Tier I Analysis Exam

January 2009

Try to work all questions. They all are worth the same amount.

1. Assume f and g are uniformly continuous functions from $\mathbb{R}^1 \to \mathbb{R}^1$. If both f and g are also bounded, show that fg is also uniformly continuous. Then give an example to show that in general, if f and g are both uniformly continuous but not both bounded, then the product is not necessarily uniformly continuous. (Verify clearly that your counter-example is not uniformly continuous.)

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are $\mathcal{C}^2$ functions, $h : \mathbb{R}^2 \to \mathbb{R}$ is a $\mathcal{C}^1$ function and assume
$$f(0) = g(0) = 0, \quad f'(0) = g'(0) = h(0,0) = 1.$$
Show that the function $H : \mathbb{R}^2 \to \mathbb{R}$ given by
$$H(x,y) := \int_0^{f(x)} \int_0^{g(y)} h(s,t)\, ds\, dt \ + \ \frac{1}{2}x^2 + by^2$$
has a local minimum at the origin provided that $b > \frac{1}{2}$ while it has a saddle at the origin if $b < \frac{1}{2}$.

3. Let $H = \{(x,y,z) \,|\, z > 0 \text{ and } x^2 + y^2 + z^2 = R^2\}$, i.e. the upper hemisphere of the sphere of radius R centered at 0 in $\mathbb{R}^3$. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field
$$F(x,y,z) = \left\{x^2(y^2 - z^3),\ xzy^4 + e^{-x^2}y^4 + y,\ x^2y(y^2x^3 + 3)z + e^{-x^2-y^2}\right\}$$
Find $\displaystyle\int_H F \cdot \hat{n}\, dS$ where $\hat{n}$ is the outward (upward) pointing unit surface normal and dS is the area element.

4. Let D be the square with vertices (2,2), (3,3), (2,4), (1,3). Calculate the improper integral
$$\int\int_D \ln(y^2 - x^2) dxdy\ .$$

5. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^1$ is a $\mathcal{C}^4$ function with the property that at some point $(x_0, y_0) \in \mathbb{R}^2$ all of the first and second order partial derivatives of f vanish. Suppose also that at least one partial derivative of third order does not vanish at (x_0, y_0). Prove that f can have neither a local maximum nor a local minimum at this critical point.

6. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{nx}{1 + n^2 \log^2(n) x^2}$ converges uniformly on $[\varepsilon, \infty)$ for any $\varepsilon > 0$.

7. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is of class $\mathcal{C}^1$, that $f(0,0,0) = 0$, and

$$f_2(0,0,0) \neq 0, \qquad f_3(0,0,0) \neq 0, \quad \text{and} \quad f_2(0,0,0) + f_3(0,0,0) \neq -1$$

where $f_k = \frac{\partial f}{\partial x_k}$. Show that the system

$$\begin{aligned} f(x_1, f(x_1, x_2, x_3), x_3) &= 0 \\ f(x_1, x_2, f(x_1, x_2, x_3)) &= 0 \end{aligned}$$

defines $\mathcal{C}^1$ functions $x_2 = \varphi(x_1)$, and $x_3 = \psi(x_1)$ for x_1 in a neighborhood of 0 satisfying

$$\begin{aligned} f(x_1, f(x_1, \varphi(x_1), \psi(x_1)), \psi(x_1)) &= 0 \\ f(x_1, \varphi(x_1), f(x_1, \varphi(x_1), \psi(x_1))) &= 0. \end{aligned}$$

8. For each $b \in [1, e]$, consider the sequence of real numbers governed by the recurrence relation

$$a_{n+1} = \left(\sqrt[b]{b}\right)^{a_n} \quad \text{for } n = 0, 1, 2 \ldots \quad \text{with } a_0 = \sqrt[b]{b} \quad i.e. \quad \{\sqrt[b]{b}, \sqrt[b]{b}^{\sqrt[b]{b}}, \sqrt[b]{b}^{\sqrt[b]{b}^{\sqrt[b]{b}}}, \sqrt[b]{b}^{\sqrt[b]{b}^{\sqrt[b]{b}^{\sqrt[b]{b}}}}, \ldots\}.$$

Show that this sequence converges and find the limit.

9. For each positive integer n, define $x_n : [-1,1] \to \mathbb{R}$ by

$$x_n(t) = \begin{cases} -1 & \text{if } -1 \le t \le -1/n \\ nt & \text{if } -1/n < t < 1/n \\ 1 & \text{if } 1/n \le t \le 1 \end{cases}$$

(a) Show that $\{x_n\}$ is a Cauchy sequence in the metric space $(\mathcal{C}([-1,1]), d)$, where $\mathcal{C}([-1,1])$ denotes the set of continuous functions defined on $[-1,1]$ and d denotes the metric given by

$$d(x,y) = \int_{-1}^{1} |x(t) - y(t)| \, dt \, .$$

(b) Show that $(\mathcal{C}([-1,1]), d)$ is not complete.