Tier 1 Exam

January, 2001

1. Compute

$$\int_S \operatorname{curl} F \cdot N \, dA \,,$$

where S is that part of the surface $y = x^2 + z^2$ in $\mathbb{R}^3$ for which $0 \le y \le 1$, N is the unit normal to S pointing toward the y-axis, dA is the area element, and $F: \mathbb{R}^3 \to \mathbb{R}^3$ is the mapping

$$F(x, y, z) = e^{x^2+z^2}(z, y, -x)\,.$$

2. Suppose that $f: [0, \infty) \to \mathbb{R}^1$ is a nonnegative, uniformly continuous function and that

$$\int_0^\infty f(x)\, dx < \infty.$$

Prove that $\lim_{x \to \infty} f(x) = 0$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}^1$ be continuous, and define

$$g(y) = \int_0^1 f(x, y) dx\ .$$

Assume that $\dfrac{\partial f}{\partial y}$ is continuous on $\mathbb{R}^2$, and compute $g'(y)$. Prove your result.

4. The function $\frac{1}{32}x^4 + x^2y^2 - x^3 - y^3 - xy^3$ has critical points at $(24, 0)$ and $(0, 0)$. By a careful analysis, determine whether each point is a local maximum, local minimum or a point which is neither a local maximum nor a local minimum.

5. Let $f: B \to \mathbb{R}^1$ be a uniformly continuous function, where $B \subset \mathbb{R}^n$ is an open ball. Prove that there is a uniformly continuous function F defined on the closure of B such that F restricted to B is equal to f.

6. Let $g: \mathbb{R}^2 \to \mathbb{R}^2$ be defined for points $\boldsymbol{x} = (x, y)$ by

$$g(x,y) = (x^2 + y^2 - |x^2 - y^2|, x^2 + y^2 + |x^2 - y^2|).$$

   (a) Give the definition of the differential of g at $\boldsymbol{x}_0$, denoted by $dg(\boldsymbol{x}_0)$.
   (b) Determine those points $\boldsymbol{x}_0 \in \mathbb{R}^2$ where $dg(\boldsymbol{x}_0)$ exists and where it does not exist. In both cases, justify your answer. Be sure to analyze the case $\boldsymbol{x}_0 = 0$.
   (c) Find those points $\boldsymbol{x}_0 \in \mathbb{R}^2$ where g locally has a differentiable inverse and where it does not. In both cases, justify your answer.

7. Let n be an integer greater than 1, and consider the following statement: If ω is a differential 2–form on $\mathbb{R}^n$ with the property that $\omega \wedge \lambda = 0$ for every differential 1–form λ, then ω must be the zero form. For what n is the above statement true? For what n is it false? Prove your answers.

8. (a) Let $F: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$F(x,y,z) = (z^2 + xy - 1, z^2 + x^2 - y^2 - 2)$$

   and observe that $F(\boldsymbol{a}) = (0,0)$ where $\boldsymbol{a} = (-1,0,1)$. Prove that there exist an open interval (a,b), a C^1 curve of the form $\gamma(t) = (f(t), g(t), h(t))$ with $a < t < b$, and an open set $U \subset \mathbb{R}^3$ containing $\boldsymbol{a}$ such that

$$U \cap F^{-1}(0,0) = \{\gamma(t) : a < t < b\}$$

   (b) Compute $\gamma'(t_0) = (f'(t_0), g'(t_0), h'(t_0))$ where $\gamma(t_0) = \boldsymbol{a}$.

9. Let $f: [0,1] \to \mathbb{R}^1$ be defined by

$$f(x) = \begin{cases} \frac{1}{2^k} & \text{if } \frac{1}{k+1} < x \le \frac{1}{k},\ k = 1, 2, \ldots \\ 0 & \text{for } x = 0\,. \end{cases}$$

   (a) For any given $\varepsilon > 0$, show how to construct a partition P of the interval $[0,1]$ such that

$$U(P,f) - L(P,f) < \varepsilon.$$

   ($U(P,f)$ and $L(P,f)$ are the upper and lower Riemann sums for f over the partition P).
   (b) Find an expression for

$$\int_0^1 f(x)dx.$$

   and justify your answer.