Tier 1 Analysis Exam

August, 2009

Show all work, and justify all answers.

This exam has 9 problems.

$\mathbf{R}$ will denote the real numbers, and $||\cdot||$ will denote the usual Euclidean norm.

1. Define the statement: "$f : \mathbf{R}^2 \to \mathbf{R}$ *is differentiable at* $(0,0)$," and show that the function $f(x,y) = x|y|^{\frac{1}{2}}$ is differentiable at $(0,0)$.

2. Show that the series

$$2\sin\frac{1}{3x} + 4\sin\frac{1}{9x} + \cdots + 2^n \sin\frac{1}{3^n x} + \cdots$$

converges absolutely for $x \neq 0$ but does not converge uniformly on any interval $(0,\epsilon)$ with $\epsilon > 0$.

3. Let $V(n,r)$ be the volume of the ball $\{x \in \mathbf{R}^n : ||x|| \leq r\}$.

(a) Show that $V(n,r) = c_n r^n$ for some constant c_n depending only on n.

(b) Find $\lim_{n\to\infty} c_n$.

4. Suppose that $x \neq 0$. Show that

$$\lim_{n\to\infty} \frac{1 + \cos(x/n) + \cos(2x/n) + \cdots + \cos((n-1)x/n)}{n} = \frac{\sin(x)}{x}$$

5. Let $X = \{x = (x_1, x_2, x_3, x_4) \in \mathbf{R}^4 : x_1^2 + x_2^2 + x_3^2 - x_4^2 = 2, \text{ and } x_1 + x_2 + x_3 + x_4 = 2\}$. For which points $p \in X$ is it possible to find a product of open intervals $V = I_1 \times I_2 \times I_3 \times I_4$ containing p such that $X \cap V$ is the graph of a function expressing some of the variables x_1, x_2, x_3, x_4 in terms of the others? If there are any points in X where this is not possible, explain why not.

6. Let a and b be two points of $\mathbf{R}^2$. Let $\sigma_n : [0,1] \to \mathbf{R}^2$ be a sequence of continuously differentiable constant speed curves with $||\sigma_n'(t)|| = L_n$ for all $t \in [0,1]$ and $\sigma_n(0) = a$ and $\sigma_n(1) = b$ for all n. Suppose that $\lim_{n\to\infty} L_n = ||b-a||$. Show that σ_n converges uniformly to σ, where $\sigma(t) = a + t(b-a)$ for $t \in [0,1]$.

7. Let $f : \mathbf{R} \to \mathbf{R}$ be a function; and let its n-th derivative, denoted $f^{(n)}$, exist for all n. Suppose that the sequence $f^{(n)}$, $n = 1, 2, 3, \ldots$ converges uniformly on compact subsets to a function g. Show that there is a constant c such that $g(x) = c\, e^x$.

8. Let $M = \{(x,y,z) \in \mathbf{R}^3 : y = 9 - x^2, y \geq 0, \text{ and } 0 \leq z \leq 1\}$. Orient M so that the unit normal $\vec{n}$ is in the positive y-direction along the line $x = 0, y = 3$. Let $\vec{F}$ be the vector field on $\mathbf{R}^3$ given by $\vec{F} = (2x^3yz, y + 3x^2y^2z, -6x^2yz^2)$.

(a) What is $\operatorname{div} \vec{F}$?

(b) Use the Divergence Theorem to express the flux of $\vec{F}$ across M (that is, $\int_M \vec{F} \cdot \vec{n}\, dS$, where dS is the surface area element) in terms of some other (easier) integrals.

(c) Calculate $\int_M \vec{F} \cdot \vec{n}\, dS$ by evaluating the integrals in part (b).

9. Let (X,d) be a compact metric space. Suppose that $h : X \to Y \subset X$ is a map which preserves d, or in other words, $d(h(x_1), h(x_2)) = d(x_1, x_2)$ for all $x_1, x_2 \in X$. Show that $Y = X$.