Tier I Analysis Exam-August 2004

1. (A) Suppose $A$ and $B$ are nonempty, disjoint subsets of $\mathbb{R}^n$ such that $A$ is compact and $B$ is closed. Prove that there exists a pair of points $a \in A$ and $b \in B$ such that
\[ \forall x \in A, \forall y \in B, \quad \|x - y\| \geq \|a - b\|. \]
Prove this fact from basic principles and results; do not simply cite a similar or more general theorem. Here and in what follows, $\|\cdot\|$ denotes the usual Euclidean norm: for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $\|x\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$.

(B) Suppose that in problem (A) above, the assumption that the set $A$ is compact is replaced by the assumption that $A$ is closed. Does the result still hold? Justify your answer with a proof or counterexample.

2. (A) Prove the following classic result of Cauchy: Suppose $r(1), r(2), r(3), \ldots$ is a monotonically decreasing sequence of positive numbers. Then $\sum_{k=1}^{\infty} r(k) < \infty$ if and only if $\sum_{n=1}^{\infty} 2^n r(2^n) < \infty$.

(B) Use the result in part (A) to prove the following theorem: Suppose $a_1, a_2, a_3, \ldots$ is a monotonically decreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n = \infty$. For each $n \geq 1$, define the positive number $c_n = \min\{a_n, 1/n\}$. Then $\sum_{n=1}^{\infty} c_n = \infty$.

3. Suppose $g : [0, \infty) \to [0,1]$ is a continuous, monotonically increasing function such that $g(0) = 0$ and $\lim_{x \to \infty} g(x) = 1$.

Suppose that for each $n = 1, 2, 3, \ldots$, $f_n : [0, \infty) \to [0,1]$ is a monotonically increasing (but not necessarily continuous) function. Suppose that for all $x \in [0,\infty)$, $\lim_{n \to \infty} f_n(x) = g(x)$. Prove that $f_n \to g$ uniformly on $[0,\infty)$ as $n \to \infty$. 

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4. Let \( x \in \mathbb{R}^3 \) and let \( f(x) \in C^1(\mathbb{R}^3) \). Further let \( n = x/\|x\| \) for \( x \neq 0 \). Show that the surface integral
\[
I \equiv \int_{\|x\|=1} f(x) dS_x
\]
can be expressed in the form of a volume integral
\[
I = \int_{\|x\|<1} \left( \frac{2}{\|x\|} f(x) + n \cdot \nabla f(x) \right) dx .
\]
Hint: Write the integrand in \( I \) as \( n \cdot (nf) \).

5. Let \( x_0 \in \mathbb{R} \) and consider the sequence defined by
\[
x_{n+1} = \cos(x_n) \quad (n = 0, 1, \ldots)
\]
Prove that \( \{x_n\} \) converges for arbitrary \( x_0 \).

6. Let \( \alpha > 0 \) and consider the integral
\[
J_\alpha = \int_0^{\infty} \frac{e^{-x}}{1 + \alpha x} dx .
\]
Show that there is a constant \( c \) such that
\[
\alpha^{1/2} J_\alpha \leq c .
\]

7. Consider the infinite series
\[
\sum_{n=1}^{\infty} X_n(x)T_n(t)
\]
where \((x, t)\) varies over a rectangle \( \Omega = [a, b] \times [0, \tau] \) in \( \mathbb{R}^2 \). Assume that

(i) The series \( \sum_{n=1}^{\infty} X_n(x) \) converges uniformly with respect to \( x \in [a, b] \);
(ii) There exists a positive constant \( c \) such that \( |T_n(t)| \leq c \) for every positive integer \( n \) and every \( t \in [0, \tau] \);
(iii) For every \( t \) such that \( t \in [0, \tau] \), \( T_1(t) \leq T_2(t) \leq T_3(t) \leq \ldots \)
Prove that \( \sum_{n=1}^{\infty} X_n(x)T_n(t) \) converges uniformly with respect to both variables together on \( \Omega \).

Hint: Let \( S_N = \sum_{n=1}^{N} X_n(x)T_n(t) \), \( s_N = \sum_{n=1}^{N} X_n(x) \). For \( m > n \) find an expression for \( S_m - S_n \) involving \( (s_k - s_n) \) for an appropriate range of values of \( k \).
8. Let \( v(x) \in C^\infty(\mathbb{R}) \) and assume that for each \( \gamma \) in a neighborhood of the origin there exists a function \( u(x, v, \gamma) \) which is \( C^\infty \) in \( x \) such that
\[
\gamma \frac{\partial}{\partial x}(u + v) = \sin(u - v).
\]
Assuming that
\[
u = u_0 + \gamma u_1 + \gamma^2 u_2 + \gamma^3 u_3 + \ldots
\]
where \( u_0(0) = v(0) \) and for all \( n \) the \( u_n \)'s are functions of \( v \) but are independent of \( \gamma \), find \( u_0, u_1, u_2 \) and \( u_3 \).

9. All partial derivatives \( \frac{\partial^{m+n}f}{\partial x^m \partial y^n} \) of a function \( f : \mathbb{R}^2 \to \mathbb{R} \) exist everywhere. Does it imply that \( f \) is continuous? Prove or give a counterexample.

10. Decide whether the two equations
\[
\sin(x + z) + \ln(yz^2) = 0, \quad e^{x+z} + yz = 0,
\]
implicitly define \( (x, y) \) near \( (1, 1) \) as a function of \( z \) near \(-1\).