**Name** ______________________________ **ID number** ________________________

**Analysis Qualifying Exam, Spring 2002, Indiana University**

**Instructions.** There are nine problems, each of equal value. Show your work, justifying all steps by direct calculation or by reference to an appropriate theorem. Good luck!

1. Let $a_0, a_1, ..., a_n$ be a set of real numbers satisfying

$$a_0 + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0.$$

Prove that the polynomial $P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$ has at least one root in $(0, 1)$.

2. Let $f_n : R \to R$ be differentiable, for all n, with derivative uniformly bounded (in absolute value) by 1. Further assume that $\lim_{n\to\infty} f_n(x) = g(x)$ exists for all $x \in R$. Prove that $g : R \to R$ is continuous.

3. Let $f : R^2 \to R$ have the property that for every $(x, y) \in R^2$, there exists *some* rectangular interval $[a, b] \times [c, d]$, $a < x < b$, $c < y < d$, on which $f$ is Riemann integrable. Show that $f$ is Riemann integrable on *any* rectangular interval $[e, f] \times [g, h]$.

4. Show that the sequence

$$1/2, (1/2)^{1/2}, ((1/2)^{1/2})^{1/2}, (((1/2)^{1/2})^{1/2})^{1/2}, \ldots$$

converges to a limit $L$, and determine this limit.

5. Let $f$, $g : R^2 \to R$ be functions with continuous first derivative such that the map $F : (x, y) \to (f, g)$ has Jacobian determinant

$$\det \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

identically equal to one. Show that $F$ is open, i.e., it takes open sets to open sets. If also $f$ is *linear* , i.e. $f_x$ and $f_y$ are constant, show that $F$ is one-to-one.

6. Let $f : (0, 1] \to R$ have continuous first derivative, with $f(1) = 1$ and $|f'(x)| \le x^{-1/2}$ if $|f(x)| \le 3$. Prove that $\lim_{x\to 0^+} f(x)$ exists.

7. Letting $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ denote the unit sphere in $R^3$, evaluate the surface integral

$$F = -\int\int_S P(x, y, z)\nu \, dA,$$

where $\nu(x, y, z) = (x, y, z)$ denotes the outward normal to $S$, $dA$ the standard surface element, and:

(a) $P(x, y, z) = P_0$, $P_0$ a constant.

(b) $P(x, y, z) = Gz$, $G$ a constant.

Remark (not needed for solution): $F$ corresponds to the total buoyant force exerted on the unit ball by an external, ideal fluid with pressure field $P$.

8. Compute the integral

$$\int_C y(z + 1)dx + xzdy + xydz,$$

where $C: \; x = \cos\theta, \; y = \sin\theta, \; z = \sin^3\theta + \cos^3\theta, \quad 0 \le \theta \le 2\pi.$

9. Let $X$ and $Y$ be metric spaces and $f : X \to Y$. If $\lim_{p\to x} f(p)$ exists for all $x \in X$, show that $g(x) = \lim_{p\to x} f(p)$ is continuous on $X$.

Tier 1 Analysis Examination – August, 2002

1. In the classical *false position* method to find roots of $f(x) = 0$, one begins with two approximations $x_0,\ x_1$ and generates a sequence of (hopefully) better approximations via

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_0}{f(x_n) - f(x_0)} \quad \text{for} \quad n = 1, 2, \ldots$$

Consider the following sketch in which the function $f(x)$ is to be increasing and convex:

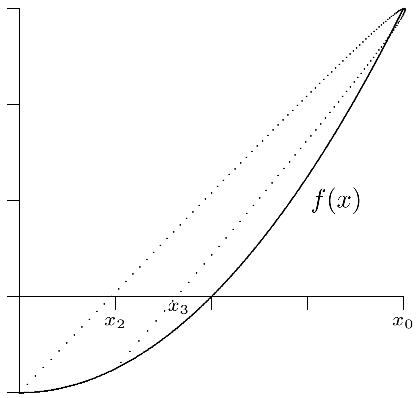


Fig. 1.2

The sequence $\{x_n\}$ is constructed as follows. We begin with the two approximations $(x_0, f(x_0))$ and $(x_1, f(x_1)) = (0, f(0))$ The chord is drawn between these two points; the point at which this chord crosses the $x$–axis is taken to be the next approximation $x_2$. One then draws the chord between the two points $(x_0, f(x_0))$ and $(x_2, f(x_2))$. The next approximation $x_3$ is that point where this chord crosses the axis, as shown. For $f$ strictly increasing and convex and for initial approximations $x_0 > 0$, $x_1 = 0$ with $f(x_0) > 0$, $f(x_1) < 0$, prove *rigorously* that this sequence must converge to the unique solution of $f(x) = 0$ over $[x_1, x_0]$.

2. (a) Show that it is possible to solve the equations

$$xu^2 + yzv + x^2z - 3 = 0$$
$$xyv^3 + 2zu - u^2v^2 - 2 = 0$$

for $(u, v)$ in terms of $(x, y, z)$ in a neighborhood of $(1, 1, 1, 1, 1)$.

(b) Given that the inverse of the matrix

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

find $\frac{\partial u}{\partial x}$ at $(1, 1, 1)$.

3. Let $X$ be a complete metric space and let $Y$ be a subspace of $X$. Prove that $Y$ is complete if and only if it is closed.

4. Suppose $f\colon K \to \mathbb{R}^1$ is a continuous function defined on a compact set $K$ with the property that $f(x) > 0$ for all $x \in K$. Show that there exists a number $c > 0$ such that $f(x) \geq c$ for all $x \in K$.

5. Let $f(x)$ be a continuous function on $[0, 1]$ which satisfies

$$\int_0^1 x^n f(x)\, dx = 0 \quad \text{for all} \quad n = 0, 1, \ldots$$

Prove that $f(x) = 0$ for all $x \in [0, 1]$.

6. Show that the Riemann integral $\int_0^\infty \frac{\sin x}{x}\, dx$ exists.

7. Let

$$G(x, y) = \begin{cases} x(1-y) & \text{if } 0 \le x \le y \le 1 \\ y(1-x) & \text{if } 0 \le y \le x \le 1 \end{cases}$$

Let $\{f_n(x)\}$ be a uniformly bounded sequence of continuous functions on $[0, 1]$ and consider the sequence

$$u_n(x) = \int_0^1 G(x, y) f_n(y)\, dy.$$

Show that the sequence $\{u_n(x)\}$ contains a uniformly convergent subsequence on $[0, 1]$.

8. Let $f$ be a real–valued function defined on an open set $U \subset \mathbb{R}^2$ whose partial derivatives exist everywhere on $U$ and are bounded. Show that $f$ is continuous on $U$.

9. For $x \in \mathbb{R}^3$ consider spherical coordinates $x = r\omega$ where $|\omega| = 1$ and $|x| = r$. Let $\omega_k$ be the $k$'th component of $\omega$ for any $k = 1, 2, 3$. Use the divergence theorem to evaluate the surface integral

$$\int_{|\omega|=1} \omega_k\, dS.$$

10. Let $\{f_k\}$ be a sequence of continuous functions defined on $[a, b]$. Show that if $\{f_k\}$ converges uniformly on $(a, b)$, then it also converges uniformly on $[a, b]$.

11. Let $f\colon \mathbb{R}^n \to \mathbb{R}^k$ be a continuous mapping. Show that $f(S)$ is bounded in $\mathbb{R}^k$ if $S$ is a bounded set in $\mathbb{R}^n$.