Try to work all questions. They all are worth the same amount.

1. Assume $f$ and $g$ are uniformly continuous functions from $\mathbb{R}^1 \to \mathbb{R}^1$. If both $f$ and $g$ are also bounded, show that $fg$ is also uniformly continuous. Then give an example to show that in general, if $f$ and $g$ are both uniformly continuous but not both bounded, then the product is not necessarily uniformly continuous. (Verify clearly that your counter-example is not uniformly continuous.)

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are $C^2$ functions, $h : \mathbb{R}^2 \to \mathbb{R}$ is a $C^1$ function and assume

$$f(0) = g(0) = 0, \quad f'(0) = g'(0) = h(0,0) = 1.$$ 

Show that the function $H : \mathbb{R}^2 \to \mathbb{R}$ given by

$$H(x,y) := \int_0^{f(x)} \int_0^{g(y)} h(s,t) \, ds \, dt + \frac{1}{2} x^2 + by^2$$

has a local minimum at the origin provided that $b > \frac{1}{2}$ while it has a saddle at the origin if $b < \frac{1}{2}$.

3. Let $H = \{(x,y,z) | z > 0 \text{ and } x^2 + y^2 + z^2 = R^2\}$, i.e. the upper hemisphere of the sphere of radius $R$ centered at 0 in $\mathbb{R}^3$. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field

$$F(x,y,z) = \left\{ x^2(y^2 - z^3), \ xy^4 + e^{-x^2}y^4 + y, \ x^2y(y^2x^3 + 3)z + e^{-x^2-y^2} \right\}$$

Find $\int_H F \cdot \hat{n} \, dS$ where $\hat{n}$ is the outward (upward) pointing unit surface normal and $dS$ is the area element.

4. Let $D$ be the square with vertices $(2,2)$, $(3,3)$, $(2,4)$, $(1,3)$. Calculate the improper integral

$$\int \int_D \ln(y^2 - x^2) \, dxdy .$$
5. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^1$ is a $C^4$ function with the property that at some point $(x_0, y_0) \in \mathbb{R}^2$ all of the first and second order partial derivatives of $f$ vanish. Suppose also that at least one partial derivative of third order does not vanish at $(x_0, y_0)$. Prove that $f$ can have neither a local maximum nor a local minimum at this critical point.

6. Prove that the series $\sum_{n=1}^{\infty} \frac{nx}{1 + n^2 \log^2(n)x^2}$ converges uniformly on $[\varepsilon, \infty)$ for any $\varepsilon > 0$.

7. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is of class $C^1$, that $f(0, 0, 0) = 0$, and

$$f_2(0, 0, 0) \neq 0, \quad f_3(0, 0, 0) \neq 0, \quad \text{and} \quad f_2(0, 0, 0) + f_3(0, 0, 0) \neq -1$$

where $f_k = \frac{\partial f}{\partial x_k}$. Show that the system

$$f(x_1, f(x_1, x_2, x_3), x_3) = 0$$

$$f(x_1, x_2, f(x_1, x_2, x_3)) = 0$$

defines $C^1$ functions $x_2 = \varphi(x_1)$, and $x_3 = \psi(x_1)$ for $x_1$ in a neighborhood of 0 satisfying

$$f(x_1, f(x_1, \varphi(x_1), \psi(x_1)), \psi(x_1)) = 0$$

$$f(x_1, \varphi(x_1), f(x_1, \varphi(x_1), \psi(x_1))) = 0.$$ 

8. For each $b \in [1, e]$, consider the sequence of real numbers governed by the recurrence relation

$$a_{n+1} = \left(\sqrt[3]{b}\right)^{a_n} \quad \text{for} \quad n = 0, 1, 2 \ldots \quad \text{with} \quad a_0 = \sqrt[3]{b} \quad i.e. \quad \{\sqrt[3]{b}, \sqrt[3]{\sqrt[3]{b}}, \sqrt[3]{\sqrt[3]{\sqrt[3]{b}}}, \sqrt[3]{\sqrt[3]{\sqrt[3]{\sqrt[3]{b}}}}, \ldots\}.$$ 

Show that this sequence converges and find the limit.
9. For each positive integer $n$, define $x_n : [-1, 1] \rightarrow \mathbb{R}$ by

$$x_n(t) = \begin{cases} 
-1 & \text{if } -1 \leq t \leq -1/n \\
nt & \text{if } -1/n < t < 1/n \\
1 & \text{if } 1/n \leq t \leq 1
\end{cases}$$

(a) Show that $\{x_n\}$ is a Cauchy sequence in the metric space $(\mathcal{C}([-1, 1]), d)$, where $\mathcal{C}([-1, 1])$ denotes the set of continuous functions defined on $[-1, 1]$ and $d$ denotes the metric given by

$$d(x, y) = \int_{-1}^{1} |x(t) - y(t)| \, dt .$$

(b) Show that $(\mathcal{C}([-1, 1]), d)$ is not complete.