Tier 1 Analysis Exam

January 2003

1. Consider a function $f : \mathbb{R} \to \mathbb{R}$. Which of the following statements is equivalent to the continuity of f at 0? (Provide justification for each of your answers.)

   a) For every $\varepsilon \geq 0$ there exists $\delta > 0$ such that $|x| < \delta$ implies $|f(x) - f(0)| \leq \varepsilon$.

   b) For every $\varepsilon > 0$ there exists $\delta \geq 0$ such that $|x| < \delta$ implies $|f(x) - f(0)| \leq \varepsilon$.

   c) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x| \leq \delta$ implies $|f(x) - f(0)| \leq \varepsilon$.

2. Consider a uniformly continuous real-valued function f defined on the interval $[0, 1)$. Show that $\lim_{t \to 1^-} f(t)$ exists. Is a similar statement true if $[0, 1)$ is replaced by $[0, \infty)$?

3. Let f be a real-valued continuous function on $[0, 1]$ such that $f(0) = f(1)$. Show that there exists $x \in [0, 1/2]$ such that $f(x) = f(x + 1/2)$.

4. If f is differentiable on $[0, 1]$ with continuous derivative f', show that

$$\int_0^1 |f(x)|dx \leq \max\left\{ \left|\int_0^1 f(x)dx\right| \, , \, \int_0^1 |f'(x)|dx \right\}$$

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous and with compact support, i.e. there exists $R > 0$ such that $f(x, y) = 0$ if $x^2 + y^2 \geq R^2$.

   a) Show that the integral

$$g(u, v) = \iint_{\mathbb{R}^2} \frac{f(x, y)}{\sqrt{(x-u)^2 + (y-v)^2}} dxdy$$

   converges for all $(u, v) \in \mathbb{R}^2$, and show that $g(u, v)$ is continuous in (u, v).

   b) Show that, if in addition f has continuous first order partial derivatives, then so does g and

$$\frac{\partial g}{\partial u}(u, v) = \iint_{\mathbb{R}^2} \frac{\frac{\partial f}{\partial x}(x, y)}{\sqrt{(x-u)^2 + (y-v)^2}} dxdy \ .$$

6. Show that for any two functions f, g which have continuous second order partial derivatives, defined in a neighborhood of the sphere $S = \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$ in $\mathbb{R}^3$, one has

$$\int_S (\nabla f \times \nabla g) \cdot \mathbf{dS} = 0$$

where ∇f, ∇g are the gradient of f, g respectively.

7. Show that if $\{x_n\}$ is a bounded sequence of real numbers such that $2x_n \leq x_{n+1}+x_{n-1}$ for all n, then $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$.

8. For a non-empty set X, let $\mathbb{R}^X$ be the set of all maps from X to $\mathbb{R}$. For $f, g \in \mathbb{R}^X$, define

$$d(f, g) = \sup_{x\in X} \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}.$$

   a) Show that $(\mathbb{R}^X, d)$ is a metric space.
   b) Show that $f_n \to f$ in $(\mathbb{R}^X, d)$ if and only if f_n converges uniformly to f.

9. Show that if $f : [0, 1] \to \mathbb{R}$ is continuous, and $\int_0^1 f(x)x^{2n}dx = 0,\ n = 0, 1, 2, \cdots$ then $f(x) = 0$ for all $x \in [0, 1]$.

10. a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Show that for any $x, y \in \mathbb{R}^n$, there exists $z \in \mathbb{R}^n$ such that

$$f(x) - f(y) = Df(z) \cdot (x - y)$$

where $Df(z)$ denotes the derivative matrix of f (in this case it is the same as the gradient of f) at z, and "$\cdot$" denotes the usual dot product in $\mathbb{R}^n$.

b) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map. Show that if f has the property that $||Df(z) - I|| < \frac{1}{2n}$ for all $z \in \mathbb{R}^n$, where I is the $n \times n$ identity matrix, then f is a diffeomorphism, i.e. f is one-to-one, onto and f^{-1} is also differentiable. (For a matrix $A = (a_{ij})$, $||A|| = (\sum_{i,j} a_{ij}^2)^{1/2}$.)