Tier 1 Analysis Exam
January 2003

1. Consider a function $f : \mathbb{R} \rightarrow \mathbb{R}$. Which of the following statements is equivalent to the continuity of $f$ at 0? (Provide justification for each of your answers.)
   a) For every $\varepsilon \geq 0$ there exists $\delta > 0$ such that $|x| < \delta$ implies $|f(x) - f(0)| \leq \varepsilon$.
   b) For every $\varepsilon > 0$ there exists $\delta \geq 0$ such that $|x| < \delta$ implies $|f(x) - f(0)| \leq \varepsilon$.
   c) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x| \leq \delta$ implies $|f(x) - f(0)| \leq \varepsilon$.

2. Consider a uniformly continuous real-valued function $f$ defined on the interval $[0,1)$. Show that $\lim_{t \to 1-} f(t)$ exists. Is a similar statement true if $[0,1)$ is replaced by $[0,\infty)$?

3. Let $f$ be a real-valued continuous function on $[0,1]$ such that $f(0) = f(1)$. Show that there exists $x \in [0,1/2]$ such that $f(x) = f(x+1/2)$.

4. If $f$ is differentiable on $[0,1]$ with continuous derivative $f'$, show that
   \[ \int_0^1 |f(x)| \, dx \leq \max \left\{ \left| \int_0^1 f(x) \, dx \right| , \int_0^1 |f'(x)| \, dx \right\} \]

5. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous and with compact support, i.e. there exists $R > 0$ such that $f(x, y) = 0$ if $x^2 + y^2 \geq R^2$.
   a) Show that the integral
      \[ g(u, v) = \iint_{\mathbb{R}^2} \frac{f(x, y)}{\sqrt{(x-u)^2 + (y-v)^2}} \, dxdy \]
      converges for all $(u, v) \in \mathbb{R}^2$, and show that $g(u, v)$ is continuous in $(u, v)$.
   b) Show that, if in addition $f$ has continuous first order partial derivatives, then so does $g$ and
      \[ \frac{\partial g}{\partial u}(u, v) = \iint_{\mathbb{R}^2} \frac{\partial f}{\partial x}(x, y) \frac{\partial f}{\partial y}(x, y)}{\sqrt{(x-u)^2 + (y-v)^2}} \, dxdy . \]
6. Show that for any two functions $f$, $g$ which have continuous second order partial derivatives, defined in a neighborhood of the sphere $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$, one has

$$\int_S (\nabla f \times \nabla g) \cdot dS = 0$$

where $\nabla f$, $\nabla g$ are the gradient of $f$, $g$ respectively.

7. Show that if $\{x_n\}$ is a bounded sequence of real numbers such that $2x_n \leq x_{n+1} + x_{n-1}$ for all $n$, then $\lim_{n \to \infty} (x_{n+1} - x_n) = 0$.

8. For a non-empty set $X$, let $\mathbb{R}^X$ be the set of all maps from $X$ to $\mathbb{R}$. For $f, g \in \mathbb{R}^X$, define

$$d(f, g) = \sup_{x \in X} \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}.$$ 

a) Show that $(\mathbb{R}^X, d)$ is a metric space.

b) Show that $f_n \to f$ in $(\mathbb{R}^X, d)$ if and only if $f_n$ converges uniformly to $f$.

9. Show that if $f : [0, 1] \to \mathbb{R}$ is continuous, and $\int_0^1 f(x)x^{2n}dx = 0$, $n = 0, 1, 2, \cdots$ then $f(x) = 0$ for all $x \in [0, 1]$.

10. a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Show that for any $x, y \in \mathbb{R}^n$, there exists $z \in \mathbb{R}^n$ such that

$$f(x) - f(y) = Df(z) \cdot (x - y)$$

where $Df(z)$ denotes the derivative matrix of $f$ (in this case it is the same as the gradient of $f$) at $z$, and $\cdot$ denotes the usual dot product in $\mathbb{R}^n$.

b) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map. Show that if $f$ has the property that $||Df(z) - I|| < \frac{1}{2n}$ for all $z \in \mathbb{R}^n$, where $I$ is the $n \times n$ identity matrix, then $f$ is a diffeomorphism, i.e. $f$ is one-to-one, onto and $f^{-1}$ is also differentiable. (For a matrix $A = (a_{ij})$, $||A|| = (\sum_{i,j} a_{ij}^2)^{1/2}$.)