Tier I Analysis Exam
August, 2005

Justify your answers. All problems carry equal weight.

1. Let $p > 0$. Evaluate

$$\lim_{n \to \infty} \frac{1^p + 3^p + \ldots + (2n-1)^p}{n^{p+1}}.$$

2. For $x > 0$, define

$$\phi(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \end{cases}$$

where $p, q$ have no common factor and $q \geq 1$.

(a) Where is $\phi$ continuous?
(b) Where is $\phi$ differentiable?

3. Let $a, b$ be real with $|b| > \max\{1, |a|\}$. For $x \in \mathbb{R}$, define

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos(a^n x)}{b^n}.$$

(a) Show that $f$ is uniformly continuous on all of $\mathbb{R}$.
(b) Let

$$\gamma = \{(x, y) : y = f(x), \ 0 \leq x \leq 1 \}$$

be the graph of $f$ over the unit interval. Show that $\gamma$ has finite length.

4. Let $M_2$ denote the set of 2-by-2 matrices with real entries, and for $A \in M_2$, define $S(A) = A^2$. Does the mapping $S : M_2 \to M_2$ have a local inverse near the identity matrix?

5. Fix $a > 0$. Let $x_1, \ldots, x_n$ be non-negative numbers with

$$\sum_{i=1}^{n} x_i = na.$$

Show that

$$\sum_{i<j} x_i x_j \leq \frac{1}{2} n(n-1)a^2.$$

Please turn over.
6. Let \( p \) be real. Suppose \( f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) is continuously differentiable, and satisfies
\[
    f(\lambda x) = \lambda^p f(x), \quad \text{for all } x \neq 0 \text{ and for all } \lambda > 0.
\]
Let \( \nabla f(x) \) denote the gradient of \( f \) at \( x \) and \( \cdot \) the Euclidean inner product. Prove that
\[
x \cdot \nabla f(x) = pf(x), \quad x \neq 0.
\]

7. A family \( \mathcal{F} \) of continuous real-valued functions of a real variable is called equicontinuous at \( x \) if for every \( \epsilon > 0 \), there is a \( \delta > 0 \) such that for every \( f \in \mathcal{F}, \)
\[
    |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.
\]
\( \mathcal{F} \) is called equicontinuous on the set \( E \) if it is equicontinuous at each point \( x \) of \( E \). (Note: the constant \( \delta \) may depend on both \( \epsilon \) and \( x \).

Now suppose \( \mathcal{F} \) is a family of continuous real-valued functions defined on an open interval \( I \subseteq \mathbb{R} \), and let \( x_0 \in I \).

(a) Suppose \( \mathcal{F} \) is equicontinuous at every point of \( I \setminus \{x_0\} \). Must \( \mathcal{F} \) also be equicontinuous at \( x_0 \)?

(b) Suppose \( \mathcal{F} \) is equicontinuous at \( x_0 \). Must \( \mathcal{F} \) also be equicontinuous at every point in some neighborhood \( J \) of \( x_0 \)?

8. Let \( U \) be an open subset of \( \mathbb{R}^n \) and \( f : U \to \mathbb{R}^n \) be differentiable. Suppose there exists \( C > 0 \) such that
\[
    |f(x) - f(y)| \geq C|x - y|
\]
for all \( x, y \in U \). Let \( df(x) \) denote the Jacobian derivative of \( f \) at \( x \) (that is, the linear mapping given by the \( n \) by \( n \) matrix of partial derivatives). Show that \( \det df(x) \neq 0 \) for all \( x \in U \).

9. Let \( m > 0 \) be a real number, let \( r = (x^2 + y^2 + z^2)^{1/2} \), and consider the vector field on \( \mathbb{R}^3 \) given by \( \vec{F} = r^m \cdot (x, y, z) = (x^2 + y^2 + z^2)^{m/2}(x, y, z). \)

(a) Compute the divergence \( \text{div}(\vec{F}) \).

(b) Using part (a) and the Divergence Theorem, calculate
\[
    \iiint_B r^m \, dV
\]
where \( B^3 = \{(x, y, z) : r \leq 1\} \) is the closed unit ball centered at the origin and \( dV = dx \, dy \, dz \) is the Euclidean volume.

End of exam.