Tier I exam in analysis - January 2005

Solve all problems. Justify your answers in detail. The exam's duration is 3 hours

1. Define

$$S = \{(x,y,z) \in R^3, \quad x^2 + 2y^2 + 3z^2 = 1\}, \quad f(x,y,z) = x + y + z.$$

   a. Prove that S is a compact set.

   b. Find the maximum and minimum of f on S.

2. Let $g : [0,1] \times [0,1] \to R$ be a continuous function, and define functions $f_n : [0,1] \to R$ by

$$f_n(x) = \int_0^1 g(x,y) y^n \, dy \quad x \in [0,1], n = 1, 2, \ldots$$

   Show that the sequence $(f_n)_{n=1}^{\infty}$ has a subsequence which converges uniformly on $[0,1]$.

3. Consider the subset $H = \{(a,b,c,d,e)\}$ of R^5 such that the polynomial

$$ax^4 + bx^3 + cx^2 + dx + e$$

   has at least one real root.

   a. Prove that $(1, 2, -4, 3, -2)$ is an interior point of H

   b. Find a point in H that is *not* an interior point. Justify your claim.

4. Consider a twice differentiable function $f : R \to R$, a number $a \in R$, and $h > 0$. Show that there exists a point $c \in R$ such that

$$f(a) - 2f(a+h) + f(a+2h) = h^2 f''(c).$$

5. Prove or give a counterexample: If $f(x)$ is differentiable for every $x \in R$, and if $f'(0) = 1$, then there exists $\delta > 0$ such that $f(x)$ is increasing on $(-\delta, \delta)$.

6. Let $f(x)$ be a bounded function on $(0,2)$. Suppose that for every $x, y \in (0,2),\ x \neq y$, there exists $z \in (0,2)$ such that

$$f(x) - f(y) = f(z)(x - y).$$

   a. Show that f need not be a differentiable function.

   b. Suppose that such a z can always be found between x and y. Show that f is twice differentiable.

7. Consider the torus

$$T = \{x = (a + r\sin u)\cos v, \quad y = (a + r\sin u)\sin v, \quad z = r\cos u,$$

$$0 \leq r \leq b,\ 0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi\},$$

   where $a > b$. Find the volume and surface area of T.

8. Let Ω be a bounded subset of R^n, and $f : \Omega \to R^n$ a uniformly continuous function. Show that f must be bounded.

Outline of Solutions:

1. a. It suffices to show that S is closed and bounded. Closeness follows since $S = \{h^{-1}(1)\}$, for a continuous function h. Boundedness follows since clearly S is contained in the cube $[-1,1]^3$.

   b. Both maximum and minimum are obtained at internal points on S, and can therefore be found by the Lagrange method. The Lagrange equations imply at once that $\lambda \neq 0$, and $\frac{1}{2\lambda} = x = 2y = 3z$. Solving from S we find that the maximal value is $\sqrt{11/6}$, and the minimal value is its negative.

2. $f_n(0) = 0$, and the functions f_n are equicontinuous because

$$|f_n(x) - f_n(x')| \leq \sup_y |g(x,y) - g(x',y)|,$$

   and this quantity tends to zero as $|x - x'| \to 0$ by the continuity of g. This Arzela-Ascoli applies.

3. Write the polynomial $x^4 + 2x^3 - 4x^2 + 3x - 2$. Obviously $x = 1$ is a root, so the triplet is indeed in H.

   Define the function $F(a,b,c,d,e,f,x) = ax^4 + bx^3 + cx^2 + ed + f$. Clearly $F(1,2,-4,3,-2,1) = 0$, while $F_x = 5 \neq= 0$ at that point. Therefore there exists an open neighborhood U of $(1,2,-4,3,-2)$ and a C^1 function g such that for all points (a,b,c,d,e) in U we have $F(a,b,c,d,e,g(a,b,c,d,e)) = 0$.

   Clearly $(0,0,1,0,0)$ is in H. But the the points $(0,0,1,0,\mu^2)$ are not in the set for $\mu \neq 0$ (Since $x^2 + \mu^2$ has no real root).

4. Apply the mean-value theorem to the function $F(x) = f(x+h) - f(x)$ to get

$$f(a) - 2f(a+h) + f(a+2h) = F(a+h) - F(a) = hF'(d) = h(f'(d+h) - f'(d))$$

   for some d, then apply MVT again to the right-hand side.

5. Counterexmaple - $f(x) = x + 2x^2 \sin(1/x)$.

6. a. Let $f = x$ for $0 \leq x \leq 1$, and $f = 1$ for $1 \leq x \leq 2$.

   Since f is bounded, $\lim_{y \to x} f(y) = f(x)$. Furthermore, $\lim_{x \to y} \frac{f(y)-f(x)}{x-y} = f(y)$. Therefore f is differentiable. Also, the last identity implies $f' = f$, thus $f(x) = ce^x$.

7. The Jacobian is given by $J = r(a + \sin u)$, and hence $V = 2\pi^2 ab^2$. Observing that the boundary is given by $r = b$, a simple computation gives $||N|| = ||T_u \times T_v|| = b(a + b \sin u)$. Therefore $S = 4\pi^2 ab$. Of course, it is also possible to solve with the slice method.

8. Choose $\delta > 0$ such that $|f(x) - f(y)| < 1$ whenever $|x - y| < \delta$. Assume that f is not bounded, and choose $x_k \in \Omega$ such that $|f(x_{k+1})| > |f(x_k)| + 1$ for all k. Observe that $|f(x_j) - f(x_k)| > 1$ whenever $j \neq k$. However, by Bolzano-Weierstrass, we must have $|x_j - x_k| < \delta$ for some $j \neq k$, which gives a contradiction.