Analysis Tier I Exam January 2006

All questions are worth 10 points. In question 7, each part is worth 5 points.

1. Show that the function given by

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable for all $x \in \mathbb{R}$, but not *continuously* differentiable at $x = 0$.

2. Let $\{f_n\}$ be the sequence of functions given by

$$f_n(x) = nxe^{-nx}.$$

Prove that $\{f_n\}$ converges to 0 pointwise but not uniformly on the interval $[0, 1]$ as $n \to \infty$.

3. Let n be a positive integer and define f on $[0, \infty)$ by $f(x) = \sqrt[n]{x}$. Give a direct ϵ, δ proof that f is continuous on $[0, \infty)$.

4. Associating any 2×2 real matrix (a_{ij}) with a point $(a_{11}, a_{12}, a_{21}, a_{22})$ $\mathbb{R}^4$, prove that the set of all invertible, real matrices is not a connected set in $\mathbb{R}^4$.

5. Define a sequence $\{r_n\}$ by $r_0 = 1$, and $r_{n+1} = (2/3)r_n + 1$ for $n \geq 0$. Let the sequence $\{c_n\}$ be defined by $c_0 = 1/4$, and

$$c_{n+1} = \frac{r_{n+1}\sqrt{c_n}}{3}$$

for $n \geq 0$. Prove that

$$\lim_{n\to\infty} c_n \quad \text{exists}$$

and determine what the limit is.

Hint: First argue that $\{r_n\}$ converges.

6. Do there exist continuous functions $f(x, y)$ and $g(x, y)$ in a neighborhood of $(0, 1)$ such that $f(0, 1) = 1$ and $g(0, 1) = -1$ and such that

$$\begin{aligned} \left[f(x,y)\right]^3 + xg(x,y) - y &= 0, \\ \left[g(x,y)\right]^3 + yf(x,y) - x &= 0\ ? \end{aligned}$$

Justify your answer.

7. Let $\epsilon > 0$ and a positive integer n be given. Let $F \subset \mathbb{Z} \times \mathbb{Z}$ be defined by $F = \{(i,j) : 1 \le i < j \le n\}$ and let E be any subset of F. Then define a real-valued function $G_{E,\epsilon}$ on $\mathbb{R}^n$ by

$$G_{E,\epsilon}(x_1, \ldots, x_n) = (n+\epsilon)\sum_{j=1}^{n} \sin^2 x_j - \sum_{(i,j)\in E} (x_i - x_j)^2.$$

a. Take $n = 3$ and $E = F$. Show that $G_{F,\epsilon}$ has a local minimum at the origin.

b. For arbitrary positive integer n and E any subset of F, show that $G_{E,\epsilon}$ has a local minimum at the origin.

8. Let $D \subset \mathbb{R}^2$ be an arbitrary bounded open set with C^1 boundary whose perimeter P is finite. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a given C^1 function satisfying the condition

$$|f(x,y)| \le 1 \quad \text{for all } (x,y) \in D.$$

Establish the inequality

$$\left| \int\int_D \frac{\partial f}{\partial y}(x,y)\, dx\, dy \right| \le P.$$

9. Suppose $K > 0$, and $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a differentiable mapping with $|\partial F_i / \partial x_j| < K$ at every point, for every $1 \le i, j \le 2$. Show that there exists $C > 0$ such that F satisfies the Lipschitz condition

$$\|F(p) - F(q)\| \le C\,\|p - q\| \quad \text{for all } p, q \in \mathbb{R}^2 \,.$$

Here $\|p - q\|$ denotes the usual Euclidean distance between p and q in $\mathbb{R}^2$.

10. A family $\mathcal{F}$ of functions is said to be *uniformly equicontinuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $g \in \mathcal{F}$,

$$|x_1 - x_2| < \delta \Rightarrow |g(x_1) - g(x_2)| < \epsilon.$$

Note: δ does not depend on g or x_1 or x_2. Now suppose that $f : \mathbb{R} \times [0,1] \to \mathbb{R}$ is a bounded continuous function. For each $y \in [0,1]$, define $g_y : \mathbb{R} \to \mathbb{R}$ by $g_y(x) = f(x,y)$. Suppose that for each y we know that

$$\lim_{x\to\infty} g_y(x) = 0 = \lim_{x\to -\infty} g_y(x).$$

Must any such family $\mathcal{F} := \{g_y : 0 \le y \le 1\}$ be uniformly equicontinuous? If so, prove it. If not, provide a counter-example.