Tier 1 Analysis Examination – August, 2002

1. In the classical *false position* method to find roots of $f(x) = 0$, one begins with two approximations x_0, x_1 and generates a sequence of (hopefully) better approximations via

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_0}{f(x_n) - f(x_0)} \quad \text{for} \quad n = 1, 2, \ldots$$

Consider the following sketch in which the function $f(x)$ is to be increasing and convex:

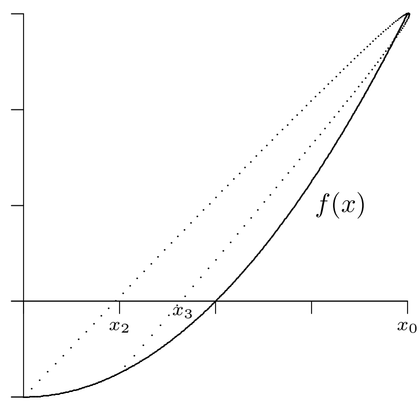

Fig. 1.2

The sequence $\{x_n\}$ is constructed as follows. We begin with the two approximations $(x_0, f(x_0))$ and $(x_1, f(x_1)) = (0, f(0))$ The chord is drawn between these two points; the point at which this chord crosses the x–axis is taken to be the next approximation x_2. One then draws the chord between the two points $(x_0, f(x_0))$ and $(x_2, f(x_2))$. The next approximation x_3 is that point where this chord crosses the axis, as shown. For f strictly increasing and convex and for initial approximations $x_0 > 0$, $x_1 = 0$ with $f(x_0) > 0$, $f(x_1) < 0$, prove *rigorously* that this sequence must converge to the unique solution of $f(x) = 0$ over $[x_1, x_0]$.

2. (a) Show that it is possible to solve the equations

$$xu^2 + yzv + x^2z - 3 = 0$$
$$xyv^3 + 2zu - u^2v^2 - 2 = 0$$

for (u, v) in terms of (x, y, z) in a neighborhood of $(1, 1, 1, 1, 1)$.

(b) Given that the inverse of the matrix

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

find $\frac{\partial u}{\partial x}$ at $(1, 1, 1)$.

3. Let X be a complete metric space and let Y be a subspace of X. Prove that Y is complete if and only if it is closed.

4. Suppose $f\colon K \to \mathbb{R}^1$ is a continuous function defined on a compact set K with the property that $f(x) > 0$ for all $x \in K$. Show that there exists a number $c > 0$ such that $f(x) \geq c$ for all $x \in K$.

5. Let $f(x)$ be a continuous function on $[0, 1]$ which satisfies

$$\int_0^1 x^n f(x)\, dx = 0 \quad \text{for all} \quad n = 0, 1, \ldots$$

Prove that $f(x) = 0$ for all $x \in [0, 1]$.

6. Show that the Riemann integral $\int_0^\infty \frac{\sin x}{x}\, dx$ exists.

7. Let

$$G(x, y) = \begin{cases} x(1-y) & \text{if } 0 \le x \le y \le 1 \\ y(1-x) & \text{if } 0 \le y \le x \le 1 \end{cases}$$

Let $\{f_n(x)\}$ be a uniformly bounded sequence of continuous functions on $[0, 1]$ and consider the sequence

$$u_n(x) = \int_0^1 G(x, y) f_n(y)\, dy.$$

Show that the sequence $\{u_n(x)\}$ contains a uniformly convergent subsequence on $[0, 1]$.

8. Let f be a real–valued function defined on an open set $U \subset \mathbb{R}^2$ whose partial derivatives exist everywhere on U and are bounded. Show that f is continuous on U.

9. For $x \in \mathbb{R}^3$ consider spherical coordinates $x = r\omega$ where $|\omega| = 1$ and $|x| = r$. Let ω_k be the k'th component of ω for any $k = 1, 2, 3$. Use the divergence theorem to evaluate the surface integral

$$\int_{|\omega|=1} \omega_k\, dS.$$

10. Let $\{f_k\}$ be a sequence of continuous functions defined on $[a, b]$. Show that if $\{f_k\}$ converges uniformly on (a, b), then it also converges uniformly on $[a, b]$.

11. Let $f\colon \mathbb{R}^n \to \mathbb{R}^k$ be a continuous mapping. Show that $f(S)$ is bounded in $\mathbb{R}^k$ if S is a bounded set in $\mathbb{R}^n$.