Tier 1 Analysis Exam

August 2006

1. Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function.

   (1) Prove that if f is uniformly continuous, and if $\{p_k\}$ is a Cauchy sequence in A, then $\{f(p_k)\}$ is a Cauchy sequence in $\mathbb{R}^m$.

   (2) Give an example of continuous f and a Cauchy sequence $\{p_k\}$ in some A (you may take $n = m = 1$) for which $\{f(p_k)\}$ is not a Cauchy sequence.

2. Let $f : (a, b) \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume f is differentiable everywhere in (a, b), except possibly at a point c. Show that, if $\lim_{x \to c} f'(x)$ exists and is equal to L, then f is differentiable at c and $f'(c) = L$.

3. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function with $g(\frac{1}{3}, \frac{2}{3}) = 3$, $\frac{\partial g}{\partial r}(\frac{1}{3}, \frac{2}{3}) = -1$, and $\frac{\partial g}{\partial s}(\frac{1}{3}, \frac{2}{3}) = -4$, where (r, s) are the coordinates for the $\mathbb{R}^2$. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = g(\frac{x}{z}, \frac{y}{z}) ,$$

   for $z \neq 0$. Show that the level surface $f^{-1}(3)$ has a tangent plane at the point $(1, 2, 3)$ and find a linear equation for it.

4. For which positive integers k does the series

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/k)}{n}$$

   converge? Justify your answer with a proof.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $x \in \mathbb{R}$ and define the sequence $\{x_n\}$ inductively by setting $x_0 = x$ and $x_{n+1} = f(x_n)$. Suppose that $\{x_n\}$ is bounded. Prove that there exists $y \in \mathbb{R}$ such that $f(y) = y$.

6. Decide whether or not the function $f : [0, 1] \times [0, 1] \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{x}{q}, & \text{if } x \notin \mathbb{Q} \text{ and } y = \frac{p}{q} \in \mathbb{Q} \\ \frac{y}{q}, & \text{if } y \notin \mathbb{Q} \text{ and } x = \frac{p}{q} \in \mathbb{Q} \\ 0, & \text{if } (x, y) \in \mathbb{Q} \times \mathbb{Q} \text{ or } (x, y) \in \mathbb{R} \setminus \mathbb{Q} \times \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

   is Riemann integrable on $[0, 1] \times [0, 1]$. Prove your decision from the definition without invoking any theorems about integrable functions. (Here all fractions $\frac{p}{q}$ are assumed to be reduced.)

7. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a C^2 function (i.e. f has continuous second order partial derivatives). Suppose $p_0 \in \mathbf{R}^n$ is a critical point of f. If

$$\det \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(p_0) \right] \neq 0 \, ,$$

show that p_0 is isolated, i.e. there is a neighborhood of p_0 in which p_0 is the only critical point of f.

8. Prove that $\sum_{n=1}^{\infty} \frac{x}{1+n^2x^2}$ converges pointwise but not uniformly on $\mathbf{R}$. Let $f(x) = \sum_{n=1}^{\infty} \frac{x}{1+n^2x^2}$. Is it true that the Riemann integral $\int_0^1 f(x)dx = \sum_{n=1}^{\infty} \int_0^1 \frac{x}{1+n^2x^2} dx$? Justify.

9. Let $\{a_n\}$ be a sequence. Show that

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \limsup_{n\to\infty} \, (a_n - a_{n-1}) \, .$$

10. Let $Q \subset \mathbf{R}^3$ be any solid rectangular box with one vertex at the origin. Show that

$$\int_{\partial Q} \frac{\mathbf{x}\cdot\hat{\mathbf{n}}}{\|\mathbf{x}\|^3} \, dS = \frac{\pi}{2}.$$

Here $\hat{\mathbf{n}}$ is the unit outer normal on ∂Q and dS is the area element. (You should notice that this integral is not an improper integral.)