Tier 1 Analysis Exam

January 2008

1. Give an example of a function $f : [0, \infty) \to \mathbb{R}$ that satisfies the three conditions:

   (i) $f(x) \geq 0$ for all $x \geq 0$,
   (ii) for every $M > 0$, $\sup_{x>M} f(x) = \infty$,
   (iii) $\int_0^\infty f(x)\,dx < \infty$,

   or else prove that no such function exists.

2. Determine whether the series
$$\sum_{n=1}^{\infty} \ln\left(n \sin \frac{1}{n}\right)$$
is convergent (conditionally or absolutely) or divergent.

3. Let S be a closed, nonempty subset of $\mathbb{R}^n$ that is convex in the sense that if q_1 and q_2 are any two points in S, then $\lambda q_1 + (1 - \lambda) q_2 \in S$ for all $\lambda \in (0, 1)$. Given any $p \in \mathbb{R}^n \setminus S$, let
$$m = \inf_{q \in S}\{\|p - q\|\}$$
where $\|\cdot\|$ denotes the usual Euclidean norm. Prove that there exists exactly one point $q \in S$ that achieves this infimum.

4. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by
$$f(x, y) = \frac{xy^2}{x^2 + y^2}$$
for $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Notice that f is C^1 on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

   (i) Show that f is continuous at $(0, 0)$.

   (ii) Show that all the directional derivatives of f at $(0, 0)$ exist by calculating the directional derivative of f at $(0, 0)$ in the direction V, for any given unit vector $V = (\cos\theta, \sin\theta)$. (Recall that the directional derivative of f at a point p in direction V is by definition, $\frac{d}{dt}\big|_{t=0} f(p + tV)$.)

   (iii) Show that f is not differentiable at $(0, 0)$.

5. Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be continuously differentiable, and assume that the 2×2 matrix $Df(x) = \left(\frac{\partial f_i}{x_j}(x)\right)$ is invertible for all $x \in \mathbb{R}^2$. Assume moreover that, for any

compact set $K \subset \mathbb{R}^2$, $f^{-1}(K)$ is compact. Prove that f is onto.

6. Let f be a continuous function on $[0, \infty)$ such that $0 \leq f(x) \leq Cx^{-1-\rho}$ for all $x > 0$, and for some constants $C, \rho > 0$. Let $f_k(x) = kf(kx)$.

   (i) Show that $\lim_{k\to\infty} f_k(x) = 0$ for any $x > 0$ and that the convergence is uniform on $[r, \infty)$ for any $r > 0$.

   (ii) Show that f_k does not converge to zero uniformly on $(0, \infty)$, unless f is identically 0.

7. Let f and f_k be defined as in the previous problem.

   (i) Show that the limit $\lim_{k\to\infty} \int_0^1 f_k(x)\, dx$ exists.

   (ii) Denote by a the limit in (i). Show that $\lim_{k\to\infty} \int_0^1 f_k(x)g(x)\, dx = ag(0)$ for any Riemann integrable function g on $[0, 1]$ that is continuous at 0.

   (Note: The result of the previous problem is not necessarily needed for solving this problem.)

8. Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function such that $|f'(x)| \leq M$ for all $x \in (0, 1)$. Show that, for any positive integer n

$$\left| \int_0^1 f(x)\, dx - \frac{1}{n} \sum_{k=1}^{n} f(\frac{k-1}{n}) \right| \leq \frac{M}{n} .$$

9. Consider the quartic equation with real coefficients

$$x^4 + a_0 x^3 + a_1 x^2 + 2a_2 x + a_3 = 0 .$$

   Show that there exists $\delta > 0$ such that if $|a_i - 1| < \delta$, $i = 0, 1, 2, 3$, then the equation above has a real solution which depends smoothly on the a_i's.

10. Compute the line integral

$$\int_C \frac{x\, dy - y\, dx}{x^2 + y^2} ,$$

   where C is a simple closed C^1 curve around the origin of the xy-plane, and oriented counterclockwise.