TIER 1 Analysis Exam January 2004

Instruction: Solve as many of these problems as you can. Be sure to justify all your answers.

1. Let $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ be strictly increasing, integer valued, sequences. Show that if for each integer $n \geq 1$,
$$p_n \cdot q_{n-1} \; - \; p_{n-1} \cdot q_n \; = \; 1,$$
then the sequence of quotients p_n/q_n converges.

2. Consider the following system of equations
$$x \cdot e^y = u,$$
$$y \cdot e^x = v.$$
(a) Show that there exists an $\epsilon > 0$ such that given any u and v with $|u| < \epsilon$ and $|v| < \epsilon$, the above system has a unique solution $(x, y) \in \mathbb{R}^2$.
(b) Exhibit a pair $(u, v) \in \mathbb{R}^2$ such that there exist two distinct solutions to this system. Justify your answer.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(a) < f'(b)$ for some $a < b$. Prove that for any $z \in (f'(a), f'(b))$, there is a $c \in (a, b)$ such that $f'(c) = z$. Note: The derivative function f' may not be continuous.

4. Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be continuously differentiable. Let $Df(x)$ denote the differential (or derivative) of f at the point $x \in \mathbb{R}^4$. Prove or provide a counter-example: The set of points x where $Df(x)$ has a null space of dimension 2 or greater is closed in $\mathbb{R}^4$.

5. Let $C([0,1])$ denote the collection of continuous real valued functions on $[0,1]$. Define $\Phi : C([0,1]) \to C([0,1])$ by
$$[\Phi(f)](t) = 1 + \int_0^t s^2 e^{-f(s)} ds \qquad t \in [0,1]$$
for $f \in C([0,1])$. Define $f_0 \in C([0,1])$ by $f_0 \equiv 1$ (i.e. the function of constant value 1). Let $f_n = \Phi(f_{n-1})$ for $n = 1, 2, \ldots$.
(a) Prove that $1 \leq f_n(t) \leq 1 + 1/3$ for all $t \in [0,1]$ and $n = 1, 2, \ldots$.
(b) Prove that
$$|f_{n+1}(x) - f_n(x)| < \frac{1}{3} \sup_{t \in [0,1]} |f_n(t) - f_{n-1}(t)|$$
for all $x \in [0,1]$ and for $n = 1, 2, \ldots$. Hint: Show that $|e^{-(x+\delta)} - e^{-x}| < \delta$ for $x > 0$ and $\delta 0$.
(c) Show that the sequence of functions $\{f_n\}$ converges uniformly to some function $f \in C([0,1])$. Be sure to indicate any theorems that you use.

6. Let I be a closed interval in $\mathbb{R}$, and let f be a differentiable real valued function on I, with $f(I) \subset I$. Suppose $|f'(t)| < 3/4$ for all $t \in I$. Let x_0 be any point in I and define a sequence x_n by $x_{n+1} = f(x_n)$ for every $n > 0$. Show that there exists $x \in I$ with $f(x) = x$ and $\lim x_n = x$.

7. Let

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y} & \text{if } x^2+y \neq 0, \\ 0 & \text{if } x^2+y=0. \end{cases}$$

   (a) Show that f has a directional derivative (in every direction) at $(0,0)$, and show that f is not continuous at $(0,0)$.
   (b) Prove or provide a counterexample: If $P_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $P_2 : \mathbb{R}^2 \to \mathbb{R}^2$ are any two functions such that $P_1(0,0) = (0,0) = P_2(0,0)$, and such that $f \circ P_i$ is differentiable at $(0,0)$, with nonvanishing derivative at $(0,0)$ for $i = 1, 2$, then $f \circ (P_1 + P_2)$ is differentiable at $(0,0)$.

8. Let $B = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1\}$ be the unit ball. Let $v = (v_1, v_2, v_3)$ be a smooth vector field on B, which vanishes on the boundary ∂B of B and satisfies

$$\text{div } v(x,y,z) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0, \qquad \forall (x,y,z) \in B.$$

   Prove that

$$\int_B x^n v_1(x,y,z)\,dxdydz = 0, \qquad \forall n = 0,1,2,\cdots,.$$

9. Suppose that $f : [0,1] \to \mathbb{R}$ is a continuous function on $[0,1]$ with

$$\int_0^1 f(x)dx = \int_0^1 f(x)(x^n + x^{n+2})dx$$

   for all $n = 0, 1, 2, \ldots$. Show that $f \equiv 0$.

10. Suppose that $f : [0,1] \to \mathbb{R}$ has a continuous second derivative, $f(0) = f(1) = 0$, and $f(x) > 0$ for all $x \in (0,1)$. Prove that

$$\int_0^1 \left| \frac{f''(x)}{f(x)} \right| dx > 4.$$