TIER I ANALYSIS EXAM

August 2008

Do all 10 problems; they all count equally.

Problem 1. Suppose that $I_1, \ldots, I_n$ are disjoint closed subintervals of $\mathbb{R}$. If f is uniformly continuous on each of the intervals, prove that f is uniformly continuous on $\bigcup_{j=1}^{n} I_j$.

Does this still hold if the intervals are open?

Problem 2. Suppose that f is a continuous function from $[0,1]$ into $\mathbb{R}$ and that $\int_0^1 f(x)\,dx = 0$.

Prove that there is at least one point, x_0, in $[0,1]$, where $f(x_0) = 0$.

Does this still hold if f is Riemann integrable but not continuous?

Problem 3. Suppose that f is a continuous function from $[a,b]$ into $\mathbb{R}$ which has the property that, for any point $x \in [a,b]$, there is another point $x' \in [a,b]$ such that $|f(x')| \leq |f(x)|/2$.

Prove that there exists a point $x_0 \in [a,b]$ where f vanishes, that is, $f(x_0) = 0$.

Problem 4. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x,y) = (\sin(y) - x, e^x - y)\,, \quad g(x,y) = (xy, x^2 + y^2)\,.$$

Compute $(g \circ f)'(0,0)$.

Problem 5. Prove that there exists a positive number θ_0 such that the following holds: For each $\theta \in [0, \theta_0]$, there exist real numbers x and y (with $xy > -1$) such that

$$2x + y + e^{xy} = \cos(\theta^3)\,, \quad \text{and} \quad \log(1+xy) + \sin(x + y^2) = \sqrt{\theta}\,.$$

(*Hint*: First evaluate the left side of each of these two equations for $x = y = 0$.)

Problem 6. If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent series of real numbers it is well-known that their *Cauchy product series* $\sum_{n=0}^{\infty} c_n$ also converges, where

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_0 b_n\,, \quad n = 0, 1, \ldots\,.$$

Show that this assertion is no longer true if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are merely conditionally convergent.

Problem 7. (a.) Let C be the line segment joining the points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$.

Prove that $\int_C x\,dy - y\,dx = x_1 y_2 - x_2 y_1$.

(b.) Suppose further that $(x_1, y_1), \ldots, (x_n, y_n)$ are vertices of a polygon in $\mathbb{R}^2$, in counterclockwise order.

Prove that the area of the polygon is equal to

$$\frac{1}{2}\left[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \cdots + (x_n y_1 - x_1 y_n)\right].$$

Problem 8. Prove that there exist a positive integer n and real numbers $a_0, a_1, \ldots, a_n$ such that

$$\left|\left(\sum_{k=0}^{n} \frac{a_k}{x^k}\right) - \exp\left(\frac{\sin(e^x)}{\sqrt{x}}\right)\right| \le\ 10^{-6} \quad \text{for all } x \in [1, \infty).$$

Problem 9. Prove that the series $\sum_{n=1}^{\infty} n^{-x}$ can be differentiated term by term on its interval of convergence.

Problem 10. Suppose that, for each positive integer n,

$$f_n : [0,1] \to \mathbb{R}$$

is a continuous function that satisfies $f_n(0) = 0$ and has a continuous derivative f_n' on $(0,1)$ such that $|f_n'(x)| \le 9000$ for all $x \in (0,1)$.

Prove that there exists a subsequence $f_{n_1}, f_{n_2}, f_{n_3}, \ldots$ such that the following holds:

For every Riemann integrable function $g : [0,1] \to \mathbb{R}$, there exists a real number L (which may depend on the function g) such that

$$\lim_{k\to\infty} \int_0^1 g(x)\, f_{n_k}(x)\, dx \ =\ L\,.$$

(*Note*. You may take for granted and freely use standard basic facts about Riemann integrals, including, e.g. the fact that a Riemann integrable function is bounded, and that linear combinations, products, and absolute values of Riemann integrable functions are Riemann integrable.)