Tier I exam in analysis - August 2003

Answer all the problems. Justify your answers.

1. Let $f(x)$ be a function that is continuous in $[-1,1]$, differentiable in $(-1,1)$, and satisfies $f(-1)=-\pi/2$, $f(1)=\pi/2$, $f'(x)\geq \frac{1}{\sqrt{1-x^2}}$ in $(-1,1)$. Prove that $f(x)=\arcsin(x)$ in $[-1,1]$.

2. Determine the values of $x\in\mathbf{R}$ such that the series
$$\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln(n+x^2)}}$$
converges.

3. Recall that a square matrix M is called orthogonal if its rows form an orthonormal set. The set of all orthogonal matrices will be denoted by O. (Note that an orthogonal matrix necessarily satisfies the condition $MM^t=I$ where M^t denotes the transpose of M.)
$$\text{Let } M_0=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$$
be a given element of O where a,b,c and d are real numbers.

   (i). Prove that, except for 4 special matrices M_0, there always exists a number $\delta>0$ and three functions f,g,h, continuously differentiable for $x\in(a-\delta,a+\delta)$, such that
$$\begin{pmatrix} x & f(x)\\ g(x) & h(x)\end{pmatrix}\in O,$$
for all $x\in(a-\delta,a+\delta)$ with $f(a)=b$, $g(a)=c$ and $h(a)=d$.

   (ii). What are the four exceptional matrices of part (i)?

4. A vector field $\vec{F}:\mathbf{R}^2\to\mathbf{R}^2$ is said to be conservative in an open set D if the line integral $\int_C \vec{F}\cdot ds=0$ for every closed curve $C\subset D$. Find all numbers a and b such that the vector field
$$\vec{F}(x,y)=\left(\frac{x+ay}{x^2+y^2},\frac{bx+y}{x^2+y^2}\right)$$
is conservative in
$$D=\{(x,y):\frac{1}{9}<x^2+y^2<\frac{1}{4}\}.$$

5. Consider the triangle with vertices $(3,0)$ $(5,0)$ and $(5,1)$ in the (x,y)-plane. Revolve it around the y-axis in (x,y,z)-space $\mathbf{R}^3$ to sweep out a "triangular torus" T, evaluate the surface integral
$$\int_T \vec{v}\cdot\vec{n}\,dS\,.$$
Here $\vec{v}:\mathbf{R}^3\to\mathbf{R}^3$ is the vector field $\vec{v}(x,y,z)=(-y,x,z)$, $\vec{n}$ is the outward unit normal field on T, and dS is the usual surface-area element on T.

6. Suppose $f: \mathbf{R}^2 \to \mathbf{R}$ is a C^∞ function that has a critical point at $(0,0)$. Suppose that $f(0,0) = 0$ and that all the first and second order partial derivatives of f vanish at $(0,0)$. Also, assume that not all third order partial derivatives vanish at $(0,0)$. Show that f can have neither a local max nor a local min at the critical point $(0,0)$.

7. Recall that a function $g: [a,b] \to \mathbf{R}$ is said to be Lipschitz if there is a constant K such that $|g(x) - g(y)| \leq K\,|x - y|$ for all $x, y \in [a,b]$.

   Assume that f is a bounded Riemann integrable function on $[a,b]$. Prove that for each $\varepsilon > 0$, there exists a Lipschitz function g such that

$$\int_a^b |f(x) - g(x)|\,dx < \varepsilon.$$

8. (a) If $B \subset \mathbf{R}^n$ is a bounded set and $f: B \to \mathbf{R}$ is uniformly continuous, show that $f(B)$ is bounded.

   (b) Give an example to show that the conclusion of part (a) is not necessarily true if f is merely continuous on B.