k x^k\big)dx \ .$$

9. Let $g : \mathbf{R}^2 \to (0, \infty)$ be C^2 and define $\Sigma \subset \mathbf{R}^3$ by $\Sigma = \{(x_1, x_2, g(x_1, x_2)) : x_1^2 + x_2^2 \leq 1\}$. Assume that Σ is contained in the ball B of radius R centered at the origin in $\mathbf{R}^3$ and that each ray through the origin intersects Σ at most once. Let E be the set of points $x \in \partial B$ such that the ray joining the origin to x intersects Σ exactly once. Derive an equation relating the area of E, R, and the integral
$$\int_{\Sigma} \nabla\Gamma(x) \cdot N(x)dS$$
where $\Gamma(x) = 1/|x|$, $N(x)$ is a unit normal vector on Σ, and dS represents surface area.