TIER I ANALYSIS EXAMINATION

August 24, 2001

NOTATION: For $x \in \mathbb{R}^n$, let $|x|$ denote the Euclidean norm of x (i.e. the Euclidean distance of x from the origin $\vec{0} \in \mathbb{R}^n$).

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that
$$\frac{\partial f}{\partial x_1}(\vec{0}) = \frac{\partial f}{\partial x_2}(\vec{0}) = 0 \, .$$
   (a) Does it follow that f differentiable at $x = \vec{0}$? Explain.
   (b) Prove or give a counterexample that f is continuous at $x = \vec{0}$.

2. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued, differentiable functions on the real line such that
   (a) $f_n(x) \to 0$ for each $x \in [0,1]$
   (b) $|f_n'(x)| \leq 1$ for all $x \in [0,1]$ and all $n = 1, 2, \ldots$.
   Show that the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[0,1]$ to 0.

3. Let $E \subset \mathbb{R}^n$ be nonempty. For $x \in \mathbb{R}^n$ define
$$D(x) \equiv \inf\{|x - y| : y \in E\} \, .$$
   (a) Show that D is a continuous function on $\mathbb{R}^n$ (under the usual topology on $\mathbb{R}$ and $\mathbb{R}^n$).
   (b) Show that $\{x \in \mathbb{R}^n : D(x) = |x|\}$ is closed in $\mathbb{R}^n$.

4. Let ω be a smooth 1-form on $\mathbb{R}^2$ that satisfies
$$\omega \wedge dx = -d(x^2) \wedge dy$$
   and
$$\omega \wedge dy = dx \wedge d(y^2).$$
   Let $\gamma : [0,1] \to \mathbb{R}^2$ be the differentiable path that joins $(0,0)$ to $(-1,4)$ given by $\gamma(t) = (-\sin \frac{\pi}{2} t, 4t^3)$ for $t \in [0,1]$. Compute $\int_\gamma \omega$.

5. a) Prove or provide a counterexample to the following statement: If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then there exists a real number L such that
$$\lim_{\epsilon \to 0} \int_{\epsilon \leq |x| \leq 1} \frac{f(x)}{x} \, dx = L$$
   b) What is your answer to part a) if there exist positive constants C and α such that for all $x \neq y \in \mathbb{R}$
$$|f(x) - f(y)| < C|x - y|^\alpha \, ?$$
   Again, prove that the limit exists or give a counterexmple.

6. Suppose that for each $j = 1, 2, \ldots, g_j : [0,1] \to \mathbb{R}$ is a continuous function such that $\int_0^1 |g_j(x)|dx \le 1000$. Suppose $h : [0,1] \to \mathbb{R}$ is a continuous function. Suppose that for each $n = 0, 1, 2, \ldots$,

$$\lim_{j\to\infty} \int_0^1 x^n g_j(x)dx = \int_0^1 x^n h(x)dx .$$

Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{j\to\infty} \int_0^1 f(x)g_j(x)dx = \int_0^1 f(x)h(x)dx .$$

7. Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f_1(x_1, x_2) = x_1$ and for $x_2 \ge 0$

$$f_2(x_1,x_2) = \begin{cases} x_2 - x_1^2 & \text{if } x_2 \ge x_1^2 \\ \frac{x_2^2}{x_1^2} - x_2, & \text{if } x_1^2 \ge x_2 > 0 \\ 0 & \text{if } x_2 = 0 \end{cases}$$

If $x_2 < 0$, define f_2 by $f_2(x_1, x_2) = -f_2(x_1, -x_2)$. This function f is differentiable at $\vec{0}$, and you may use this fact without proving it, whenever needed, below.

(a) Show that f is differentiable (at all points in $\mathbb{R}^2$). Show that $f'(\vec{0}) =$ identity.
(b) Prove that f is not one-to-one in any small neighborhood of the origin $\vec{0}$.
(c) State the inverse function theorem. In view of part b), the theorem does not apply to f near the origin $x = \vec{0}$. EXPLAIN. Explicitly what condition of the theorem is not met by the function f (at $\vec{0}$)?

8. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable.

(a) Assume that the Jacobian matrix $(\partial f_i/\partial x_j)$ has rank n everywhere. Prove that $f(\mathbb{R}^n)$ is open.
(b) Suppose that $f^{-1}(K)$ is compact whenever $K \subset \mathbb{R}^n$ is compact. Prove that $f(\mathbb{R}^n)$ is closed.
(c) Assume that the Jacobian matrix $(\partial f_i/\partial x_j)$ has rank n everywhere, and that $f^{-1}(K)$ is compact whenever $K \subset \mathbb{R}^n$ is compact. Prove that $f(\mathbb{R}^n) = \mathbb{R}^n$

9. Define the function $g : [0, \infty) \to \mathbb{R}$ as follows:

$$g(x) = \int_0^x \frac{\sin^3 u}{u} du .$$

Note that this integral is well defined, since $|\sin u| \le u$ for all $u > 0$. Prove that $\lim_{x\to\infty} g(x)$ exists in $\mathbb{R}$. (You don't have to find the limit.)