Tier I Analysis Exam-August 2004

1. (A) Suppose A and B are nonempty, disjoint subsets of $\mathbb{R}^n$ such that A is compact and B is closed. Prove that there exists a pair of points $a \in A$ and $b \in B$ such that

$$\forall x \in A,\ \forall y \in B, \quad \|x - y\| \geq \|a - b\|\,.$$

Prove this fact from basic principles and results; do not simply cite a similar or more general theorem. Here and in what follows, $\|.\|$ denotes the usual Euclidean norm: for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $\|x\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$.

(B) Suppose that in problem **(A)** above, the assumption that the set A is compact is replaced by the assumption that A is closed. Does the result still hold? Justify your answer with a proof or counterexample.

2. (A) Prove the following classic result of Cauchy: *Suppose* $r(1), r(2), r(3), \ldots$ *is a monotonically decreasing sequence of positive numbers. Then* $\sum_{k=1}^{\infty} r(k) < \infty$ *if and only if* $\sum_{n=1}^{\infty} 2^n r(2^n) < \infty$.

(B) Use the result in part **(A)** to prove the following theorem: *Suppose* $a_1, a_2, a_3, \ldots$ *is a monotonically decreasing sequence of positive numbers such that* $\sum_{n=1}^{\infty} a_n = \infty$. *For each* $n \geq 1$, *define the positive number* $c_n = \min\{a_n, 1/n\}$. *Then* $\sum_{n=1}^{\infty} c_n = \infty$.

3. Suppose $g : [0, \infty) \to [0, 1]$ is a continuous, monotonically increasing function such that $g(0) = 0$ and $\lim_{x \to \infty} g(x) = 1$.

Suppose that for each $n = 1, 2, 3, \ldots$, $f_n : [0, \infty) \to [0, 1]$ is a monotonically increasing (but not necessarily continuous) function. Suppose that for all $x \in [0, \infty)$, $\lim_{n \to \infty} f_n(x) = g(x)$. Prove that $f_n \to g$ uniformly on $[0, \infty)$ as $n \to \infty$.

4. Let $x \in \mathbb{R}^3$ and let $f(x) \in C^1(\mathbb{R}^3)$. Further let $n = x/\|x\|$ for $x \neq 0$. Show that the surface integral

$$I \equiv \int_{\|x\|=1} f(x)\, dS_x$$

can be expressed in the form of a volume integral

$$I = \int_{\|x\|<1} \left(\frac{2}{\|x\|} f(x) + n \cdot \nabla f(x) \right) dx\,.$$

Hint: Write the integrand in I as $n \cdot (nf)$.

5. Let $x_0 \in \mathbb{R}$ and consider the sequence defined by

$$x_{n+1} = \cos(x_n) \qquad (n = 0, 1, \ldots)$$

Prove that $\{x_n\}$ converges for arbitrary x_0.

6. Let $\alpha > 0$ and consider the integral

$$J_\alpha = \int_0^\infty \frac{e^{-x}}{1 + \alpha x}\, dx\,.$$

Show that there is a constant c such that

$$\alpha^{1/2} J_\alpha \leq c\,.$$

7. Consider the infinite series

$$\sum_{n=1}^{\infty} X_n(x) T_n(t)$$

where $(x,\, t)$ varies over a rectangle $\Omega = [a, b] \times [0, \tau]$ in $\mathbb{R}^2$. Assume that

(i) The series $\sum_{n=1}^{\infty} X_n(x)$ converges uniformly with respect to $x \in [a, b]$;
(ii) There exists a positive constant c such that $|T_n(t)| \leq c$ for every positive integer n and every $t \in [0, \tau]$;
(iii) For every t such that $t \in [0, \tau]$, $T_1(t) \leq T_2(t) \leq T_3(t) \leq \ldots$

Prove that $\sum_{n=1}^{\infty} X_n(x) T_n(t)$ converges uniformly with respect to both variables together on Ω.

Hint: Let $S_N = \sum_{n=1}^{N} X_n(x) T_n(t)$, $s_N = \sum_{n=1}^{N} X_n(x)$. For $m > n$ find an expression for $S_m - S_n$ involving $(s_k - s_n)$ for an appropriate range of values of k.

8. Let $v(x) \in C^\infty(\mathbb{R})$ and assume that for each γ in a neighborhood of the origin there exists a function $u(x, v, \gamma)$ which is C^∞ in x such that

$$\gamma \frac{\partial}{\partial x}(u + v) = \sin(u - v)\,.$$

Assuming that

$$u = u_0 + \gamma u_1 + \gamma^2 u_2 + \gamma^3 u_3 + \ldots$$

where $u_0(0) = v(0)$ and for all n the u_n's are functions of v but are independent of γ, find u_0, u_1, u_2 and u_3.

9. All partial derivatives $\partial^{m+n} f / \partial x^m \partial y^n$ of a function $f : \mathbb{R}^2 \to \mathbb{R}$ exist everywhere. Does it imply that f is continuous? Prove or give a counterexample.

10. Decide whether the two equations

$$\sin(x + z) + \ln(yz^2) = 0\,, \qquad e^{x+z} + yz = 0\,,$$

implicitly define (x, y) near $(1, 1)$ as a function of z near -1.