1. Let \( f : A \subset \mathbb{R}^n \to \mathbb{R}^m \) be a function.

   (1) Prove that if \( f \) is uniformly continuous, and if \( \{p_k\} \) is a Cauchy sequence in \( A \),
   then \( \{f(p_k)\} \) is a Cauchy sequence in \( \mathbb{R}^m \).

   (2) Give an example of continuous \( f \) and a Cauchy sequence \( \{p_k\} \) in some \( A \) (you may take \( n = m = 1 \)) for which \( \{f(p_k)\} \) is not a Cauchy sequence.

2. Let \( f : (a, b) \subset \mathbb{R} \to \mathbb{R} \) be a continuous function. Assume \( f \) is differentiable everywhere in \( (a, b) \), except possibly at a point \( c \). Show that, if \( \lim_{x \to c} f'(x) \) exists and is equal to \( L \),
   then \( f \) is differentiable at \( c \) and \( f'(c) = L \).

3. Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be a \( C^1 \) function with \( g(\frac{1}{2}, \frac{2}{3}) = 3 \), \( \frac{\partial g}{\partial x}(\frac{1}{2}, \frac{2}{3}) = -1 \), and \( \frac{\partial g}{\partial y}(\frac{1}{2}, \frac{2}{3}) = -4 \),
   where \( (r, s) \) are the coordinates for the \( \mathbb{R}^2 \). Define \( f : \mathbb{R}^3 \to \mathbb{R} \) by

   \[
   f(x, y, z) = g\left(\frac{x}{z}, \frac{y}{z}\right),
   \]

   for \( z \neq 0 \). Show that the level surface \( f^{-1}(3) \) has a tangent plane at the point \((1, 2, 3)\)
   and find a linear equation for it.

4. For which positive integers \( k \) does the series

   \[
   \sum_{n=1}^{\infty} \frac{\sin(n\pi/k)}{n}
   \]

   converge? Justify your answer with a proof.

5. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function. Let \( x \in \mathbb{R} \) and define the sequence \( \{x_n\} \)
   inductively by setting \( x_0 = x \) and \( x_{n+1} = f(x_n) \). Suppose that \( \{x_n\} \) is bounded. Prove
   that there exists \( y \in \mathbb{R} \) such that \( f(y) = y \).

6. Decide whether or not the function \( f : [0, 1] \times [0, 1] \to \mathbb{R} \) defined by

   \[
   f(x, y) = \begin{cases} 
   \frac{x}{q}, & \text{if } x \notin \mathbb{Q} \text{ and } y = \frac{p}{q} \in \mathbb{Q} \\
   \frac{y}{q}, & \text{if } y \notin \mathbb{Q} \text{ and } x = \frac{p}{q} \in \mathbb{Q} \\
   0, & \text{if } (x, y) \in \mathbb{Q} \times \mathbb{Q} \text{ or } (x, y) \in \mathbb{R} \setminus \mathbb{Q} \times \mathbb{R} \setminus \mathbb{Q}
   \end{cases}
   \]

   is Riemann integrable on \([0, 1] \times [0, 1]\). Prove your decision from the definition without
   invoking any theorems about integrable functions. (Here all fractions \( \frac{p}{q} \) are assumed
to be reduced.)
7. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a \( C^2 \) function (i.e. \( f \) has continuous second order partial derivatives). Suppose \( p_0 \in \mathbb{R}^n \) is a critical point of \( f \). If
\[
\det \left[ \frac{\partial^2 f}{\partial x_i \partial x_j}(p_0) \right] \neq 0,
\]
show that \( p_0 \) is isolated, i.e. there is a neighborhood of \( p_0 \) in which \( p_0 \) is the only critical point of \( f \).

8. Prove that \( \sum_{n=1}^{\infty} \frac{x^n}{1+n^2x^2} \) converges pointwise but not uniformly on \( \mathbb{R} \). Let \( f(x) = \sum_{n=1}^{\infty} \frac{x^n}{1+n^2x^2} \). Is it true that the Riemann integral \( \int_0^1 f(x)dx = \sum_{n=1}^{\infty} \int_0^1 \frac{x^n}{1+n^2x^2}dx \)? Justify.

9. Let \( \{a_n\} \) be a sequence. Show that
\[
\limsup_{n \to \infty} \frac{a_n}{n} \leq \limsup_{n \to \infty} (a_n - a_{n-1}) .
\]

10. Let \( Q \subset \mathbb{R}^3 \) be any solid rectangular box with one vertex at the origin. Show that
\[
\int_{\partial Q} \frac{x \cdot \hat{n}}{\|x\|^2} dS = \frac{\pi}{2}.
\]
Here \( \hat{n} \) is the unit outer normal on \( \partial Q \) and \( dS \) is the area element. (You should notice that this integral is not an improper integral.)